# A Tutte decomposition for matrices and bimatroids 

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#### Abstract

We develop a Tutte decomposition theory for matrices and their combinatorial abstractions, bimatroids. As in the graph or matroid case, this theory is based on a deletion-contraction decomposition. The contribution from the deletion, derived by an inclusion-exclusion argument, consists of three terms. With one more term contributed from the contraction, the decomposition has four terms in general. There are universal decomposition invariants, one of them being a corank-nullity polynomial. Under a simple change of variables, the corank-nullity polynomial equals a weighted characteristic polynomial. This gives an analog of an identity of Tutte. Applications to counting and critical problems on matrices and graphs are given. © 2005 Elsevier Inc. All rights reserved.


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## 1. Counting non-singular minors

We begin with an example. Let $M[T \mid S]$ be a matrix over a field, with rows indexed by $T$ and columns indexed by $S$. Suppose we wish to find the number $h(M[T \mid S])$ of non-singular minors, that is, non-singular square submatrices (of any size), in $M[T \mid S]$. One approach is to imitate the deletion-contraction method used in matroid theory.

The definition of deletion is easy. If $b \in T$, then the deletion $M[T \backslash b \mid S]$ is the matrix obtained by deleting the row indexed by $b$. Similarly, $M[T \mid S \backslash a]$ is obtained by deleting the column indexed by $a$. Defining contraction is a little harder. Let $b \in T$ and $a \in S$ be a pair of a row and a column such that the ba-entry of $M[T \mid S]$ is non-zero. The contraction $M[T / b \mid S / a]$ is constructed in two steps. First, by row or column operations, reduce the

[^0]matrix $M[T \mid S]$ so that the $b a$-entry is 1 and all other entries in row $b$ are zero, or, all other entries in column $a$ are zero, or both. Second, delete both row $b$ and column $a$. (Because matrix multiplication is associative, the matrix obtained is uniquely determined.) The contraction $M[T / b \mid S / a]$ has the important property: a minor with rows $J$ and columns $I$, with $b \in J$ and $a \in I$, is non-singular if and only if the minor indexed by $J \backslash b$ and $I \backslash a$ is non-singular in $M[T / b \mid S / a]$.

Let $b, a$ be a row-column pair such that the $b a$-entry is non-zero. Then, by inclusionexclusion, the number of non-singular minors in $M[T \mid S]$ not containing the row $b$, not containing the column $a$, or not containing both row $b$ and column $a$, equals

$$
h(M[T \mid S \backslash a])+h(M[T \backslash b \mid S])-h(M[T \backslash b \mid S \backslash a])
$$

Since the number of non-singular minors containing both row $b$ and column $a$ equals $h(M[T / b \mid S / a])$, the numbers $h(M[T \mid S])$ satisfy the following decomposition:

$$
h(M[T \mid S])=h(M[T \mid S \backslash a])+h(M[T \backslash b \mid S])-h(M[T \backslash b \mid S \backslash a])+h(M[T / b \mid S / a])
$$

This decomposition is an analog of the two-term deletion-contraction decomposition for graphs, arrangements of hyperplanes, and matroids.

In this paper, we shall develop a Tutte decomposition theory for matrices based on fourterm deletion-contraction decompositions. As we shall see, the numerical or algebraic invariants of this decomposition theory depend only on the rank function on submatrices. In Section 2, we extract the essential properties of matrix rank functions and use them as the axioms for bimatroids. Bimatroids are the natural structures for this decomposition theory. So as not to deter prospective readers, the theory is developed for matrices, but it holds with no changes for bimatroids.

In Section 3, we define deletion-contraction invariants (or dc-invariants) for matrices and prove the fundamental result that the generating polynomial for pairs of subsets by corank and nullity is a universal dc-invariant. In the next section, we evaluate this polynomial to obtain specific dc-invariants. In Section 5, we define the characteristic polynomial for matrices and prove an analog of an identity of Tutte which allows one to obtain a coboundary or weighted characteristic polynomial from the corank-nullity polynomial. In Section 6, we apply our theory to critical problems on matrices and show that Rédei functions of matrices can be derived algebraically from the corank-nullity polynomial. Concrete examples of critical problems on matrices include finding vertex covers, edge covers, and perfect matchings in graphs. Our theory gives inclusion-exclusion formulas for the number of vertex covers, edge covers, and perfect matchings. In addition, we show that the stable set generating polynomial and the rank generating function of a 2-polymatroid defined by Oxley and Whittle [15] are derivable from the corank-nullity polynomial of the vertex-edge incidence matrix of a graph. These connections with graph theory are presented in Section 7. An earlier version of Sections 2 and 3 appeared in the author's doctoral thesis [8].

Except for Section 5, an elementary knowledge of matroid theory (see, for example, $[6,14,22]$ ) suffices to read this paper. However, some knowledge of Tutte polynomials and critical problems (see $[3,11]$ ) will be helpful as motivation.

## 2. Matrices and bimatroids

Let $M[T \mid S]$ be a matrix. The matrix $M$ defines a rank function rk : $2^{T} \times 2^{S} \rightarrow$ $\{0,1,2, \ldots\}$ as follows: if $B \subseteq T$ and $A \subseteq S$, then $\operatorname{rk}(B, A)$ is the rank of the submatrix $M[B \mid A]$ with rows $B$ and columns $A$. The rank function satisfies three properties:

R1. Normalization: $\operatorname{rk}(B, \emptyset)=\operatorname{rk}(\emptyset, A)=0$.
R2. Unit increase: $\operatorname{rk}(B \cup\{b\}, A)=\operatorname{rk}(B, A)+i, i=0$ or 1 , $\operatorname{and} \operatorname{rk}(B, A \cup\{a\})=$ $\operatorname{rk}(B, A)+i, i=0$ or 1 .

R3. Submodularity:

$$
\operatorname{rk}(B, A)+\operatorname{rk}(D, C) \geqslant \operatorname{rk}(B \cup D, A \cap C)+\operatorname{rk}(B \cap D, A \cup C)
$$

In analogy with the rank-function axiomatization of matroids, a bimatroid $M[T \mid S]$ on the row set $T$ and column set $S$ can be specified by a rank function rk satisfying axioms R1, R2, and R3. Other axiomatizations exist (see, $[9,16]$ ). For our decomposition theory, there is no essential difference between matrices and bimatroids. When described in terms of rank functions, the theory is valid for both.

Two matrices $M[T \mid S]$ and $M^{\prime}\left[T^{\prime} \mid S^{\prime}\right]$ are isomorphic if they are the same up to a relabeling of the row and columns, in other words, if there exist bijections $\pi: T \rightarrow T^{\prime}$ and $\sigma: S \rightarrow S^{\prime}$ so that row $b$ in $M$ equals row $\pi(b)$ in $M^{\prime}$ and column $a$ in $M$ equal $\sigma(a)$ in $M^{\prime}$. Two bimatroids $M[T \mid S]$ and $M^{\prime}\left[T^{\prime} \mid S^{\prime}\right]$ with rank functions rk and $\mathrm{rk}^{\prime}$ are isomorphic if there exist bijections $\pi: T \rightarrow T^{\prime}$ and $\sigma: S \rightarrow S^{\prime}$ so that $\operatorname{rk}(B, A)=\operatorname{rk}^{\prime}(\pi(B), \sigma(A))$ for all subsets $B \subseteq T$ and $A \subseteq S$.

A matrix or bimatroid $M[T \mid S]$ determines two families of matroids. Given a row set $U$, the column matroid $G[U]$ is the matroid on the column set $S$ defined by the rank function: for $A \subseteq S$

$$
\operatorname{rk}(A)=\operatorname{rk}(U, A)
$$

The matroid $G[T]$ is called the full column matroid. The row matroids $H[X]$ are defined similarly. In particular, the rank of the row subset $B \subseteq T$ in the row matroid $H[X]$ is $\operatorname{rk}(B, X)$. The column matroids depend heavily on the closure in the full row matroid $H[S]$.
2.1. Lemma. If $\bar{U}$ is the closure of $U$ in the full row matroid $H[S]$, then the column matroids $G[U]$ and $G[\bar{U}]$ are equal. In addition, if $U \subseteq V \subseteq T$, then $G[U]$ is a quotient or strong map image of $G[V]$, that is to say, for every subset $A$ in $S$, the closure of $A$ in $G[U]$ contains the closure of $A$ in $G[V]$.

A similar result holds for row matroids. A proof of Lemma 2.1 can be found in [9]. If $M$ comes from a matrix, then this lemma says that adding dependent rows does not change the rank function on columns, and deleting rows gives a linear transformation on the column vectors.

Small matrices and the bimatroids they define play essential roles in decomposition theory. The smallest matrix is the $0 \times 0$ matrix. We shall denote it by o. Next in line are the $1 \times 0$ matrix $\uparrow$, and its transpose, the $0 \times 1$ matrix $\rightarrow$. The matrices $\uparrow$ and $\rightarrow$ are defined only up to isomorphism.

A row-column pair $b, a$ is said to be typical $\operatorname{if} \operatorname{rk}(\{b\},\{a\})=1$ (or, equivalently for matrices, the $b a$-entry is non-zero). The operations of deletion and contraction for matrices were defined in Section 1. The rank functions of deletions of a bimatroid are restrictions of its rank function. The contraction $M[T / b \mid S / a]$ is defined only when the row-column pair $b, a$ is typical. The rank function of the contraction $M[T / b \mid S / a]$ is given by the formula: for subsets $B \subseteq T \backslash b$ and $A \subseteq S \backslash a$,

$$
\operatorname{rk}_{M[T / b \mid S / a]}(B, A)=\operatorname{rk}_{M}(B \cup\{b\}, A \cup\{a\})-1 .
$$

The direct sum $M[S \mid T] \oplus M^{\prime}\left[S^{\prime} \mid T^{\prime}\right]$ of two bimatroids (with $S \cap S^{\prime}=\emptyset$ and $T \cap T^{\prime}=\emptyset$ ) is the bimatroid on the row set $T \cup T^{\prime}$ and the column set $S \cup S^{\prime}$ with rank function defined by

$$
\mathrm{rk}_{M \oplus M^{\prime}}(B, A)=\mathrm{rk}_{M}(B \cap T, A \cap S)+\mathrm{rk}_{M^{\prime}}\left(B \cap T^{\prime}, A \cap S^{\prime}\right)
$$

For matrices $M$ and $M^{\prime}$, the direct sum is the matrix

$$
\left(\begin{array}{cc}
M & 0 \\
0 & M^{\prime}
\end{array}\right) .
$$

For example, if [0] is the $1 \times 1$ zero matrix,

$$
\begin{equation*}
[0]=\uparrow \oplus \rightarrow . \tag{1}
\end{equation*}
$$

A row $b$ in a matrix $M[T \mid S]$ is a zero row if $\operatorname{rk}(\{b\}, S)=0$, or, equivalently, every entry in the row $b$ is zero. A column $a$ in $M[T \mid S]$ is a zero column if $\operatorname{rk}(T,\{a\})=0$. If $b$ is a zero row in $M[T \mid S]$, then

$$
M[T \mid S]=\uparrow \oplus M[T \backslash b \mid S]
$$

where the row in the $1 \times 0$ matrix $\uparrow$ is indexed by $b$. A similar decomposition holds for zero columns.

A matrix or bimatroid $M^{\prime}$ is a deletion-contraction minor or dc-minor of $M$ if $M^{\prime}$ can be obtained from $M$ by a sequence of deletions or contractions. A class of matrices or bimatroids is closed under deletions and contractions or dc-closed if it is closed under isomorphisms and taking dc-minors. For example, the class of all matrices with entries in a fixed field is dc-closed. However, the class of vertex-edge incidence matrices of graphs over GF(2) is not dc-closed, but the bigger class of all GF(2)-matrices in which every column has at most two non-zero entries is dc-closed. The notion of orthogonal duality does not go over well into matrices or bimatroids. Possible analogs include the transpose (obtained by switching rows and columns) and the inverse (which works only for square non-singular matrices and has a combinatorial definition using an identity of Jacobi).

If $G$ is a rank- $r$ matroid on the set $S$ with rank function rk and $X \subseteq S$, the characteristic polynomial $\chi(G / X ; \lambda)$ of the contraction $G / X$ is the polynomial in the variable $\lambda$ defined by the formula

$$
\begin{equation*}
\chi(G / X ; \lambda)=(-1)^{|X|} \sum_{A: X \subseteq A \subseteq S}(-1)^{|A|} \lambda^{r-\mathrm{rk}(A)} \tag{2}
\end{equation*}
$$

We shall use the following known result.
2.2. Lemma. Let $G$ be a rank-r matroid. The characteristic polynomial $\chi(G ; \lambda)$ is a (nonzero) polynomial of degree $r$ if and only if $G$ has no loops. It is identically zero if $G$ has loops. In particular, $\chi(G / X ; \lambda)$ is non-zero if and only if $X$ is closed in $G$.

## 3. The corank-nullity polynomial

If $b, a$ is a row-column pair, we define the integers $i, j, l$ as follows:

$$
\begin{aligned}
& \operatorname{rk}(T \backslash b, S)=\operatorname{rk}(T, S)-i, \\
& \operatorname{rk}(T, S \backslash a)=\operatorname{rk}(T, S)-j, \\
& \operatorname{rk}(T \backslash b, S \backslash a)=\operatorname{rk}(T, S)-l .
\end{aligned}
$$

The integers $i$ and $j$ can be 0 or 1 . The integer $l$ can be 0,1 , or 2 .
A function $f$ defined on matrices taking values in a ring is said to be a deletion-contraction invariant or dc-invariant with parameter $\phi$ if $f(\circ)=1$ and $f$ satisfies the following axioms:

T0. Isomorphism invariance: If $M$ and $M^{\prime}$ are isomorphic, then $f(M)=f\left(M^{\prime}\right)$.
T1. The deletion-contraction decomposition: If $b, a$ is a typical row-column pair in a matrix $M[T \mid S]$, then

$$
\begin{aligned}
f(M[T \mid S])= & \phi^{i} f(M[T \backslash b \mid S])+\phi^{j} f(M[T \mid S \backslash a])-\phi^{l} f(M[T \backslash b \mid S \backslash a]) \\
& +f(M[T / b \mid S / a])
\end{aligned}
$$

T2. The zero-row or zero-column decomposition: If $b$ is a zero row in $M[T \mid S]$, then

$$
f(M[T \mid S])=f(\uparrow) f(M[T \backslash b \mid S])
$$

and if $a$ is a zero column in $M[T \mid S]$, then

$$
f(M[T \mid S])=f(\rightarrow) f(M[T \mid S \backslash a]) .
$$

For example, using T 1 on the first row and the first column, and then using T 0 and T 2 , we have

$$
\begin{aligned}
f\left(\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right]\right) & =\phi f([10])+\phi f\left(\left[\begin{array}{l}
1 \\
0
\end{array}\right]\right)-\phi^{2} f([0])+f([-1]) \\
& =\phi f([1])(f(\rightarrow)+f(\uparrow))-\phi^{2} f(\rightarrow) f(\uparrow)+f([1])
\end{aligned}
$$

The corank-nullity polynomial $R(M ; \lambda, x, y)$ of the rank- $r$ matrix $M[T \mid S]$ is the polynomial in three variables $\lambda, y$, and $x$ defined by

$$
R(M ; \lambda, y, x)=\sum_{B, A: B \subseteq T, A \subseteq S} \lambda^{r-\mathrm{rk}(B, A)} y^{|B|-\mathrm{rk}(B, A)} x^{|A|-\mathrm{rk}(B, A)} .
$$

As was noted in [3], there are disadvantages in using the term "corank-nullity polynomial", but we do not see a better alternative.

It is easy to check that the corank-nullity polynomial $R(M ; \lambda, y, x)$ is a dc-invariant with parameter $\lambda$ and

$$
R(\uparrow ; \lambda, y, x)=y+1, \quad R(\rightarrow ; \lambda, y, x)=x+1
$$

The next theorem says that the corank-nullity polynomial is a "universal" dc-invariant. It implies that the value of a dc-invariant on a matrix depends only on the rank function or bimatroid structure of the matrix.
3.1. Theorem. The corank-nullity polynomial is a dc-invariant with parameter $\lambda$. Every dc-invariant is an evaluation of the corank-nullity polynomial in the following way: iff is a dc-invariant with parameter $\phi$, then

$$
\begin{equation*}
f(M)=R(M ; \phi, f(\uparrow)-1, f(\rightarrow)-1) . \tag{3}
\end{equation*}
$$

The proof is an easy induction, using the fact that both sides of Eq. (3) agree on $\rightarrow$ and $\uparrow$ by T0 and satisfy decompositions T1 and T2 with the same parameter $\phi$.
3.2. Corollary. [The product formula] Let f be a dc-invariant. Then

$$
f\left(M[T \mid S] \oplus M\left[T^{\prime} \mid S^{\prime}\right]\right)=f(M[T \mid S]) f\left(M\left[T^{\prime} \mid S^{\prime}\right]\right)
$$

One proof is to show that the product formula holds for the corank-nullity polynomial, and hence, by universality, for all dc-invariants. The second is by induction on the size of one of the summands.

By Eq. (1) and Corollary 3.2,

$$
R([0] ; \lambda, y, x)=(y+1)(x+1)
$$

The $1 \times 1$ matrix [1] is not a direct sum. Its corank-nullity polynomial is

$$
\lambda+\lambda y+\lambda x+1
$$

A row-column pair $b, a$ such that

$$
\operatorname{rk}(B, A)+1=\operatorname{rk}(B \cup\{b\}, A \cup\{a\})
$$

for all subsets $B \subseteq T \backslash b$ and $A \subseteq S \backslash a$ or, equivalently, the $b a$-entry is the only non-zero entry in both the row $b$ and the column $a$, is said to be a matrix (or bimatroid) isthmus. Matrix isthmuses are direct summands. Hence, if the pair $b, a$ is a matrix isthmus,

$$
R(M[S \mid T] ; \lambda, y, x)=(\lambda+\lambda y+\lambda x+1) R(M[T \backslash b \mid S \backslash a] ; \lambda, y, x)
$$

This equation suggests that the matrix [1] plays the same role as the isthmus in the Tutte decomposition theory for matroids. Can the parameter $\phi$ for a dc-invariant $f$ be replaced by the value $f([1])$ ? By T1, the parameter $\phi$ is related to the value $f([1])$ by the equation

$$
\begin{equation*}
\phi(f(\rightarrow)+f(\uparrow)-1)=f([1])-1 \tag{4}
\end{equation*}
$$

Over a field, this equation can be solved uniquely for $\phi$ if $f([1]) \neq 1$ or when $f([1])=1$ and $f(\rightarrow)+f(\uparrow)-1 \neq 0$. However, in the truly exceptional case when both conditions fail, $\phi$ is not determined by Eq. (4) and it is necessary to specify it directly.

As in traditional Tutte decomposition theory (see, for example, [3,17]), we can set up a Tutte-Grothendieck ring over a category of matrices or bimatroids using versions of T1 and T 2 as exact sequences. When all is said and done, this amounts to suppressing the function $f$ by writing $M[T \mid S]$ instead of $f(M[T \mid S])$ when using $\mathrm{T} 0, \mathrm{~T} 1$, and T 2 . Theorem 3.1 says that the Tutte-Grothendieck ring is isomorphic to the polynomial ring $\mathbb{C}[\lambda, Y, X]$, where $Y$ is $\uparrow$ and $X$ is $\rightarrow$. In the Tutte-Grothendieck ring, the matrix $M[T \mid S]$ equals $R(M ; \lambda, Y-$ $1, X-1$ ).

We end by remarking that a bimatroid "links" the row and column matroids. A linking polynomial between two matroids on the same set has been studied in [21]. There seems to be no deep connection between the linking polynomial and the corank-nullity polynomial.

## 4. Some deletion-contraction invariants

We begin with several simple evaluations of the corank-nullity polynomial. Let $M$ be a rank- $r$ matrix. Then

$$
R\left(M ;(y x)^{-1}, y, x\right)=(y x)^{-r}(1+y)^{|T|}(1+x)^{|S|}
$$

and, in particular, $R(M ; 1,1,1)$ equals $2^{|T|+|S|}$, the number of submatrices of $M$. In addition, $R(M ; 0,1,1)$ equals the number of submatrices having maximum rank $r$. Let $i_{k}(M)$ be the number of $k \times k$ non-singular minors, that is, $k \times k$ submatrices having rank $k$. Then

$$
R(M ; \lambda, 0,0)=\sum_{k=0}^{r} i_{k}(M) \lambda^{r-k}
$$

In particular, the number $h(M)$ of non-singular minors of any size equals $R(M ; 1,0,0)$ and the number $i_{r}(M)$ of maximum-rank non-singular minors equals $R(M ; 0,0,0)$.

The next dc-invariant is inspired by reliability theory (see [4] or [3, Example 6.2.7 and Section 6.3.E].) Let $M$ be a rank- $r$ matrix. Suppose that a "random" submatrix is obtained by deleting a row with probability $1-q$ or a column with probability $1-p$, independently of other rows or columns. Suppose in addition that $0<p<1$ and $0<q<1$. Let $\operatorname{Pr}(M)$ be the probability that a random submatrix has the same rank $r$ as the original matrix $M$.

### 4.1. Theorem.

$$
\operatorname{Pr}(M)=(1-q)^{|T|-r} q^{r}(1-p)^{|S|-r} p^{r} R(M ; 0, q /(1-q), p /(1-p))
$$

Proof. We begin by deriving decompositions for $\operatorname{Pr}(M[T \mid S])$. Let $b, a$ be a typical rowcolumn pair. If $\operatorname{rk}(T, S)=\operatorname{rk}(T \backslash b, S \backslash a)$, then by conditioning on whether the pair $b, a$ is chosen,

$$
\begin{aligned}
\operatorname{Pr}(M[T \mid S])= & \operatorname{Pr}(M[T \backslash b \mid S])(1-q)+\operatorname{Pr}(M[T \mid S \backslash a])(1-p) \\
& -\operatorname{Pr}(M[T \backslash b \mid S \backslash a])(1-q)(1-p)+\operatorname{Pr}(M[T / b \mid S / a]) q p .
\end{aligned}
$$

On the other hand, if $\operatorname{rk}(T, S)>\operatorname{rk}(T \backslash b, S \backslash a)$, then

$$
\operatorname{Pr}(M[T \mid S])=\operatorname{Pr}(M[T / b \mid S / a]) q p .
$$

Further, if $b$ is a zero row,

$$
\operatorname{Pr}(M[T \mid S])=\operatorname{Pr}(M[T \backslash b \mid S]),
$$

while if $a$ is a zero column,

$$
\operatorname{Pr}(M[T \mid S])=\operatorname{Pr}(M[T \mid S \backslash a]) .
$$

From this, we conclude that

$$
\frac{\operatorname{Pr}(M[T \mid S])}{(1-q)^{|T|-\operatorname{rk}(T, S)} q^{\mathrm{rk}(T, S)}(1-p)^{|S|-\operatorname{rk}(T, S)} p^{\mathrm{rk}(T, S)}}
$$

is a dc-invariant with parameter 0 . Since $\operatorname{Pr}(\circ)=1, \operatorname{Pr}(\rightarrow)=\operatorname{Pr}(\uparrow)=1$, we conclude that

$$
\frac{\operatorname{Pr}(M)}{(1-q)^{|T|-r} q^{r}(1-p)^{|S|-r} p^{r}}=R\left(M ; 0, \frac{q}{1-q}, \frac{p}{1-p}\right) .
$$

This completes the proof.
We note that when $q=1$, then every row is retained with probability 1 . This case is precisely Example 6.2.7 on [3, p. 128].

## 5. Tutte's identity

In this section, we prove an identity for the corank-nullity polynomial inspired by an identity of Tutte. We begin by defining characteristic polynomials of matrices in analogy with Eq. (2). Let $M[T \mid S]$ be a rank- $r$ matrix and let $U \subseteq T$ and $X \subseteq S$. The characteristic polynomial $\chi(M, U, X ; \lambda)$ (based at the subsets $U$ and $X$ ) of the matrix $M$ is the polynomial defined by the formula

$$
\chi(M, U, X ; \lambda)=(-1)^{|U|+|X|} \sum_{B, A: U \subseteq B \subseteq T, X \subseteq A \subseteq S}(-1)^{|A|+|B|} \lambda^{r-\mathrm{rk}(B, A)} .
$$

When $U=X=\emptyset$, the characteristic polynomial is a dc-invariant. Indeed,

$$
\chi(M, \emptyset, \emptyset ; \lambda)=R(M ; \lambda,-1,-1)
$$

5.1. Lemma. The characteristic polynomial $\chi(M, U, X ; \lambda)$ is identically zero unless $U$ is a closed set in the full row matroid $H[S]$ and $X$ is a closed set in the full column matroid $G[T]$. If $U$ and $X$ are closed, then

$$
\begin{aligned}
\chi(M, U, X ; \lambda) & =\sum_{V \in L(H / U)} \mu_{L(H)}(U, V) \lambda^{r-\mathrm{rk}(V, S)} \chi(G[V] / X ; \lambda) \\
& =\sum_{Y \in L(G / X)} \mu_{L(G)}(X, Y) \lambda^{r-\mathrm{rk}(T, X)} \chi(H[Y] / U ; \lambda),
\end{aligned}
$$

where $\mu_{L(H)}$ and $\mu_{L(G)}$ are Möbius functions in the lattices $L(H)$ and $L(G)$ of flats of the full row and column matroids.

Proof. Summing over all subsets $A$ in $S$, we have

$$
\begin{aligned}
& \chi(M, U, X ; \lambda) \\
& \quad=\sum_{B: U \subseteq B \subseteq T}(-1)^{|B \backslash U|} \lambda^{r-\mathrm{rk}(B, S)}\left(\sum_{A: X \subseteq A \subseteq S}(-1)^{|A \backslash X|} \lambda^{\mathrm{rk}(B, S)-\mathrm{rk}(B, A)}\right) \\
& \quad=\sum_{B: U \subseteq B \subseteq T}(-1)^{|B \backslash U|} \lambda^{r-\mathrm{rk}(B, S)} \chi(G[B] / X ; \lambda) .
\end{aligned}
$$

If $X$ is not closed in the full column matroid $G[T]$, then $X$ is not closed in $G[U]$ for any subset $U \subseteq T$ (by Lemma 2.3) and $G[U] / X$ contains as loops those columns in $\bar{X} \backslash X$. Thus, all the characteristic polynomials $\chi(G[U] / X ; \lambda)$ are identically zero and hence, $\chi(M, U, X ; \lambda)$ is identically zero. Switching rows and columns in the above argument, we conclude that if $U$ is not closed in the full row matroid $H[S]$, then $\chi(M, U, X ; \lambda)$ is identically zero.

We will now suppose that both $U$ and $X$ are closed. By Lemma 2.1, we can group terms in the preceding equation to obtain

$$
\chi(M, U, X ; \lambda)=\sum_{V \in L(H / U)}\left(\sum_{B: \bar{B}=V}(-1)^{|B \backslash U|}\right) \lambda^{r-\mathrm{rk}(V, S)} \chi(G[V] / X ; \lambda) .
$$

Observing that

$$
\mu_{L(H)}(U, V)=\sum_{B: \bar{B}=V}(-1)^{|B \backslash U|},
$$

we obtain the first half of the equation in Lemma 5.1. The second half is obtained by switching rows and columns in the above argument.

### 5.2. Lemma. Let $B \subseteq T$ and $A \subseteq S$. Then

$$
\begin{equation*}
\sum_{U, X: B \subseteq U, A \subseteq X} \chi(M, U, X ; \lambda)=\lambda^{r-\mathrm{rk}(B, A)}, \tag{5}
\end{equation*}
$$

the sum ranging over all subsets $U, B \subseteq U \subseteq T$, and $X, A \subseteq X \subseteq S$. The equation remains valid if the sum ranges over all flats $U$ in $L(H)$ containing $B$ and $X$ in $L(G)$ containing $A$.

Proof. The final statement follows from Lemma 5.1. To show the main assertion, consider the left-hand side of Eq. (5). It equals

$$
\sum_{U, X: B \subseteq U, A \subseteq X}\left(\sum_{D, C: U \subseteq D, X \subseteq C}(-1)^{|D|+|U|+|C|+|X|} \lambda^{r-\mathrm{rk}(D, C)}\right)
$$

Changing the order of summation, we obtain

$$
\begin{equation*}
\sum_{D, C: B \subseteq D, A \subseteq C}\left(\sum_{U, X: B \subseteq U \subseteq D, A \subseteq X \subseteq C}(-1)^{|D|+|U|+|C|+|X|}\right) \lambda^{r-\mathrm{rk}(D, C)} . \tag{6}
\end{equation*}
$$

The inner sum factors into the product

$$
\left((-1)^{|D|+|B|} \sum_{U: B \subseteq U \subseteq D}(-1)^{|U \backslash B|}\right)\left((-1)^{|C|+|A|} \sum_{X: A \subseteq X \subseteq C}(-1)^{|X \backslash A|}\right)
$$

By the binomial theorem, the inner sum equals 1 when both $D=B$ and $C=A$, and 0 in all other cases. Hence, the sum (6) has exactly one non-zero term, $\lambda^{r-\operatorname{rk}(B, A)}$, the right-side of Eq. (5).

Let $M[T \mid S]$ be a rank- $r$ matrix and let $\left\{y_{b}: b \in T\right\}$ and $\left\{x_{a}: a \in S\right\}$ be two sets of variables. The corank-subset polynomial $\mathbf{R}\left(M ; \lambda, x_{a}, y_{b}\right)$ of $M$ is the polynomial in the variables $\lambda, x_{a}$, and $y_{b}$ defined by the formula

$$
\mathbf{R}\left(M ; \lambda, y_{b}, x_{a}\right)=\sum_{B, A: B \subseteq T, A \subseteq S} \lambda^{r-\mathrm{rk}(B, A)}\left(\prod_{b \in B} y_{b}\right)\left(\prod_{a \in A} x_{a}\right) .
$$

The corank-subset polynomial and the corank-nullity polynomial are related by the equation

$$
\mathbf{R}(M ; \lambda, y, x)=(y x)^{r} R(M ; \lambda / y x, y, x),
$$

where the left-hand side is obtained by substituting $y_{b}=y$ and $x_{a}=x$ in the coranksubset polynomial. We remark that generic versions of decompositions T1 and T2, as well as Theorem 4.1, hold for the corank-subset polynomial.

The following theorem is a generic version of an identity of Tutte (see [18,5,3, Section 6.3.F], and particularly, the last section of [13] dealing with Tugger polynomials).

### 5.3. Theorem.

$$
\mathbf{R}\left(M ; \lambda, y_{b}-1, x_{a}-1\right)=\sum_{U, X: U \in L(H), X \in L(G)} \chi(M, U, X ; \lambda)\left(\prod_{b \in U} y_{b}\right)\left(\prod_{a \in X} x_{a}\right)
$$

Proof. We use the following elementary identity:

$$
\begin{equation*}
\left(\prod_{b \in U} y_{b}\right)\left(\prod_{a \in X} x_{a}\right)=\sum_{B, A: B \subseteq U, A \subseteq X}\left(\prod_{b \in B}\left(y_{b}-1\right)\right)\left(\prod_{a \in A}\left(x_{a}-1\right)\right) . \tag{7}
\end{equation*}
$$

Consider the right-hand side of the identity in Theorem 5.3. Using Eq. (7) and Lemma 5.2, and changing the order of summation, we have

$$
\begin{aligned}
& \sum_{U, X: U \subseteq T, X \subseteq S} \chi(M, U, X ; \lambda)\left(\prod_{b \in U} y_{b}\right)\left(\prod_{a \in X} x_{a}\right) \\
= & \sum_{U, X: U \subseteq T, X \subseteq S} \chi(M, U, X ; \lambda)\left(\sum_{B: B \subseteq U} \prod_{b \in U}\left(y_{b}-1\right)\right)\left(\sum_{A: A \subseteq X} \prod_{a \in X}\left(x_{a}-1\right)\right) \\
= & \sum_{B, A: B \subseteq T, A \subseteq S}\left(\prod_{b \in U}\left(y_{b}-1\right) \prod_{a \in X}\left(x_{a}-1\right)\right)_{U, X: B \subseteq U, A \subseteq X} \chi(M, U, X ; \lambda)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{B, A: B \subseteq T, A \subseteq S}\left(\prod_{b \in U}\left(y_{b}-1\right) \prod_{a \in X}\left(x_{a}-1\right)\right) \lambda^{r-\mathrm{rk}(B, A)} \\
& =\mathbf{R}\left(M ; \lambda, y_{b}-1, x_{a}-1\right) .
\end{aligned}
$$

### 5.4. Corollary. Let $M$ be a rank-r matrix. Then

$$
\begin{aligned}
& (y-1)^{r}(x-1)^{r} R(M ; \lambda /(y-1)(x-1), y-1, x-1) \\
& \quad=\sum_{U, X: U \in L(H), X \in L(G)} \chi(M, U, X ; \lambda) y^{|U|} x^{|X|} .
\end{aligned}
$$

Note that as a polynomial in the variable $y$, the "weighted characteristic polynomial" in Corollary 5.4 has degree $|T|$ and its leading coefficient is the weighted characteristic polynomial of the full column matroid.

## 6. The critical problem for matrices

Let $M[T \mid S]$ be a matrix. An $s$-tuple $\left(b_{1}, b_{2}, \ldots, b_{s}\right)$ of rows distinguishes the column set $S$ if for every column $a$, there exists a row $b_{i}$ such that $\operatorname{rk}\left(\left\{b_{i}\right\},\{a\}\right)=1$, or, equivalently, the $b_{i} a$-entry is non-zero. A set $B$ of rows distinguishes $S$ if it distinguishes $S$ when arranged in some order as a $|B|$-tuple. The critical exponent $c(M)$ is the smallest positive integer $s$ such that there exists an $s$-tuple of rows in $M$ distinguishing $S$ if such an $s$-tuple exists; otherwise, it is defined to be infinite. The latter case occurs if and only if $M$ contains a zero column.

One way to find the critical exponent is to find the number $\zeta(M ; s)$ of $s$-tuples distinguishing $S$ for each positive integer $s$. The function $\zeta(M ; s)$ is called the Rédei function of $M$. If $A \subseteq S$, let the orthogonal $A^{\perp}$ of $A$ be the subset of zero rows in the submatrix $M[T \mid A]$. By deletion-contraction or Möbius inversion (see [10,11, Theorem 3.1]), one can show that

$$
\zeta(M ; s)=\sum_{A: A \subseteq S}(-1)^{|A|}\left|A^{\perp}\right|^{s}
$$

The Rédei function is a Dirichlet polynomial (in the variable $s$ ) rather than an ordinary polynomial. Closely related to the Rédei function is the function $\xi(M ; s)$ whose value at the non-negative integer $s$ is defined to be the number of $s$-element subsets distinguishing $S$. One can prove that

$$
\xi(M ; s)=\sum_{A: A \subseteq S}(-1)^{|A|}\binom{\left|A^{\perp}\right|}{s}
$$

For more on Rédei functions, see [10].
6.1. Theorem. The functions $\zeta(M ; s)$ and $\xi(M ; s)$ (and hence, the critical exponent $c(M)$ ) can be derived from the corank-nullity polynomial of $M$.

Proof. Let

$$
\begin{aligned}
Z(M ; \lambda, y, x) & =\lambda^{r} y^{r} x^{r} R\left(M ;(\lambda y x)^{-1}, y, x\right) \\
& =\sum_{B, A: B \subseteq T, A \subseteq S} \lambda^{\mathrm{rk}(B, A)} y^{|B|} x^{|A|} .
\end{aligned}
$$

Then,

$$
Z(M ; 0, y-1, x)=\sum_{A: A \subseteq S} x^{|A|}\left(\sum_{B: B \subseteq T, \mathrm{rk}(B, A)=0}(y-1)^{|B|}\right) .
$$

Since $\operatorname{rk}(B, A)=0$ if and only if every entry of the submatrix $M[B \mid A]$ is zero, we conclude that $\operatorname{rk}(B, A)=0$ if and only if $B \subseteq A^{\perp}$. Hence, the inner sum can be simplified using the binomial identity

$$
\sum_{B: B \subseteq A^{\perp}}(y-1)^{|B|}=y^{\left|A^{\perp}\right|}
$$

to obtain

$$
\begin{equation*}
Z(M ; 0, y-1, x)=\sum_{A: A \subseteq S} y^{\left|A^{\perp}\right|} x^{|A|} \tag{8}
\end{equation*}
$$

and in particular,

$$
Z(M ; 0, y-1,-1)=\sum_{A: A \subseteq S}(-1)^{|A|} y^{\left|A^{\perp}\right|}
$$

From this, we conclude that

$$
\begin{equation*}
\zeta(M ; s)=\left[\left(y \frac{\partial}{\partial y}\right)^{s} Z(M ; 0, y-1,-1)\right]_{y=1} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi(M ; s)=\left[\left(\frac{\partial}{\partial y}\right)^{s} Z(M ; 0, y-1,-1)\right]_{y=1} \tag{10}
\end{equation*}
$$

This completes the proof.
We remark that $Z(M ; 0, y-1, x)$, as a polynomial in $x$, has degree $|S|$. Its leading coefficient is $y$ raised to the power $\left|S^{\perp}\right|$. Hence, the number $\left|S^{\perp}\right|$ of zero rows (and by transposing, the number of zero columns) in $M$ can be derived from the corank-nullity polynomial of $M$.

Let $M[T \mid S]$ be a matrix. The support of a column $a$ is the set of rows $b$ such that the $b a$-entry is non-zero, or, equivalently, $\operatorname{rk}(\{b\},\{a\})=1$. A subset $B$ of rows is stable if it does not contain the support of any non-zero column. If $M[T \mid S]$ has no zero columns, a row subset $B$ is stable if and only if its complement $T \backslash B$ distinguishes $S$. From Lemma 2.2, we obtain the following result.
6.2. Lemma. Suppose that the matrix $M[T \mid S]$ has no zero column. Then the following are equivalent: (1) the characteristic polynomial $\chi(G[U] ; \lambda)$ of the column matroid $G[U]$ is not identically zero, (2) $U$ distinguishes $S$, and (3) the complement $T \backslash U$ is a stable set.
6.3. Theorem. Let $M[T \mid S]$ be a rank-r matrix with no zero columns. Then the coefficient of $\lambda^{r}$ in $(-y)^{r} R(M ;-\lambda / y, y,-1)$ is the generating polynomial $D(M ; y)$ for the number of row subsets distinguishing $S$, defined by

$$
D(M ; y)=\sum_{B: B \text { distinguishes } S} y^{|B|},
$$

where the sum ranges over all row subsets $B$ distinguishing $S$.
Proof. The theorem follows from Lemma 6.2 and the following calculation

$$
\begin{aligned}
(-y)^{r} R(M ;-\lambda / y, y,-1) & =\sum_{B: B \subseteq T} y^{|B|}\left(\sum_{A: A \subseteq S}(-1)^{|A|} \lambda^{r-\operatorname{rk}(B, A)}\right) \\
& =\sum_{B: B \subseteq T} y^{|B|} \lambda^{r-\operatorname{rk}(B, S)} \chi(G[B] ; \lambda) .
\end{aligned}
$$

Lemma 6.2 and Theorem 6.3 imply that the number of stable sets of a given size can be derived from the corank-nullity polynomial.

## 7. Incidence matrices of graphs

Let $\Gamma$ be a graph on the vertex set $V$ and the edge set $E$ in which loops and multiple edges may occur. A matrix $M[V \mid E]$ with row set $V$ and column set $E$ is said to be a vertex-edge matrix of $\Gamma$ if (1) when $a$ is a loop, all the entries in column $a$ are zero, and (2) when $a$ is an edge linking the two vertices $u$ and $v$, column $a$ has exactly two non-zero entries, at rows $u$ and $v$. Vertex-edge matrices represent various matroids on edge sets of graphs. One extreme is when the entries are in the two-element field $\mathrm{GF}(2)$. In this case, the matrix represents the cycle matroid. The other extreme is when the non-zero entries are algebraically independent variables. In this case, the matrix represents the bicircular matroid.

Let $I[V \mid E]$ be a vertex-edge matrix of the graph $\Gamma$. Then a vertex subset $\left\{v_{1}, v_{2}, \ldots, v_{s}\right\}$ distinguishes the edge set $E$ if every edge in $\Gamma$ is incident on a vertex $v_{i}$, that is, in graphtheoretic terminology, if the vertex subset is a vertex cover of $\Gamma$. By Eq. (10), the number of vertex covers of size $s$ equals

$$
\sum_{B: B \subseteq E}(-1)^{|B|}\binom{\left|B^{\perp}\right|}{s},
$$

where $B^{\perp}$ is the complement of the subset of vertices incident on the edges $B$. By Theorem 6.1, the number of vertex covers of a given size is derivable from the corank-nullity polynomial of a vertex-edge matrix.

In graph theory, a vertex subset $B$ in the graph $\Gamma$ is stable if no edge has both of its vertices in $B$, or, equivalently, it is a stable set of rows in the vertex-edge matrix $I[V \mid E]$ (as defined in Section 6). Since the generating polynomial for the number of stable subsets equals $y^{|V|} D(I ; 1 / y)$, the proof of Theorem 6.3 gives a derivation of this polynomial from the corank-nullity polynomial of a vertex-edge matrix. (The stable set generating polynomial for a graph has been studied. See, for example, [7].)

In [15], Oxley and Whittle define a 2-polymatroid on a graph in the following way. Let $\Gamma$ be a graph without loops. If $B$ is an edge subset, let $V(B)$ be the set of vertices incident on some edge in $B$. In the notation of critical problems, $V(B)$ is the complement of $B^{\perp}$. The function $f_{\Gamma}$ defined by

$$
\begin{equation*}
f_{\Gamma}(B)=|V(B)|=|V|-\left|B^{\perp}\right| \tag{11}
\end{equation*}
$$

is submodular. It defines a 2-polymatroid on the edge set $E$. Oxley and Whittle define the 2-polymatroid rank generating function $S\left(f_{\Gamma} ; u, v\right)$ by the formula

$$
S\left(f_{\Gamma} ; u, v\right)=\sum_{B: B \subseteq E} u^{|V(E)|-|V(B)|} v^{2|B|-|V(B)|}
$$

Much enumerative information is contained in the 2-polymatroid rank generating function. For example,

$$
u^{|V(E)| / 2} S\left(f_{\Gamma} ; 1 / \sqrt{u}, 0\right)=\sum_{k: k \geqslant 0} m_{k} u^{k}
$$

where $m_{k}$ is the number of partial matchings in $\Gamma$ with $k$ edges. Other examples can be found in [15].
7.1. Theorem. The polynomial $S\left(f_{\Gamma} ; u, v\right)$ can be derived from the corank-nullity polynomial of a vertex-edge matrix $I[V \mid E]$ of the graph $\Gamma$.

Proof. By Eq. (11),

$$
S\left(f_{\Gamma} ; u, v\right)=\sum_{B: B \subseteq E} u^{-\left|E^{\perp}\right|+\left|B^{\perp}\right|} v^{2|B|-|V|+\left|B^{\perp}\right|} .
$$

Combining this with Eq. (8), we obtain

$$
\begin{equation*}
S\left(f_{\Gamma} ; u, v\right)=u^{-\left|E^{\perp}\right|} v^{-|V|} Z\left(I ; 0, u v-1, v^{2}\right) \tag{12}
\end{equation*}
$$

Since we can derive $Z(I ; \lambda, y-1, x),|V|$, and $\left|E^{\perp}\right|$ from the corank-nullity polynomial of the matrix $I[V \mid E]$, we can also derive $S\left(f_{\Gamma} ; u, v\right)$.

When $B$ is an edge subset, $B^{\perp}$ is the set of isolated vertices in the subgraph $\Gamma \mid B$ on the same vertex set with edge set $B$. Thus, $Z(M ; 0, y-1, x)$ equals the generating polynomial of edge subsets $B$ by the number of isolated vertices in $\Gamma \mid B$ and $|B|$. This polynomial is a specialization of a three-variable polynomial for graphs of Borzacchini and Pulito [1].

An edge-vertex matrix is the transpose of a vertex-edge matrix. Let $J[E \mid V]$ be an edgevertex matrix of a graph $\Gamma$ with no loops. Then an $s$-tuple $\left(e_{1}, e_{2}, \ldots, e_{s}\right)$ of edges distinguishes $V$ if every vertex is incident on an edge $e_{i}$, that is, if the set $\left\{e_{1}, e_{2}, \ldots, e_{s}\right\}$ is an edge cover. The critical exponent $c(J)$ is the minimum size of a edge cover. Since we are assuming that an edge has two incident vertices, $c(J) \geqslant\lceil|V| / 2\rceil$. If $|V|$ is even and equals $2 k$, then an edge cover of size $k$ (if it exists) is a perfect matching or 1 -factor of $\Gamma$.
7.2. Proposition. Let $\Gamma$ be a graph with no loops and an even number $2 k$ of vertices. Then the number of perfect matchings of $\Gamma$ equals

$$
\frac{1}{k!} \sum_{A: A \subseteq V}(-1)^{|A|}\left|A^{\perp}\right|^{k}
$$

where $A^{\perp}$ is the set of edges not incident on any vertex in $A$.
By Theorem 6.1, the number of perfect matchings can be derived from the corank-nullity polynomial. For a graph with no loops or isolated vertices, the number of perfect matchings equals $S\left(f_{\Gamma} ; 0,0\right)$. To calculate $S\left(f_{\Gamma} ; 0,0\right)$ from Eq. (12) requires a limiting process such as L'Hôpital's rule. The formula so obtained is equivalent to the formula in Proposition 7.2. We also note that Tutte has studied the number of perfect matchings for cubic graphs using decomposition methods (see [19,20]).

Proposition 7.2 for bipartite graphs (or relations) yields the following result.
7.3. Proposition. Let $R$ be a bipartite graph between two sets $S^{\prime}$ and $S$ of equal size $k$. Then the number of perfect matchings in $R$ equals

$$
\frac{1}{k!} \sum_{B, A: B \subseteq S^{\prime}, A \subseteq S}(-1)^{|A|+|B|} e\left(S^{\prime} \backslash B, S \backslash A\right)^{k},
$$

where $e\left(S^{\prime} \backslash B, S \backslash A\right)$ is the number of edges between $S^{\prime} \backslash B$ and $S \backslash A$, the sets complementary to $B$ and $A$.

The formula in Proposition 7.3 can be applied to counting permutations with restricted positions (see, for example, [2, Section 7.2]). When $R$ is the complete bipartite graph $K_{n, n}$, Theorem 7.3 yields the elementary inclusion-exclusion identity

$$
n!=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}(n-j)^{n}
$$

However, it seems harder to derive recognizable formulas from other classes of bipartite graphs.

The results in this section extend without any difficulty to hypergraphs. Recall that a hypergraph $\Gamma$ is defined by a vertex set $V$ and a multiset of subsets of $V$ called edges. A perfect matching of $\Gamma$ is a subset of edges such that every vertex is incident on exactly one edge. A hypergraph is e-uniform if every edge has exactly $e$ vertices. With these definitions, Proposition 7.2 has the following generalization.
7.4. Proposition. Let $\Gamma$ be an e-uniform hypergraph with ek vertices. Then the number of perfect matchings of $\Gamma$ equals

$$
\frac{1}{k!} \sum_{B: B \subseteq V}(-1)^{|B|}\left|B^{\perp}\right|^{k},
$$

where $B^{\perp}$ is the set of edges not incident on any vertex in $B$.

Similarly, an $e$-uniform hypergraph defines an $e$-polymatroid and Proposition 7.1 extends to $e$-uniform hypergraphs.

## 8. Bilinear forms

We end with two remarks. A bilinear system consists of two ground sets $T \subseteq \mathcal{V}_{1}$ and $S \subseteq \mathcal{V}_{2}$, where $\mathcal{V}_{1}$ and $\mathcal{V}_{2}$ are two vector spaces with a bilinear form $\langle$,$\rangle between them. The$ Gram matrix $M$ is the matrix with rows $T$, columns $S$, and ba-entry equal to $\langle b, a\rangle$. Every matrix can be thought of as a Gram matrix of some bilinear system. A Tutte decomposition theory with a four-term deletion-contraction decomposition and a five-variable coranknullity polynomial can be developed for bilinear systems. However, the details are quite tedious and there is, as yet, no application where this more general theory is needed.

Four-term decompositions also occur in counting problems connected with subsets of vector spaces equipped with a bilinear form and their matroid analogs. For some indication and applications of the associated decomposition theory, see [12].

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