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Factors for $|A|_k$ summability of infinite series

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Abstract

In this paper we generalize Bor's result by using the correct definition of absolute summability.

Keywords: Absolute summability; Weighted mean matrix; Cesáro matrix; Summability factor

Recently Bor [1] generalized a result of Sulaiman [3]. Unfortunately he used an incorrect definition of absolute summability. In this paper we obtain the corresponding result for a lower triangular matrix using the correct definition (see, e.g. [2]). We obtain the correct form of [1] as a corollary.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence. Then we put

$$T_n := \sum_{\nu=0}^{n} t_{n\nu} s_\nu .$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{1}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\hat{T}$ defined as follows:

$$\bar{t}_{n\nu} = \sum_{r=\nu}^{n} t_{nr}, \quad n, \nu = 0, 1, 2, \ldots,$$

and

$$\hat{t}_{n\nu} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \quad n = 1, 2, 3, \ldots .$$

We may write

$$T_n = \sum_{\nu=0}^{n} a_{n\nu} \sum_{i=0}^{\nu} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{\nu=i}^{n} a_\nu = \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i .$$

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Thus

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} a_i \lambda_i \\
&= \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n} \bar{a}_{n-1,i} a_i \lambda_i \\
&= \sum_{i=0}^{n} (\bar{a}_{ni} - \bar{a}_{n-1,i}) a_i \lambda_i \\
&= \sum_{i=0}^{n} \hat{a}_{ni} a_i \lambda_i = \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i (s_i - s_{i-1}) \\
&= \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_i - \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + \hat{a}_{nn} \lambda_n s_n - \sum_{i=1}^{n} \hat{a}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni} \lambda_i s_i + a_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_{i+1} s_i \\
&= \sum_{i=1}^{n} (\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) s_i + a_{nn} \lambda_n s_n.
\end{aligned}$$

We may write

$$\begin{aligned}
(\hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1}) &= \hat{a}_{ni} \lambda_i - \hat{a}_{n,i+1} \lambda_{i+1} - \hat{a}_{n,i+1} \lambda_i + \hat{a}_{n,i+1} \lambda_i \\
&= (\hat{a}_{ni} - \hat{a}_{n,i+1}) \lambda_i + \hat{a}_{n,i+1} (\lambda_i - \lambda_{i+1}) \\
&= \lambda_i \Delta_i \hat{a}_{ni} + \hat{a}_{n,i+1} \Delta \lambda_i.
\end{aligned}$$

Therefore

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n-1} \Delta_i \hat{a}_{ni} \lambda_i s_i + \sum_{i=1}^{n-1} \hat{a}_{n,i+1} \Delta \lambda_i s_i + a_{nn} \lambda_n s_n \\
&= T_{n1} + T_{n2} + T_{n3}, \quad \text{say.}
\end{aligned}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

Theorem 1. *Let A be a lower triangular matrix with nonnegative entries satisfying*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \ldots,$
(ii) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$, *and*
(iii) $na_{nn} = O(1)$.

Let $\{X_n\}$ be given sequence of positive numbers and let $s_n = O(X_n)$ as $n \to \infty$. If $(\lambda_n)_{n \geq 0}$ is a sequence of complex numbers such that

(iv) $\sum_{n=1}^{\infty} a_{nn} (|\lambda_n| X_n)^k < \infty$, *and*
(v) $\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty$,

then the series $\sum a_n \lambda_n$ is summable $|A|_k, k \geq 1$.

Proof. To prove the theorem it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \quad \text{for } r = 1, 2, 3.$$

Using Hólder's inequality, (iii), and (iv),

$$I_1 := \sum_{n=1}^{m+1} n^{k-1}|T_{n1}|^k \le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i||s_i|\right)^k$$
$$\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i|^k(X_i)^k\right)\left(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}|\right)^{k-1}.$$

From (ii)

$$\Delta_i\hat{a}_{n\nu} = \hat{a}_{ni} - \hat{a}_{n,i+1}$$
$$= \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1}$$
$$= a_{ni} - a_{n-1,i} \le 0.$$

Thus, using (i),

$$\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}| = \sum_{i=0}^{n-1}(a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nn} = a_{nn}.$$

Using (iv),

$$I_1 := O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i|^k(X_i)^k$$
$$= O(1)\sum_{i=0}^{m}(X_i|\lambda_i|)^k\sum_{n=i+1}^{m+1}(na_{nn})^{k-1}|\Delta_i\hat{a}_{ni}|$$
$$= O(1)\sum_{i=0}^{m}a_{ii}(|\lambda_i|X_i)^k$$
$$= O(1).$$

By Hólder's inequality, (iii) and (v),

$$I_2 := \sum_{n=1}^{m+1} n^{k-1}|T_{n2}|^k \le \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{i=0}^{n-1}\hat{a}_{n,i+1}s_i\Delta\lambda_i\right|^k$$
$$\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i||s_i|\right)^k$$
$$\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i|X_i\right)\left(\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i|X_i\right)^{k-1}.$$

From the definition of $\hat{A}$ and $\bar{A}$, and using (i) and (ii),

$$\hat{a}_{n,i+1} = \bar{a}_{n,i+1} - \bar{a}_{n-1,i+1}$$
$$= \sum_{\nu=i+1}^{n} a_{n\nu} - \sum_{\nu=i+1}^{n-1} a_{n-1,\nu}$$
$$= 1 - \sum_{\nu=0}^{i} a_{n\nu} - 1 + \sum_{\nu=0}^{i} a_{n-1,\nu}$$
$$= \sum_{\nu=0}^{i}\left(a_{n-1,\nu} - a_{n,\nu}\right) \ge 0. \quad (2)$$

Using (i)

$$\begin{aligned}\hat{a}_{n,i+1} &= \sum_{\nu=0}^{i}\left(a_{n-1,\nu} - a_{n,\nu}\right)\\ &\le \sum_{\nu=0}^{n-1}\left(a_{n-1,\nu} - a_{n,\nu}\right)\\ &= 1 - 1 + a_{nn}. \end{aligned} \tag{3}$$

Therefore

$$\begin{aligned} I_2 &:= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=0}^{n-1}\hat{a}_{n,i+1}|\Delta\lambda_i|X_i\\ &= O(1)\sum_{i=1}^{m}|\Delta\lambda_i|X_i\sum_{n=i+1}^{m+1}(na_{nn})^{k-1}\hat{a}_{n,i+1},\\ &= O(1)\sum_{i=1}^{m}|\Delta\lambda_i|X_i\sum_{n=i+1}^{m+1}\hat{a}_{n,i+1}. \end{aligned}$$

From (2)

$$\begin{aligned}\sum_{n=i+1}^{m+1}\left(\sum_{\nu=0}^{i}(a_{n-1,\nu} - a_{n\nu})\right) &= \sum_{\nu=0}^{i}\sum_{n=i+1}^{m+1}(a_{n-1,\nu} - a_{n\nu})\\ &= \sum_{\nu=0}^{i}(a_{i,\nu} - a_{m+1,\nu})\\ &\le \sum_{\nu=0}^{i}a_{i,\nu} = 1. \end{aligned} \tag{4}$$

Hence

$$\begin{aligned} I_2 &:= O(1)\sum_{i=1}^{m}|\Delta\lambda_i|X_i,\\ &= O(1). \end{aligned}$$

Using (iii) and (iv),

$$\begin{aligned}\sum_{n=1}^{m+1}n^{k-1}|T_{n3}|^k &\le \sum_{n=1}^{m+1}n^{k-1}|a_{nn}\lambda_n s_n|^k\\ &= O(1)\sum_{n=1}^{m}(na_{nn})^{k-1}a_{nn}(|\lambda_n|X_n)^k\\ &= O(1)\sum_{n=1}^{m}a_{nn}(X_n|\lambda_n|)^k,\\ &= O(1). \quad \square \end{aligned}$$

Corollary 1. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfies*

(i) $np_n = O(P_n)$.
Let $\{X_n\}$ be given sequence of positive numbers and let $s_n = O(X_n)$ as $n \to \infty$. If $(\lambda_n)_{n\ge 0}$ is a sequence of complex numbers such that

(ii) $\sum_{n=1}^{\infty}\frac{p_n}{P_n}(|\lambda_n|X_n)^k < \infty$, *and*

(iii) $\sum_{n=1}^{\infty} X_n |\triangle \lambda_n| < \infty$,

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

Proof. Condition (iii) of Corollary 1 is condition (v) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automatically satisfied for any weighted mean method. Conditions (iii) and (iv) of Theorem 1 become, respectively, conditions (i) and (ii) of Corollary 1. □

It should be noted that, in [1], an incorrect definition of absolute summability was used. Corollary 1 gives the correct version of Bor's theorem.

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