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Non-finitely generated projective modules over generalized Weyl algebras ☆

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ABSTRACT

We classify infinitely generated projective modules over generalized Weyl algebras. For instance, we prove that over such algebras every projective module is a direct sum of finitely generated modules.

1. Introduction

The theory of finitely generated projective modules is a classical topic in ring theory inspired by rich connections with K-theory, geometry and algebraic topology. However, it is often difficult to classify finitely generated projective modules over a given ring up to isomorphism, and one should be usually content with finding coarser invariants of this class of modules such as its Grothendieck group. For instance, this is certainly the case for projective modules over the first Weyl algebra; and calculating ideal class groups of commutative Dedekind domains is a core problem in algebraic number theory.

On the other hand the theory of infinitely generated projective modules is often essentially easier. For instance, Kaplansky's classical result says that every non-finitely generated projective module over a commutative Dedekind domain is free and later Bass [2] extended this to any indecomposable

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commutative noetherian ring as a consequence of his theory of big projectives. For instance, it follows from his theory that every non-finitely generated projective module over a simple noetherian ring is free. Thus it is quite often that the theory of infinitely generated projectives is 'trivial', which partly justifies Bass' remark [2, p. 24] that it 'invites little interest'. However, this is not always the case: non-finitely generated projective modules could be truly 'big'. For example, extending early results by Akasaki [1] and Linnell [14], Příhoda [19] found a superdecomposable (that is, without indecomposable direct summands) projective module over a certain localization of the integral group ring of the alternating group A_5.

In fact this result is a consequence of a far reaching development by Příhoda [19] of Bass' theory of big projectives, that leads to a 'rough' classification of infinitely generated projective modules over noetherian rings satisfying one mild additional condition $(*)$; for instance, $(*)$ holds true for any noetherian ring with the d.c.c. on two-sided ideals. Namely, he showed that projective modules over a noetherian ring R with $(*)$ are classified by pairs (I, P), where I is an idempotent ideal of R and P is a finitely generated projective R/I-module. The only drawback of his classification is that it is usually very difficult to understand the structure of the projective module Q corresponding to a given pair (I, P); for instance, to decide whether Q is finitely generated or isomorphic to a direct sum of finitely generated modules.

In this paper we will apply Příhoda's theory to obtain a satisfactory classification of non-finitely generated projective modules over the so-called generalized Weyl algebras (GWAs). This class of algebras was introduced and investigated by Bavula [3], but also was studied by Hodges [9] who called the rings in this class deformations of type-A Kleinian singularities; and by Rosenberg [20] under the name of hyperbolic rings. For instance, every GWA is a noetherian domain of Krull dimension 1, and this class of algebras includes the first Weyl algebra and all infinite dimensional primitive quotients of the universal enveloping algebra Usl_2 over a field of characteristic zero. In particular, the global dimension of any GWA is 1, 2 or ∞, and there is a good understanding of the finitely generated projective modules—the Grothendieck group of projectives has been calculated (see [7,9,11,18]) for most GWAs.

Recall that an old result of Kaplansky says that every projective left module over a left hereditary ring is a direct sum of finitely generated modules isomorphic to left ideals. In this paper we will show that something similar is true for projective modules over GWAs. In fact, the result is even more precise: in each GWA we will find finitely many homogeneous left ideals such that every non-finitely generated projective (left) module is a direct sum of copies of those.

In detail, in Section 2 we discuss some basic properties of idempotent ideals and will gather, in Section 3, certain (mostly folklore) statements on the structure of projective modules and their trace ideals. We will overview, in Section 4, the theory of (countably generated) projective modules (called fair-sized projectives in [19]) over noetherian rings with $(*)$, and draw some consequences of this theory. For example, in Theorem 4.7 we will give a general criterion for when every projective module over a noetherian ring with $(*)$ is a direct sum of finitely generated modules. For instance, for this to be true, finitely generated projective modules over factors of R by idempotent ideals must lift to finitely generated projectives over R. We also collect in this section some nice examples illustrating the power of the aforementioned theory. For instance (see Example 4.4) we will classify non-finitely generated projective modules over the ring of differential operators of n-dimensional projective space.

In Section 5 we will discuss some (mostly known) facts on the structure of generalized Weyl algebras, the main sources of information being Bavula [3] and Hodges [9]. Note that every GWA A is a noetherian domain with finitely many two-sided ideals (so $(*)$ holds true) and A has a least nonzero ideal $I_{\min}$. We also recall the structure of maximal ideals of GWAs and their simple finite dimensional modules. We will prove that the nonzero idempotent ideals of a GWA A form a finite Boolean algebra $B(A)$ and describe its coatoms.

Finally, in Section 7 we will classify infinitely generated projective modules over any GWA A. Using a description of idempotent ideals of A we will show that every such ideal is the trace of a finitely generated projective module; moreover, finitely generated projectives can be lifted modulo idempotent ideals of A. This is the crucial point of the paper, and our choice of finitely generated projective modules (to cover all finitely generated projectives over factor rings) is a bare guess. Certainly we had in mind a family of finitely generated projective modules constructed by Hodges [9], but our situation is essentially more demanding. For instance, the construction of a finitely generated projective A-module whose trace equals $I_{\min}$ (see Lemma 7.1) is quite involved. Even more this is true for the construction (in Lemma 7.2) of finitely generated projectives whose traces are atoms in $B(A)$.

Having spent a lot of time and space on these technicalities, we are awarded with a relative easy proof of two final results (Theorem 7.5 and Proposition 7.6). Namely, Theorem 7.5 states that every infinitely generated projective module over a GWA A is a direct sum of homogeneous left ideals of A from a prescribed finite family. In Proposition 7.6 we will improve this result by finding a canonical form for every infinitely generated projective module over any GWA, thus classifying projectives over GWAs by means of cardinal invariants.

2. Idempotent ideals

Most modules in this paper will be left modules over rings with unity. An element e of a ring R is said to be an *idempotent* if $e = e^2$. For instance, $0, 1 \in R$ are trivial idempotents. We say that an ideal I of R is *idempotent* if $I = I^2$, for which $\{0\}$ and R are trivial examples. Furthermore, the (two-sided) ideal ReR generated by an idempotent e (or by any set of idempotents) is idempotent. By [12, Corollary 2.43], every finitely generated idempotent ideal of a commutative ring is generated by an idempotent. However, if I is the augmentation ideal of the integral group ring $\mathbb{Z}A_5$, then (see [1]) I is idempotent, but $\mathbb{Z}A_5$ has no nontrivial idempotents.

If R is a semisimple artinian ring, then every two-sided ideal of R is generated by a central idempotent, therefore idempotent. Furthermore, in this case the set of (idempotent) ideals of R ordered by inclusion forms a finite Boolean algebra whose atoms correspond to minimal (two-sided) ideals of R, therefore to isomorphism classes of simple R-modules.

Note that the sum of any set of idempotent ideals is idempotent. For instance, every ideal I of R contains a largest idempotent ideal $I_{\mathrm{idem}} \subseteq I$. Furthermore, when ordered by inclusion, the set of idempotent ideals of R forms a lattice. The join in this lattice is the usual sum, but the meet of two idempotent ideals I and J equals $(I \cap J)_{\mathrm{idem}}$, which could be a proper subset of $I \cap J$ (see some examples below).

It is often important to describe the lattice of idempotent ideals of a given ring R. For this the following reductions will be useful. Suppose that $I \subseteq J$ are ideals of R such that I is idempotent. Then J is idempotent iff its image J/I is an idempotent ideal of the factor ring R/I. For instance, assume that R has a least nonzero ideal $I_{\min}$ (that is, R is subdirectly irreducible) such that $I_{\min}^2 \neq 0$, therefore $I_{\min}$ is idempotent. It follows from the above remark that the description of idempotent ideals of R boils down to the description of idempotent ideals of $R/I_{\min}$.

To make some further reductions we need the following result.

Fact 2.1. (See [22, L. 1].) If I, K are distinct idempotent ideals of R and $J^m \subseteq I$, $J^n \subseteq K$ for some ideal J of R, then I and K have distinct images in R/J.

Another way to say this is that $I + J = K + J$ yields $I = K$, that is, every idempotent ideal containing some power of J is uniquely determined by its image in R/J. One obvious instance of this situation is when J is a nilpotent ideal of R, and more can be said in this case. Recall that a ring R is said to be *semiperfect*, if the factor of R by its Jacobson radical J is a semisimple artinian ring and idempotents can be lifted modulo J. A semiperfect ring with a nilpotent Jacobson radical is called *semiprimary*. For instance, every one-sided artinian ring is semiprimary.

Lemma 2.2. *Every idempotent ideal of a semiprimary ring R is generated by an idempotent. Furthermore, the lattice of idempotent ideals of R is a finite Boolean algebra with m atoms, where m is the number of simple R-modules.*

Proof. If J denotes the Jacobson radical of R, then J is nilpotent and R/J is a semisimple artinian ring.

Let I be an idempotent ideal of R. Then $\bar{I}=(I+J)/J$ is an idempotent ideal of the semisimple ring R/J, hence $\bar{I}$ is generated by a central idempotent $\bar{e}$. Since J is nilpotent, one can lift $\bar{e}$ modulo J—there exists an idempotent $e \in R$ with $e+J=\bar{e}$. Then $K=ReR$ is an idempotent ideal of R such that $\overline{K}=\bar{e}(R/J)=\bar{I}$, therefore $K=I$ by Fact 2.1. Thus every idempotent ideal of R is generated by an idempotent.

Now the canonical projection $\pi : R \to R/J$ induces a map (also denoted by π) from the poset of idempotent ideals of R into the poset of idempotent ideals of R/J that preserves sums, hence preserves ordering. Since R/J is semisimple, the latter poset is a Boolean algebra with m atoms. Because J is nilpotent, Fact 2.1 yields that π is an injection. Furthermore, by the proof of the first part, π is a surjection, and it is easily seen that π reflects sums, hence reflects the ordering. Thus π is an isomorphism of posets, therefore an isomorphism of lattices. □

The following corollary is exactly what we need for further applications.

Corollary 2.3. *Suppose that R is a ring with a least nonzero ideal $I_{\min}$, $I_{\min}^2 \neq 0$, such that $R/I_{\min}$ is a semiprimary ring. Then the lattice of nonzero idempotent ideals of R is a finite Boolean algebra with m atoms, where m is the number of simple (non-isomorphic) $R/I_{\min}$-modules.*

3. Projective modules

One explanation why idempotent ideals are important is that they are intimately connected with projective modules. Recall that a module P over a ring R is said to be *free* if P is isomorphic to a module $R^{(I)}$ for some set I; and P is called *projective* if it is isomorphic to a direct summand of a free module. For instance, every free module is projective, as is the module Re for an idempotent e; but below we will see less obvious examples of projective modules.

If P is a projective module, then the *trace* of P, $\mathrm{Tr}(P)$, will denote the sum of images of all morphisms from P to ${}_RR$. For instance, if $P=Re$ for an idempotent e, then $\mathrm{Tr}(P)=ReR$ is an idempotent ideal. In fact it is always the case.

Fact 3.1. If P is a projective module, then $\mathrm{Tr}(P)$ is an idempotent ideal such that $P=\mathrm{Tr}(P)P$. Furthermore, $\mathrm{Tr}(P)$ is the least among ideals I such that $P=IP$.

Proof. The first part is a common knowledge (see [12, Proposition 2.40]). The second part is also well known, but somehow avoids any written account. □

Clearly (say, from Fact 3.1) $\mathrm{Tr}(P) \neq 0$ for any nonzero projective module P and P is said to be a *generator* if $\mathrm{Tr}(P)=R$ (the maximal possible value of the trace). If P is a direct summand of a free module $R^{(I)}$, then P is isomorphic to the module generated by the columns of a column-finite idempotent $I \times I$ matrix E over R, therefore $\mathrm{Tr}(P)$ is a two-sided ideal generated by entries of E.

Given projective modules P and Q, we say that *P generates Q* if, for some α, there is an epimorphism $P^{(\alpha)} \to Q$. Since Q is projective, this is the same as Q being isomorphic to a direct summand of $P^{(\alpha)}$. The following lemma is also folklore, but should be put on the paper, at least once.

Lemma 3.2. *Let P and Q be projective modules. Then the following are equivalent*:

(1) *P generates Q*;
(2) $Q=\mathrm{Tr}(P)Q$;
(3) $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$.

Proof. (1) $\Rightarrow$ (2). Let $f : P^{(\alpha)} \to Q$ be an epimorphism. Applying f to $P^{(\alpha)}=\mathrm{Tr}(P)P^{(\alpha)}$ (see Fact 3.1) we obtain $Q=\mathrm{Tr}(P)Q$.

(2) $\Rightarrow$ (3). By Fact 3.1, $\mathrm{Tr}(Q)$ is the least ideal I such that $Q=IQ$, therefore $Q=\mathrm{Tr}(P)Q$ yields $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$.

(3) $\Rightarrow$ (2). Since $\mathrm{Tr}(Q)Q = Q$ and $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$, we conclude that $\mathrm{Tr}(P)Q = Q$.

(2) $\Rightarrow$ (1). It suffices to prove that every $q \in Q$ is in the image of a morphism $P^k \to Q$, for some (finite) k. From $Q = \mathrm{Tr}(P)Q$ it follows that $q = \sum_{i=1}^n r_i q_i$ for some $r_i \in \mathrm{Tr}(P)$, $q_i \in Q$. Clearly we may assume that $n = 1$, that is, $q = rq'$, $r \in \mathrm{Tr}(P)$, $q' \in Q$. Furthermore, $r \in \mathrm{Tr}(P)$ yields that $r = \sum_{j=1}^k f_j(p_j)$, where $p_j \in P$ and $f_j : P \to {}_R R$ are morphisms. Let $g = \sum_{j=1}^k f_j : P^k \to R$ and let $h : R \to Q$ be given by $h(1) = q'$. Then hg maps P^k into Q and $hg(\sum_{j=1}^k p_j) = h(r) = rh(1) = rq' = q$, as desired. $\square$

A module M is said to be *countably generated* if it has a finite or infinite countable set of generators. By Kaplansky's theorem (see [8, Corollary 2.48]) every projective module is a direct sum of countably generated modules, thus most (but not all) questions on the structure of projective modules can be reduced to the countably generated case.

The following lemma, which is a version of Eilenberg's trick (see [2, p. 24] or [12, p. 22]), shows that a projective module with a larger trace 'absorbs' another 'smaller' projective module.

Lemma 3.3. *Let P and Q be countably generated projective modules with $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$. If $\alpha \geqslant \beta, \omega$, then $P^{(\alpha)} \cong P^{(\alpha)} \oplus Q^{(\beta)}$.*

Proof. By Lemma 3.2 and because Q is countably generated, Q, hence $Q^{(\beta)}$ is isomorphic to a direct summand of $P^{(\alpha)}$. If $P^{(\alpha)} \cong Q^{(\beta)} \oplus T$ for some module T, then

$$P^{(\alpha)} \cong \big(P^{(\alpha)}\big)^{(\omega)} \cong \big(Q^{(\beta)} \oplus T\big)^{(\omega)} \cong Q^{(\beta)} \oplus \big(T \oplus Q^{(\beta)}\big)^{(\omega)} \cong Q^{(\beta)} \oplus P^{(\alpha)}. \qquad \square$$

As we have seen in Fact 3.1 the trace of a projective module is always an idempotent ideal. Unfortunately, given an idempotent ideal I, it is usually quite difficult to decide whether I is a trace of some projective module. The following is a rare case that provides such an answer.

Fact 3.4. (See [25, Corollary 2.7].) Let I be an idempotent ideal of a ring R such that I is finitely generated as a right ideal. Then there exists a countably generated projective left R-module whose trace equals I.

However, we do not know much about the structure of this projective module, for instance, whether it can be chosen to be finitely generated or not.

In the next section we will discuss the property of a projective module to decompose into a direct sum of finitely generated modules. Thus the following result of Kaplansky will be useful in this discussion.

Fact 3.5. (See [12, 2.24].) Every projective left module over a left hereditary ring is a direct sum of modules isomorphic to finitely generated left ideals.

Recall that Kaplansky proved that every projective module over a local ring is free. One more result along this line is worth mentioning.

Fact 3.6. (See [2, Corollary 3.4].) Every infinitely generated projective left module over a left noetherian simple ring is free.

4. The theory of fair-sized projectives

In this section we recall (from [19]) a classification of (countably generated) projective modules over certain classes of noetherian rings. One can consider this theory as a far reaching generalization of Bass' theory of big projectives (see [2]).

We say that a ring R satisfies the condition $(*)$ if the following holds.

$$\text{Every (descending) chain } I_1, I_2, \ldots \text{ of ideals of } R, \text{ with } I_{k+1}I_k = I_{k+1} \text{ for any } k, \text{ stabilizes.} \qquad (*)$$

For instance, if the lattice of (two-sided) ideals of R is finite then R satisfies $(*)$.

Remark 4.1. Sakhaev [21] characterized rings R with the following property: Any projective left R-module finitely generated modulo its Jacobson radical is finitely generated. He showed that this condition is connected with the stabilization of the (descending) sequence of left principal ideals of the matrix ring $M_n(R)$ generated by $n \times n$ matrices A_i, where $A_{i+1}A_i = A_{i+1}$ for every i (see condition (t6) in his Theorem 3) for every positive integer n. If I_i denotes the two-sided ideal generated by entries of A_i then we obtain that $I_{i+1}I_i = I_{i+1}$, as in $(*)$. However, it is easy to see that Sakhaev's condition is satisfied in any (left) noetherian ring while there are noetherian rings not satisfying $(*)$. Therefore in this paper we will not pursue this analogy any further.

Proposition 4.2. *(See [19].) Suppose that R is a noetherian ring satisfying $(*)$. Then there is a natural one-to-one correspondence between countably generated projective R-modules and pairs (I, P), where I is an idempotent ideal of R and P is a finitely generated projective R/I-module.*

One direction in this correspondence is easy to describe. If Q is a countably generated projective R-module, then $(*)$ implies (see [19] for a proof) that there exists a least ideal $I = I(Q)$ of R such that $P = Q/IQ$ is a finitely generated (projective) R/I-module. Thus we assign to Q the pair (I, P). The opposite direction in the above correspondence is rather an existence theorem. For example, it is usually quite difficult to decide whether the (countably generated) projective module corresponding to a given pair (I, P) is a direct sum of finitely generated modules or not.

Note that the pairs $(0, P)$ in the above classification correspond to finitely generated projective R-modules, so Proposition 4.2 says nothing new about them. Furthermore, if Q is a countably generated projective module, then, using Fact 3.1, it is easily seen that $Q^{(\omega)}$ corresponds to the pair $(\mathrm{Tr}(Q), 0)$. In particular, the pair $(R, 0)$ corresponds to the free module $R^{(\omega)}$. For example, it follows that every infinitely generated projective module over a simple noetherian ring is free, a slightly weaker form of Bass' result in Fact 3.6.

Now we will show how this theory works in a slightly more elaborate situation.

Proposition 4.3. *Suppose that R is a noetherian ring with a unique nonzero proper ideal J and such that $D = R/J$ is a skew field. Further assume that there exists a finitely generated projective module Q such that $\mathrm{Tr}(Q) = J$. Then every infinitely generated projective module is either free or isomorphic to $R^{(\alpha)} \oplus Q^{(\beta)}$, where $\alpha < \beta$, $\beta \geqslant \omega$, and α, β are uniquely determined by Q.*

Proof. Since $D = R/J$ is a skew field, every finitely generated projective R/J-module is of the form $(R/J)^k$ for some $k < \omega$. If P is a countably infinitely generated projective module, then $I(P) \neq 0$, hence either $I(P) = R$, and then P is free, or $I(P) = J$. In the latter case P goes to $(J, (R/J)^k)$ in the correspondence of Proposition 4.2. But clearly $R^k \oplus Q^{(\omega)}$ also corresponds to this pair, therefore $P \cong R^k \oplus Q^{(\omega)}$.

If P is uncountably generated, then (using Kaplansky's theorem) decompose it into a direct sum of countably infinitely generated modules $P = \bigoplus_{i \in I} P_i$. By what we have already proved each P_i is either free or isomorphic to $R^{k_i} \oplus Q^{(\omega)}$ for some $k_i < \omega$. Gathering the copies of R and Q together, we obtain $P \cong R^{(\alpha)} \oplus Q^{(\beta)}$. If $\alpha \geqslant \beta, \omega$ then P is isomorphic to $R^{(\alpha)}$ by Lemma 3.3. Otherwise, since P is not finitely generated, $\alpha < \beta$ and $\beta \geqslant \omega$.

Now $\alpha = \dim_D P/JP$ is uniquely determined by P and the same is true for $\beta = \alpha + \beta$ which equals the uniform dimension of P. $\square$

Note that (at least in some cases—see below) a finitely generated projective module Q is not unique. However, if Q' is another finitely generated projective module with $\mathrm{Tr}(Q') = J$, then Proposition 4.2 implies that $Q^{(\omega)} \cong Q'^{(\omega)}$, because both modules correspond to the pair $(J, 0)$.

Now we will give some examples showing that the situation described in Proposition 4.3 occurs naturally.

Example 4.4. Let $R = \mathcal{D}(\mathbb{P}^n)$ denote the ring of differential operators on the projective space $\mathbb{P}^n(k)$, where k is an algebraically closed field of characteristic zero. By [6, pp. 213–214] R is a noetherian domain of Krull dimension n (and global dimension $n+1$) with a unique nonzero proper ideal J and $R/J \cong k$ holds true. Thus to apply Proposition 4.3 it suffices to find a finitely generated projective module Q such that $\mathrm{Tr}(Q) = J$. Indeed, let $Q = \mathcal{D}(1)$ as in [6, p. 215]. Then, by [6, Cor. 4.8], Q is a finitely generated projective (left) $\mathcal{D}(\mathbb{P}^n)$-module such that $JQ = Q$, hence $\mathrm{Tr}(Q) = J$.

Thus Proposition 4.3 gives a classification of infinitely generated projective modules over $\mathcal{D}(\mathbb{P}^n)$.

Example 4.5. Let k be a field of characteristic 2 containing a nonzero element λ which is not a root of unity. Let S be obtained by factoring the ring of Laurent polynomials $k\{X^{\pm 1}, Y^{\pm 1}\}$ by the ideal generated by $XY - \lambda YX$. Let σ be an automorphism of S of order 2 given by $\sigma(X) = X^{-1}$, $\sigma(Y) = Y^{-1}$; and set $R = S^\sigma$, the subring of S fixed by σ.

Then (see [15, Example 1.8] or [10, pp. 140–141]) R has a unique (nonzero proper) two-sided ideal J such that $R/J \cong k$ and S is an indecomposable rank 2 projective module whose trace is equal to J.

Thus, by Proposition 4.3 again, we obtain a classification of non-finitely generated projective R-modules.

As one more example let us consider the subring $R = k + xA_1(k)$ of the first Weyl algebra over a field k of characteristic zero. By [16, 1.3.10, 5.5.11], R is a hereditary noetherian domain with a unique nonzero proper two-sided ideal $J = xA_1(k)$. Then J is a finitely generated projective module coinciding with its trace. Thus taking $Q = J$ and applying Proposition 4.3 we obtain a classification of infinitely generated projective R-modules (though one should be able to extract this from the classification of infinitely generated projective modules over hereditary noetherian prime rings in Levy and Robson [13]).

In this case the finitely generated projective module Q is not unique. Indeed it is well known that $A_1(k)$ has infinitely many non-isomorphic left ideals. Using $\mathrm{End}(J) = xA_1(k)x^{-1} \cong A_1(k)$ one concludes that there are infinitely many non-isomorphic (projective) left ideals of R with trace J.

Next we will investigate an even more advanced example of Stafford [23]. To keep the notation of his paper, in this example we will consider right modules.

Example 4.6. Let k be a field of characteristic zero, $C = k[x_1, \dots, x_n]$ be the ring of polynomials, and δ is a derivation of C given by $\delta(x_1) = 1$ and $\delta(x_i) = x_i x_{i-1} - 1$ for $i > 1$. Let $S = C[y, \delta]$ be the ring of differential polynomials, and take $R = C + x_1 S$. Then (see [23, pp. 384–385]) R is a noetherian domain with a least nonzero proper ideal $J = x_1 S$ and $R/J \cong k[x_1, \dots, x_{n-1}]$. It follows easily that J is the only nonzero proper idempotent ideal of R; and every finitely generated projective R/J-module is isomorphic to $(R/J)^k$ (because every projective $k[x_1, \dots, x_{n-1}]$-module is free). Furthermore, it is not difficult to check that S is a finitely generated projective R-module whose trace equals J. Thus arguing as in Proposition 4.3 we conclude that every infinitely generated projective R-module is either free or isomorphic to $R^{(\alpha)} \oplus S^{(\beta)}$, $\alpha < \beta$, $\beta \geqslant \omega$.

Note that over rings in Examples 4.4–4.6 every projective module is a direct sum of finitely generated modules, but this is not always the case. Indeed, let $R = \mathbb{Z}A_5$ be the integral group ring of the alternating group A_5 and let I be the augmentation ideal of R. Since (see [1]) I is idempotent, by Fact 3.4 there exists a countably generated projective module P whose trace is equal to I. But P cannot contain a finitely generated direct summand because (as follows from [24, Theorem 8.1]—see [1, Corollary 14] for arguments) every finitely generated projective R-module is a generator.

In the next proposition we characterize in the framework of the theory of fair-sized projectives the rings whose projective modules are direct sums of finitely generated modules. As we have already mentioned (see Fact 3.5) this holds true for left hereditary rings; for a more thorough treatment of this question see [17].

Theorem 4.7. *Let R be a noetherian ring satisfying $(*)$. Then the following are equivalent*:

(1) *Every projective module is a direct sum of finitely generated modules*;
(2) (a) *every idempotent ideal of R is the trace of a finitely generated projective module and*
(b) *if I is an idempotent ideal of R and P is a finitely generated projective R/I-module, then there exists a finitely generated projective module Q such that $Q/IQ \cong P$.*

Proof. (1) $\Rightarrow$ (2). (a) Suppose that I is an idempotent ideal of R. By Fact 3.4, there exists a countably generated projective module Q whose trace is equal to I. By the assumption, $Q = \bigoplus_{j\in J} Q_j$ is a direct sum of finitely generated modules. Then $\mathrm{Tr}(Q)$ is a directed union of traces of finitely generated projectives $Q_{j_1} \oplus \cdots \oplus Q_{j_k}$, $j_1, \ldots, j_k \in J$. Since R is noetherian, I is the trace of one of such finitely generated modules.

(b) Suppose that P is a finitely generated projective R/I-module, where I is an idempotent ideal of R. Let Q be a countably generated projective module that corresponds to the pair (I, P) in Proposition 4.2. By the assumption, $Q = \bigoplus_{j\in J} Q_j$ is a direct sum of finitely generated modules. From the definition of $I = I(Q)$ it follows that $Q_j \neq IQ_j$ for only finitely many $j \in J$. Adding up the Q_j from this finite subset we obtain a finitely generated projective module Q' such that $Q'/IQ' \cong P$.

(2) $\Rightarrow$ (1). Let Q be a countably generated projective module and set $I = I(Q)$, $P = Q/IQ$ (see Proposition 4.2), therefore I is an idempotent ideal of R and P is a finitely generated projective R/I-module. By the assumption, there are finitely generated projective modules P_1 and P_2 such that $\mathrm{Tr}(P_1) = I$ and $P_2/IP_2 \cong P$. It is easily seen that the module $P_1^{(\omega)} \oplus P_2$ also corresponds to the pair (I, P), therefore $Q \cong P_1^{(\omega)} \oplus P_2$ by Proposition 4.2. $\square$

Note that (2)(b) of the above theorem says that one can 'lift' finitely generated projective modules modulo idempotent ideals.

5. Generalized Weyl algebras

Let k be an algebraically closed field of characteristic zero and let σ be an automorphism of the ring of polynomials $k[H]$. In this paper we will consider only the case when $\sigma(H) = H - 1$ (and σ fixes k pointwise); for the case when σ is arbitrary see for example [5]. Let $a(H) \in k[H]$ be a nonconstant polynomial. We say that a k-algebra $A = A(a)$ is a *generalized Weyl algebra*, GWA, if A is generated over $k[H]$ by (noncommuting) variables X, Y subject the following relations.

$$YX = a(H), \qquad XY = \sigma(a) = a(H-1) \quad \text{and} \quad HY = Y(H-1), \qquad HX = X(H+1).$$

Thus for every polynomial $b(H) \in k[H]$ we obtain

$$b(H)\cdot Y = Y\sigma(b) = Y\cdot b(H-1) \quad \text{and} \quad b(H)\cdot X = X\sigma^{-1}(b) = X\cdot b(H+1).$$

For instance, consider the first Weyl algebra $A_1(k)$ as an algebra of differential operators acting on the ring of polynomials $k[x]$ on the left; therefore A_1 is generated by x and ∂ subject to the relation $\partial x - x\partial = 1$. It is easily checked that the map $X \to x$, $Y \to \partial$ and $H \to \partial x$ provides an isomorphism from the generalized Weyl algebra $A(H)$ onto $A_1(k)$.

Furthermore (see [9] or [5, p. 522]) if G is a cyclic group of order m acting on $A_1(k)$ via $\partial \to \omega\partial$, $x \to \omega^{-1}x$, where ω is a primitive mth root of unity, then the fixed ring $A_1^G = k\langle \partial^m, \partial x, x^m\rangle$ is a GWA with $a(H) = m^m H(H + 1/m)\cdot \cdots \cdot (H + (m-1)/m)$, where $X \to x^m$, $Y \to \partial^m$ and $H \to \partial x/m$.

Finally, let U be the universal enveloping algebra $Usl_2(k)$ with the usual generators e, f, h (thus $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$). If $C = 4fe + h^2 + 2h$ is the Casimir element, then all infinite dimensional primitive factors of U are of the form $U_\lambda = U/(C-\lambda)U$, $\lambda \in k$. It is straightforward to verify that U_λ is a GWA with $a(H) = \lambda/4 - (H+1)H$, where $X \to e$, $Y \to f$ and $H \to h/2$.

By [4, Theorem 3.28], one can multiply the polynomial $a(H)$ by a nonzero constant and 'shift' it to the left or right without changing the isomorphism type of A. It follows that every GWA with a linear polynomial $a(H)$ is isomorphic to $A_1(k)$, and every GWA with a quadratic polynomial is isomorphic to one of primitive factors U_λ.

Note that some rings we have already considered are GWAs. For instance, from [6, p. 205] it follows that the ring of differential operators $\mathcal{D}(\mathbb{P}^1)$ is isomorphic to U_0, that is, to a GWA with $a(H) = -(H+1)H$. By what we have just said this GWA is isomorphic to the GWA with $a(H) = H(H-1)$. Using [4, Theorem 3.28], it is easily checked that the latter GWA is not isomorphic to the GWA with $a(H) = H(H-2)$. However, using translation functors from [9, Theorem 2.3], one concludes that the last two GWAs are Morita equivalent.

The first crucial fact about GWAs is that they are noetherian.

Fact 5.1. (See [3, Proposition 1.3, Theorem 2.5].) Every GWA is a noetherian domain of Krull dimension 1.

Furthermore, looking at the roots of $a(H)$ one can decide whether a given GWA is simple and calculate its global dimension. We say that $\lambda, \mu \in k$ are *comparable* if $\lambda - \mu \in \mathbb{Z}$.

Fact 5.2. (See [3, Theorem 5].) Let $A = A(a)$ be a GWA.

(1) A is simple iff $a(H)$ has no comparable (distinct) roots;
(2) A is hereditary iff $a(H)$ has neither comparable nor repeated (= multiple) roots;
(3) A has global dimension 2 iff $a(H)$ has comparable roots but no repeated roots;
(4) A is of infinite global dimension iff $a(H)$ has a repeated root.

Thus every GWA has global dimension 1, 2 or ∞. For instance, if $a(H) = H^2$, then A is a simple algebra of infinite global dimension; and if $a(H) = H(H-1)$, then A has global dimension 2 and is not simple.

Recall that every GWA A has a standard $\mathbb{Z}$-grading: setting $\deg(X) = 1$, $\deg(Y) = -1$ and $\deg(H) = 0$, we obtain $A = \bigoplus_{n\in\mathbb{Z}} A_n$, where $A_n = k[H]Y^n = Y^n k[H]$ if $n < 0$, $A_0 = k[H]$, and $A_n = k[H]X^n = X^n k[H]$ if $n > 0$. Note also that $\mathrm{ad}(H)X^n = [H, X^n] = nX^n$ and $\mathrm{ad}(H)Y^m = -mY^m$. It follows easily that every (two-sided) ideal I of A is homogeneous, $I = \bigoplus_{n\in\mathbb{Z}} I_n$, where $I_n = I \cap A_n$ is the nth homogeneous component of I; therefore the lattice of two-sided ideals of A is distributive (because the lattice of ideals of $k[H]$ is distributive). In fact more can be said.

Fact 5.3. (See [3, Proposition 2.2].) If I is a nonzero ideal of a GWA A, then the factor A/I is finite dimensional. Furthermore, the lattice of ideals of A is finite and there is a least nonzero ideal $I_{\min}$.

In the following lemma we will pinpoint this ideal. Note that, for every $n \geqslant 1$, $X^nY^n = a(H-1)\cdot \cdots \cdot a(H-n)$ is a polynomial $c_n(H)$ such that $Y^nX^n = a(H+n-1)\cdot \cdots \cdot a(H) = c_n(H+n)$.

Lemma 5.4. *Let n be the maximum of $|\lambda - \mu|$, where λ and μ are comparable roots of $a(H)$. Then $I_{\min}$ is generated by the polynomial $d_n(H) = \gcd(X^nY^n, Y^nX^n) = \gcd(c_n(H), c_n(H+n))$ and $X^n, Y^n \in I_{\min}$.*

Proof. Let I be a nonzero ideal of A. Since I is homogeneous, it contains a nonzero polynomial $f(H)$, and we may assume that $\deg f \geqslant 1$. Choose $k \geqslant n$ such that $f(H)$ and $f(H-k)$ are coprime. Then $f(H)X^k \in I$ and $X^k f(H) = f(H-k)X^k \in I$ implies $X^k \in I$ (and similarly $Y^k \in I$). It follows that $X^kY^k = c_k(H) \in I$ and $Y^kX^k = c_k(H+k) \in I$, therefore $d_k(H) = \gcd(c_k(H), c_k(H+k)) \in I$.

If λ is a root of $d_k(H)$ then $\lambda - i$ and $\lambda + j$ are roots of $a(H)$ for some $1 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant k-1$. By the assumption, $i + j = |(\lambda - i) - (\lambda + j)| \leqslant n$, in particular $i \leqslant n$ and $j \leqslant n-1$. It follows easily that $d_n(H) = d_k(H) \in I$. Thus $d_n(H)$ belongs to every nonzero ideal of A, therefore d_n generates $I_{\min}$.

Suppose that $\lambda \leqslant \mu$ are roots of $d_n(H)$. Then $\lambda - i$ and $\mu + j$ are roots of $a(H)$ for some $1 \leqslant i$ and $j \geqslant 0$. By the assumption $|(\mu + j) - (\lambda - i)| = \mu - \lambda + i + j \leqslant n$, hence $\mu - \lambda \leqslant n-1$. Now it is easily checked that $d_n(H)$ and $d_n(H-n)$ are coprime, therefore, by the first part of the proof, $X^n, Y^n \in I$. □

For instance, if A is a GWA with $a(H) = H(H-1)$, then $n = 1$, hence $d_1(H) = \gcd(XY, YX) = \gcd((H-1)(H-2), H(H-1)) = H-1$; and $\langle H-1\rangle$ is the unique nonzero proper ideal of A. If A is a GWA with $a(H) = H(H-1)(H-2)$, then $n = 2$ and $d_2(H) = \gcd(X^2Y^2, Y^2X^2) = (H-1)(H-2)$.

Since every maximal ideal of $k[H]$ is generated by $H-\lambda$, $\lambda \in k$, the action of σ on the set of maximal ideals can be identified with the action $\lambda \to \lambda + 1$ on k. The orbits of this action are of the form $\lambda + \mathbb{Z}$, $\lambda \in k$, therefore $\lambda, \mu \in k$ are on the same orbit iff they are comparable. If B is an orbit and $\lambda, \mu \in B$, then we set $\lambda \leqslant_B \mu$ if $\mu - \lambda \geqslant 0$, that is $\mu - \lambda$ is a nonnegative integer; clearly $\leqslant_B$ is a linear ordering.

Let S be the (finite) set of all roots of $a(H)$, and let U denote the set of all orbits containing at least two roots of $a(H)$. If $B \in U$, then $S \cap B$ contains a smallest element x_B and a largest element $y_B \neq x_B$ (with respect to $\leqslant_B$). Denote by $T_B = (x_B, y_B]$ the semi-interval $\{z \in B \mid x_B < z \leqslant y_B\}$ and set $T = \bigcup_{B\in U} T_B$. For instance, if $a(H) = H(H-2)(H-5)$, then $U = \{0 + \mathbb{Z}\}$ and $T = \{1, 2, 3, 4, 5\}$, in particular, $1 \in T$ is not a root of $a(H)$. By Fact 5.2, T is nonempty iff A is not simple.

For every $\lambda \in T$ let $L_\lambda = \{\mu \in S \mid \lambda - \mu \in \mathbb{N}\}$ and $R_\lambda = \{\mu \in S \mid \mu - \lambda \in \mathbb{N}_0\}$, where $\mathbb{N}$ stands for the set of positive integers and $\mathbb{N}_0$ for the set of nonnegative integers. Thus $\mu \in L_\lambda$ iff μ is strictly to the left of λ within the equivalence class of λ; and $\mu \in R_\lambda$ iff μ is strictly to the right of λ in the equivalence class of λ or $\mu = \lambda$. Let m_μ denote the multiplicity of $H - \lambda$ in $a(H)$ and set $k_\lambda = \min(\sum_{\mu\in L_\lambda} m_\mu, \sum_{\tau\in R_\lambda} m_\tau)$.

We will give an even more algorithmic way (compare with Lemma 5.4) to compute $d_n(H)$.

Lemma 5.5. $d_n(H) = \prod_{\lambda\in T}(H-\lambda)^{k_\lambda}$.

Proof. By definition, $d_n(H)$ is the greatest common divisor of $c_n(H) = X^nY^n = a(H-1)\cdot \cdots \cdot a(H-n)$ and $c_n(H+n) = Y^nX^n = a(H+n-1)\cdot \cdots \cdot a(H)$. First notice that $H-\lambda$ divides $d_n(H)$ iff it divides both $c_n(H)$ and $c_n(H+n)$, that is, $\lambda - i$ and $\lambda + j$ are roots of $a(H)$ for some $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-1$. By the definition of T, it follows that $H-\lambda$ divides $d_n(H)$ iff $\lambda \in T$.

Suppose that $\lambda \in T$ and let us calculate the multiplicity of $H-\lambda$ in $c_n(H)$. Notice that $H-\lambda$ has multiplicity $m_{\lambda-1}$ in $a(H-1), \ldots,$ and multiplicity $m_{\lambda-n}$ in $a(H-n)$. Thus $H-\lambda$ has multiplicity $\sum_{\mu\in L_\lambda} m_\mu$ in $c_n(H)$. By similar arguments $H-\lambda$ has multiplicity $\sum_{\tau\in R_\lambda} m_\tau$ in $c_n(H+n)$, and the result follows immediately. □

For instance, let $a(H) = H^2(H-1)^3(H-2)^4$. Then $T = \{1, 2\}$, $L_1 = \{0\}$, $R_1 = \{1, 2\}$, hence $\sum_{\mu\in L_1} m_\mu = 2$, $\sum_{\tau\in R_1} m_\tau = 3 + 4 = 7$ and $k_1 = \min(2, 7) = 2$. Similarly $L_2 = \{0, 1\}$, $R_2 = \{2\}$, therefore $\sum_{\mu\in L_2} m_\mu = 2 + 3 = 5$, $\sum_{\tau\in R_2} m_\tau = 4$ and $k_2 = \min(5, 4) = 4$. Thus $d_n(H) = (H-1)^2(H-2)^4$ is a generator for $I_{\min}$.

6. Idempotent ideals of GWAs

As one may see from Proposition 4.2, the description of idempotent ideals is an important ingredient in the classification of projective modules. In this section we describe the idempotent ideals of any GWA. But first we should recall description of maximal ideals of GWAs.

Recall (see Fact 5.3) that every GWA A has a least nonzero ideal $I_{\min}$ such that $A/I_{\min}$ is a finite dimensional algebra. It follows that every maximal ideal of A contains $I_{\min}$ and is the annihilator of a simple finite dimensional A-module. A classification of such simples is available from [3]. Suppose that $\lambda < \mu$ are roots of $a(H)$ lying on the same orbit B. We say that λ and μ are *adjacent* if the interval $(\lambda, \mu) = \{\tau \in B \mid \lambda < \tau < \mu\}$ contains no roots of $a(H)$. For instance, if $a(H) = H(H-2)(H-4)$, then $0 < 2$ and $2 < 4$ are the only pairs of adjacent roots. If $\lambda < \mu$ are adjacent roots of $a(H)$, then $S_{\lambda,\mu}$ will denote the cyclic module $A/A(Y^n, X, H-\mu)$. It is easily calculated that this module is n-dimensional with a k-basis given by $Y^{n-1}, \ldots, 1$. Note also that Y^i spans a one-dimensional eigenspace for the action of H, with eigenvalue $\mu - i$.

Lemma 6.1. *(See [3, Theorem 3.2].) $S_{\lambda,\mu}$ is a simple (finite dimensional) A-module, and every simple finite dimensional A-module is isomorphic to a module of this form.*

In particular, the number of simple finite dimensional A-modules is the cardinality of $T \cap S$.

Thus if $I_{\lambda,\mu}$ denotes the annihilator of $S_{\lambda,\mu}$, then these ideals form a complete list of maximal ideals of A. Furthermore (see [3, Lemma 3.3]) if $\mu - \lambda = m$, then the factor $A/I_{\lambda,\mu}$ is isomorphic to the full matrix ring $M_m(k)$, therefore $A/I_{\lambda,\mu}$ is a direct sum of m copies of $S_{\lambda,\mu}$.

In fact one can give a precise formula for a generator of $I_{\lambda,\mu}$. If $\lambda < \mu$ are adjacent roots on an orbit B, then $T_{\lambda,\mu}$ will denote the semi-interval $(\lambda, \mu] = \{\tau \in B \mid \lambda < \tau \leqslant \mu\}$. For instance, if $a(H) = H(H-2)(H-4)$, then $T_{0,2} = \{1, 2\}$ and $T_{2,4} = \{3, 4\}$. Clearly $T = \bigcup T_{\lambda,\mu}$, where the union runs over all pairs of adjacent roots of $a(H)$.

Fact 6.2. (See [3, Lemma 3.3].) $I_{\lambda,\mu}$ is generated by $d_{\lambda,\mu}(H) = \prod_{\tau \in T_{\lambda,\mu}} (H - \tau)$.

For instance, if $a(H) = H(H-2)(H-4)$, then $I_{0,2}$ is generated by $(H-1)(H-2)$, in particular, $X^2, Y^2 \in I_{0,2}$, but $X, Y \notin I_{0,2}$.

Let $J \supseteq I_{\min}$ denote the ideal of A whose image $J/I_{\min}$ is the Jacobson radical of $A/I_{\min}$. It follows that J is the intersection of the ideals $I_{\lambda,\mu}$ when $\lambda < \mu$ run over all pairs of adjacent roots of $a(H)$. Since $T = \bigcup T_{\lambda,\mu}$, we obtain the following.

Corollary 6.3. *The zeroth homogeneous component J_0 of J is generated by $f(H) = \prod_{\tau \in T} (H - \tau)$.*

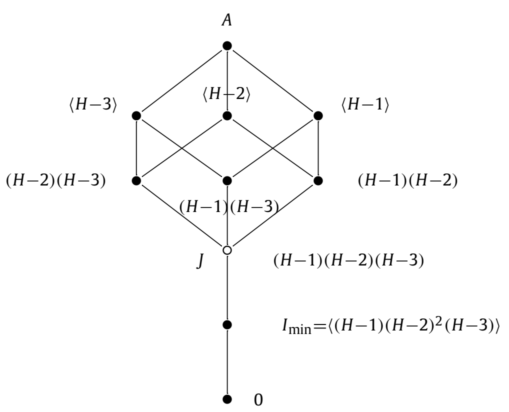
The remaining homogeneous components of J can be calculated using Fact 6.2. For instance, if $a(H) = H(H-2)(H-4)$, then $X^2, Y^2 \in J$ (since the maximum of differences between adjacent roots of $a(H)$ is 2), but (see Lemma 5.4) X^4 is the first power of X in $I_{\min}$.

Since every GWA has the least nonzero ideal $I_{\min}$ and $A/I_{\min}$ is a finite dimensional algebra, by Corollary 2.3 we obtain the following.

Corollary 6.4. *Let A be a GWA with m nonisomorphic simple finite dimensional modules. Then the lattice of nonzero idempotent ideals of A is a (finite) Boolean algebra $B(A)$ with m atoms.*

Note that $I_{\min}$ is the least element of $B(A)$, and every element of $B(A)$ but $I_{\min}$ is a sum of atoms (since the join in $B(A)$ is usual sum).

But first let us look at the following example. Let A be a GWA with $a(H) = H(H-1)(H-2)(H-3)$. Then the following is a fragment of the lattice of two-sided ideals of A containing $B(A)$, where idempotent ideals are marked by bullets.



For instance, $I_{0,1} = \langle H-1\rangle$ (that is, generated by $H-1$), $I_{1,2} = \langle H-2\rangle$ and $I_{2,3} = \langle H-3\rangle$ are the only maximal ideals of A, and they are idempotent. However, J, the intersection of all these ideals, is not idempotent and is strictly larger than $I_{\min}$. Indeed, the zeroth component of J is generated by

$(H-1)(H-2)(H-3)$ and is it possible to check (it is not so obvious as it seems!) that it is larger than the zeroth component of $I_{\min}$, which is generated by $(H-1)(H-2)^2(H-3)$.

If $\lambda < \mu$ are adjacent roots of $a(H)$, then $m(\lambda,\mu) = \min(m_\lambda, m_\mu)$ will denote the common multiplicity of λ and μ as roots of $a(H)$. The following lemma describes coatoms in $B(A)$, that is, maximal idempotent ideals.

Lemma 6.5. *$I_{\lambda,\mu}^{m(\lambda,\mu)}$, where $\lambda < \mu$ run over adjacent roots of $a(H)$, is a complete list of maximal idempotent ideals of A.*

Proof. If I is a maximal idempotent ideal of A, then I is contained in a maximal ideal L; and $L = I_{\lambda,\mu}$ for some adjacent roots $\lambda < \mu$ of $a(H)$, by the description of maximal ideals. Since I is idempotent, it follows that $I \subseteq L^m$ for all m. But, by [3, Proof of Theorem 3.3], $m = m(\lambda,\mu)$ is the smallest number such that the ideal $I_{\lambda,\mu}^m$ is idempotent. □

Since $B(A)$ is a Boolean algebra, every idempotent ideal of A is a (unique) intersection (in $B(A)$) of maximal idempotent ideals $I_{\lambda,\mu}^{m(\lambda,\mu)}$. However, since the intersection in $B(A)$ may differ from set-theoretic intersection, this description is not very constructive. In the next section we will list the atoms of $B(A)$, hence obtain another, more handy, description of the idempotent ideals of GWAs.

7. Projective modules over GWAs

In this section we will classify projective modules over any given GWA. Recall that (by Bass' result) if A is a simple GWA, then every infinitely generated projective module is free. Thus the only interesting case is when A is not simple, hence (by Fact 5.2) $a(H)$ has distinct comparable roots (that is, $T \neq \emptyset$). In most statements of this section we will make a default assumption that A is not simple.

Let us make a general (well known) remark. Suppose that I is a left ideal of a GWA A and let $Q = Q(A)$ denote the skew field of quotients of A. Since A is a noetherian domain, every morphism from I to ${}_AA$ is given by right multiplication by some $q \in Q$. Using the dual basis lemma (see [12, Lemma 2.9]) we conclude that I is projective iff there are $p_1, \ldots, p_m \in I$ and $q_1, \ldots, q_m \in \mathrm{Hom}(I, A) \subseteq Q$ such that $\sum_{i=1}^m q_i p_i = 1$. In this case right multiplication by the row $(q_1, \ldots, q_m)$ defines a morphism from I to ${}_AA^m$ whose one-sided inverse is given by right multiplication by the column $(p_1, \ldots, p_m)^t$. Thus I is represented by the idempotent $m \times m$ matrix $(p_i q_j)$, therefore the trace of I is generated by the $p_i q_j$. Moreover, $\mathrm{Tr}(I)$ is also generated by $p_i' q_j$, where $p_1', \ldots, p_l'$ is any set of generators for I, for instance this is the case when $l = m$ and $p_i = r_i p_i'$ for some $r_i \in A$.

First we construct a projective homogeneous left ideal of A whose trace is equal to $I_{\min}$. We will use the notation introduced before Lemma 5.5. Recall that if $\lambda \in T$, then R_λ denotes the set of all roots of $a(H)$ that are comparable with λ and lie to the right of λ (including λ). Let $n_\lambda = \sum_{\mu \in R_\lambda} m_\mu$, where m_μ denotes the multiplicity of μ as a root of $a(H)$; and we set $q(H) = \prod_{\lambda \in T} (H-\lambda)^{n_\lambda}$. It is easily seen (see Lemma 5.5 for a similar proof) that $H - \lambda$ has multiplicity n_λ in $c_n(H+n) = Y^n X^n$, therefore $q(H) = Y^n X^n|_T$, the restriction of $c_n(H+n)$ to T. For instance, if $a(H) = H(H-1)(H-2)(H-3)$, then $q(H) = (H-1)^3(H-2)^2(H-3)$.

Recall that n denotes the maximum of $|\lambda - \mu|$, where λ and μ are comparable roots of $a(H)$. For instance, if $\lambda \in T$, then $\lambda - n \notin T$.

Lemma 7.1. *$P_{\min} = Aq(H) + AX^n$ is a projective homogeneous left ideal of A whose trace is equal to $I_{\min}$.*

Proof. Recall that Q denotes the classical ring of quotients of A, and let the morphism $f : A \to {}_AQ^{|T|+1}$ be given by right multiplication by the row $(q_0, \ldots, q_n) = (1, \ldots, Y^n(H-\lambda-n)^{-n_\lambda}, \ldots)$, where each $\lambda \in T$ gives an entry. We claim that, when restricted to $P_{\min}$, f provides a morphism from $P_{\min}$ to ${}_AA$. Indeed $f(q(H)) = (q(H), \ldots, q(H)Y^n(H-\lambda-n)^{-n_\lambda}, \ldots)$. Since $(H-\lambda)^{n_\lambda}$ is a factor of $q(H)$ for each $\lambda \in T$, therefore $q(H)Y^n(H-\lambda-n)^{-n_\lambda} = q(H)(H-\lambda)^{-n_\lambda}Y^n \in A$. It remains to check that each component of $f(X^n)$ belong to A. Indeed, as we have already noticed, $(H-\lambda)^{n_\lambda}$ divides $c_n(H+n) = Y^nX^n$, hence $(H-\lambda-n)^{n_\lambda}$ divides $c_n(H) = X^nY^n$. Thus $X^n \cdot Y^n(H-\lambda-n)^{-n_\lambda} = c_n(H)(H-\lambda-n)^{-n_\lambda} \in A$.

Now we consider the following polynomials: $q(H)$ and $Y^n(H-\lambda-n)^{-n_\lambda}X^n = Y^nX^n(H-\lambda)^{-n_\lambda}$, $\lambda \in T$. Because $q(H) = \prod_{\lambda \in T}(H-\lambda)^{n_\lambda} = Y^nX^n|_T$, therefore $H-\lambda$ does not divide $Y^nX^n(H-\lambda)^{-n_\lambda}$ for any $\lambda \in T$, and the above polynomials are coprime. Thus there are polynomials $p(H)$, $p_\lambda(H)$, $\lambda \in T$ such that $q(H)p(H) + \sum_{\lambda \in T} Y^n(H-\lambda-n)^{-n_\lambda}X^np_\lambda(H) = 1$. Now $(p_0, \dots, p_n)^t = (q(H)p(H), \dots, X^np_\lambda(H), \dots)^t$ is the column of $|T|+1$ elements of $P_{\min}$ such that the right multiplication by this column defines a morphism $g : A^{|T|+1} \to P$ with $gf = 1_{P_{\min}}$, therefore $P_{\min}$ is projective.

It remains to show that $\operatorname{Tr}(P_{\min}) = I_{\min}$. By what we have said at the beginning of the section, the trace of $P_{\min}$ is generated by the images of $q(H)$ and X^n when multiplying them by the q_i on the right. Since $I_{\min}$ is a minimal nonzero ideal, it suffices to check that $q(H), q(H)Y^n(H-\lambda-n)^{-n_\lambda} \in I_{\min}$ and $X^n, X^nY^n(H-\lambda-n)^{-n_\lambda} \in I_{\min}$. But (see Lemma 5.4) $X^n, Y^n \in I_{\min}$, therefore $q(H)Y^n(H-\lambda-n)^{-n_\lambda} = q(H)(H-\lambda)^{-n_\lambda}Y^n \in I_{\min}$, because $(H-\lambda)^{n_\lambda}$ divides $q(H)$. Further, from Lemma 5.5 and the definition of $q(H)$ it follows that $d_n(H)$ divides $q(H)$, therefore $q(H) \in I_{\min}$.

Now consider $X^nY^n(H-\lambda-n)^{-n_\lambda} = c_n(H)(H-\lambda-n)^{-n_\lambda}$. As we have already seen, $(H-\lambda-n)^{n_\lambda}$ divides $c_n(H)$, hence it can be canceled. Recall (see Lemma 5.4) that $d_n(H)$ also divides $c_n(H)$ and is a product of polynomials $H-\mu$, $\mu \in T$. If $\lambda \in T$, then $\lambda - n \notin T$, hence $d_n(H)$ still divides $c_n(H)(H-\lambda-n)^{-n_\lambda}$. By Lemma 5.4, the latter polynomial belongs to $I_{\min}$, as desired. □

For example if $a(H) = H(H-1)(H-2)(H-3)$, then $P_{\min} = A(H-1)^3(H-2)^2(H-3) + AX^3$.

In the next lemma we will extend our supply of projective modules, hence of idempotent ideals. For $\lambda \in T$ we define $q_\lambda(H) = q(H)/(H-\lambda)^{n_\lambda} = \prod_{\mu \in T, \mu \neq \lambda}(H-\mu)^{n_\mu}$ and set $P_\lambda = Aq_\lambda(H) + AX^n$.

Lemma 7.2. *If $\lambda \in T$, then P_λ is a projective homogeneous left ideal of A whose trace is generated by $q_\lambda(H)$.*

Proof. As in Lemma 7.1, let $f : A \to {}_AQ^{|T|}$ be given by right multiplication by the row $(1, \dots, Y^n(H-\mu-n)^{-n_\mu}, \dots)$, where each $\mu \in T$, $\mu \neq \lambda$ gives one entry. We claim that the restriction of f to P_λ gives a morphism from P_λ to ${}_AA$. It suffices to check that $q_\lambda(H) \cdot Y^n(H-\mu-n)^{-n_\mu} \in A$ and $X^n \cdot Y^n(H-\mu-n)^{-n_\mu} \in A$. Indeed $q_\lambda(H)Y^n(H-\mu-n)^{-n_\mu} = q_\lambda(H)(H-\mu)^{-n_\mu}Y^n \in A$, because $\mu \neq \lambda$ yields that $(H-\mu)^{n_\mu}$ divides $q_\lambda(H)$. Since $(H-\mu)^{n_\mu}$ divides $c_n(H+n)$, it follows that $(H-\mu-n)^{n_\mu}$ divides $c_n(H)$, therefore $X^nY^n(H-\mu-n)^{-n_\mu} = c_n(H)(H-\mu-n)^{-n_\mu} \in A$.

Now we consider the following $|T|$ polynomials: $q_\lambda(H)$ and $Y^n(H-\mu-n)^{-n_\mu}X^n$, where each $\mu \in T$, $\mu \neq \lambda$ gives one polynomial. Since $Y^n(H-\mu-n)^{-n_\mu}X^n = c_n(H+n)(H-\mu)^{-n_\mu}$, from the definition of $q_\lambda(H)$ it follows that these polynomials are coprime. (Indeed, $H-\mu$ is not a root of $c_n(H+n)(H-\mu)^{-n_\mu}$ for every $\lambda \neq \mu \in T$.) Thus $q_\lambda(H)p_\lambda(H) + \sum_{\mu \in T, \mu \neq \lambda} Y^n \times (H-\mu-n)^{-n_\mu}X^np_\mu(H) = 1$ for some polynomials $p_\tau(H)$, $\tau \in T$. Now right multiplication by the column $(q_\lambda(H)p_\lambda(H), \dots, X^np_\mu(H), \dots)^t$ of elements of P_λ defines a morphism $g : A^{|T|} \to P$ such that $gf = 1_{P_\lambda}$, therefore P_λ is projective.

It remains to calculate the trace of P_λ. By the remark at the beginning of the section, $\operatorname{Tr}(P_\lambda)$ is generated by the images of $q_\lambda(H)$ and X^n when multiplying them by 1 or $Y^n(H-\mu-n)^{-n_\mu}$, $\lambda \neq \mu \in T$. Thus $q_\lambda(H) = q_\lambda(H) \cdot 1 \in \operatorname{Tr}(P_\lambda)$, and clearly $X^n, Y^n \in \operatorname{Tr}(P_\lambda)$ (because X^n, Y^n belong to every nonzero ideal—see Lemma 5.4). Furthermore, $q_\lambda(H)Y^n(H-\mu-n)^{-n_\mu} = q_\lambda(H)(H-\mu)^{-n_\mu}Y^n$ is a multiple of Y^n, hence belongs to $\langle q_\lambda(H)\rangle$. Thus it remains to look at $X^nY^n(H-\mu-n)^{-n_\mu} = c_n(H)(H-\mu-n)^{-n_\mu}$. But in the proof of Lemma 7.1 we showed that this polynomial is in $I_{\min} \subseteq \langle q_\lambda(H)\rangle$. □

For instance, if $a(H) = H(H-1)(H-2)(H-3)$ and $\lambda = 1 \in T = \{1, 2, 3\}$, then $q_1(H) = (H-2)^2(H-3)$, hence $P_1 = A(H-2)^2(H-3) + AX^3$ is a projective module whose trace is generated by $(H-2)^2(H-3)$.

Now we are in a position to describe the atoms of $B(A)$.

Lemma 7.3. *If $\tau \in T$, then $\langle q_\tau\rangle = Aq_\tau A$ is an atom in $B(A)$, and every atom of $B(A)$ is of this form.*

Proof. By Lemma 7.2, $\langle q_\tau \rangle$ is the trace of the projective module P_τ, hence idempotent. Since $\tau \in T$, it follows that $\tau \in T_{\lambda,\mu}$ for the only pair $\lambda < \mu$ of adjacent roots of $a(H)$. From Fact 6.2 it follows that $q_\tau \notin I_{\lambda,\mu}$ and $q_\tau \in I_{\rho,\pi}$ for all remaining pairs of adjacent roots $\rho < \pi$ of $a(H)$. Since $\langle q_\tau \rangle$ is idempotent, it equals the intersection in $B(A)$ of maximal idempotent ideals $I_{\rho,\pi}^{m(\rho,\pi)}$ (see Lemma 6.5). It follows easily that $\langle q_\tau \rangle$ is an atom in $B(A)$, and every atom of $B(A)$ is of this form. $\square$

Thus we have obtained a somehow better (see a remark after Lemma 6.5) description of the idempotent ideals of GWAs. Since every nonzero idempotent ideal of A either equals $I_{\min}$ or is a (finite) sum of atoms, it follows from Lemmas 7.1 and 7.2 that every idempotent ideal of A is the trace of a finitely generated projective module, hence (2)(a) of Theorem 4.7 holds true. Instead of verifying (2)(b) of this theorem, we will proceed directly to the classification of projective modules. But first we need the following lemma.

Lemma 7.4. *If $\tau \in T_{\lambda,\mu}$, then P_τ/JP_τ is a simple module isomorphic to $S_{\lambda,\mu}$.*

Proof. First we will show that P_τ/JP_τ is a cyclic module generated by $\bar{q}_\tau = q_\tau + JP_\tau$. For this it suffices to prove that X^n, the second generator of P_τ, belongs to JP_τ. Indeed, from $X^n \in J$ we obtain $X^n q_\tau(H) = q_\tau(H-n)X^n \in JP_\tau$. Further, if $f(H) = \prod_{\eta \in T}(H-\eta)$ is a generator of the zeroth component of J (see Corollary 6.3), then $f(H)X^n \in JP_\tau$. Since all the roots of q_τ are in T (and n is the maximum of differences of comparable roots), it follows that $q_\tau(H-n)$ and $f(H)$ are coprime, hence $X^n \in JP_\tau$.

From the description of maximal ideals of A (see after Fact 6.2) we conclude that $q_\tau \notin I_{\lambda,\mu}$ and $q_\tau \in I_{\rho,\pi}$ for all remaining maximal ideals of A. It follows easily that $I_{\lambda,\mu}\bar{q}_\tau = \bar{0}$. Since $I_{\lambda,\mu}$ is the annihilator of $S_{\lambda,\mu}$, this implies that P_τ/JP_τ is a direct sum of copies of $S_{\lambda,\mu}$.

Recall (see before Lemma 6.1) that the τ-eigenspace of $S_{\lambda,\mu}$ (when acting by H) is 1-dimensional. Thus to prove that P_τ/JP_τ is simple it suffices to show that its τ-eigenspace is also 1-dimensional. Moreover, since $\bar{q}_\tau$ is a generator for this module, it is enough to check that $(H-\tau)\bar{q}_\tau = \bar{0}$, that is, $(H-\tau)q_\tau \in JP_\tau$. If $f(H) = \prod_{\eta \in T}(H-\eta)$, then (as above) $f(H) \in J$, hence $f(H)q_\tau \in JP_\tau$. Furthermore, $Y^n \in J$ implies $Y^n X^n = c_n(H+n) \in JP_\tau$, therefore $g(H) = \gcd(f(H)q_\tau(H), c_n(H+n)) \in JP_\tau$. Since every root of $f(H)q_\tau(H)$ belongs to T and $Y^n X^n|_T = q(H) = (H-\tau)^{n_\tau} q_\tau$, it follows that $g(H) = \gcd(f(H)g_\tau, q) = (H-\tau)q_\tau \in JP_\tau$, as desired. $\square$

Note that we have some excess of projective modules 'covering' the same simple module: if $\tau, \eta \in T_{\lambda,\mu}$, then both P_τ/JP_τ and P_η/JP_η are isomorphic to $S_{\lambda,\mu}$. To get uniqueness one can choose one representative τ in each set $T_{\lambda,\mu}$; and the most natural choice would be to take $\tau = \mu$, the utmost right end of $T_{\lambda,\mu}$, which is a root of $a(H)$. Thus simple finite dimensional A-modules, hence the corresponding projective ideals, are parameterized by $T \cap S$.

Let $\lambda_1, \dots, \lambda_m$ be a complete list of elements of $T \cap S$ (that is, of elements of T which are roots of $a(H)$), where we may assume that $i < j$ implies $\lambda_i <_B \lambda_j$, if λ_i and λ_j are on the same orbit B. Let $S_1, \dots, S_m$ be the corresponding (complete) list of finite dimensional A-modules. Thus, if $\lambda_i < \lambda_{i+1}$ are adjacent roots of $a(H)$, then $S_{i+1} = S_{\lambda_i,\lambda_{i+1}}$ (in notation before Lemma 6.1). For example, if $a(H) = H(H-2)(H-4)$, then $T \cap S = \{2, 4\}$, therefore we set $\lambda_1 = 2 < \lambda_2 = 4$ and $S_1 = S_{0,2}$, $S_2 = S_{2,4}$. By what we have just noticed, then $P_{\lambda_1}, \dots, P_{\lambda_m}$ are projective homogeneous left ideals of A such that $P_{\lambda_i}/JP_{\lambda_i} \cong S_i$.

Now we are in a position to prove the main result of the paper.

Theorem 7.5. *Every infinitely generated projective module Q over a generalized Weyl algebra A is a direct sum of copies of homogeneous left ideals $P_{\min}$ and $P_{\lambda_1}, \dots, P_{\lambda_m}$.*

Proof. By Kaplansky's theorem we may assume that Q is countably (infinitely) generated. Let $I = I(Q)$ be a two-sided ideal of A corresponding to Q in Proposition 4.2; in particular, I is idempotent and $P = Q/IQ$ is a finitely generated projective A/I-module. Since Q is infinitely generated, therefore $I \neq 0$.

Suppose first that $I = I_{\min}$. Since $I_{\min} \subseteq J$ and J is nilpotent modulo $I_{\min}$, therefore the canonical projection $P/I_{\min}P \to P/JP$ is a projective cover of P/JP as an $A/I_{\min}$-module. Furthermore, because A/J is a semisimple artinian ring, we conclude that P/JP is a direct sum of simple finite dimensional A-modules, $P/JP \cong S_1^{k_1} \oplus \cdots \oplus S_m^{k_m}$. Then $P' = P_{\lambda_1}^{k_1} \oplus \cdots \oplus P_{\lambda_m}^{k_m}$ is a projective left A-module with $P'/JP' \cong P/JP$. Thus $P/I_{\min}P$ and $P'/I_{\min}P'$ are projective covers of P/JP as an $A/I_{\min}$-module, therefore these modules are isomorphic.

Now it is easy to calculate that the pair corresponding to the projective module $P_{\min}^{(\omega)} \oplus P'$ is $(I_{\min}, P'/I_{\min}P')$, therefore Q is isomorphic to this module by Proposition 4.2.

Now assume that $I \supset I_{\min}$ is an idempotent ideal of A. If I_i denotes $\langle q_{\lambda_i} \rangle$, then, by Lemma 7.3, $I_1, \dots, I_m$ is a complete list of atoms of $B(A)$, therefore I admits a (unique) representation as a sum of atoms, $I = \sum_{j \in \Lambda} I_j$, where Λ is a subset of $\{1, \dots, m\}$ (for instance, if $I = A$, then $\Lambda = \{1, \dots, m\}$); and let $\Lambda' = \{1, \dots, m\} \setminus \Lambda$ be the complement of Λ.

Since A/J is semisimple, we conclude that P/JP is a direct sum of copies of simple modules $S_1, \dots, S_m$. Furthermore, because $I(P) = I$, it follows easily that $Q/JQ \cong \bigoplus_{j \in \Lambda} S_j^{(\omega)} \oplus \bigoplus_{l \in \Lambda'} S_l^{k_l}$, $k_l < \omega$, therefore $Q/IQ \cong \bigoplus_{l \in \Lambda'} S_l^{k_l}$. Let us consider the following projective A-module $Q' = \bigoplus_{j \in \Lambda} P_{\lambda_j}^{(\omega)} \oplus \bigoplus_{l \in \Lambda'} P_{\lambda_l}^{k_l}$. Clearly $I(Q) = \sum_{j \in \Lambda} \mathrm{Tr}(P_{\lambda_j}) = \sum_{j \in \Lambda} I_j = I$ and $Q'/JQ' \cong Q/JQ$. Using projective covers (as in the first part of the proof) we conclude that $Q'/I_{\min}Q' \cong Q/I_{\min}Q$. Since $I_{\min} \subseteq I$, it follows that $Q'/IQ' \cong Q/IQ$, therefore $Q' \cong Q$ by Proposition 4.2. □

Note that Hodges [9, Lemma 2.4] constructed a family of finitely generated projective modules over a GWA A as follows. Suppose that $a(H) = b(H)c(H)$, where the polynomials $b(H)$ and $c(H)$ are coprime. Then $P_b = Ab(H) + AX$ is a projective homogeneous left ideal of A. It is not difficult to check that $\mathrm{Tr}(P_b)$ is generated by $X, Y, b(H)$ and $c(H-1)$. For instance, if $a(H) = H(H-1)(H-2)$ and $b(H) = H - 1$, then $\mathrm{Tr}(P_b) = \langle H - 1 \rangle$, therefore $\mathrm{Tr}(P_b)$ is a maximal (idempotent) ideal of A. However, $\mathrm{Tr}(P_b)$ is always situated close to the top of $B(A)$, for instance, in most cases one cannot obtain $I_{\min}$ as $\mathrm{Tr}(P_b)$. Thus our approach to idempotent ideals 'from below' seems to have a crucial advantage.

If we take a GWA with $a(H) = H(H-2)$, set $b(H) = H - 2$ and apply Hodges' construction, then (see [9, Theorem 2.3]) $P = A(H-2) + AX$ is a projective generator whose endomorphism ring is isomorphic to the GWA with $a(H) = H(H-1)$, therefore these algebras are Morita equivalent. This is an example of a translation functor we mentioned before Fact 5.1.

As one can see from the proof of Theorem 7.5, some direct summands of the projective module Q are clearly redundant. For instance, executing this proof for $Q = A^{(\omega)}$, we will end up with representation $Q \cong \bigoplus_{i=1}^m P_{\lambda_i}^{(\omega)}$. In the next proposition we will get rid of these repetitions, therefore obtain a canonical form for each infinitely generated projective module over a GWA. This also allows us to include uncountably generated projectives.

Proposition 7.6. *Let Q be an infinitely generated projective module over a GWA A. Then exactly one of the following holds true.*

(1) *Q is free;*
(2) *$Q \cong A^{(\alpha)} \oplus \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus P_{\min}^{(\beta)}$, where $\omega \leqslant \alpha < \alpha_i < \beta$ and Λ is a proper (maybe empty) subset of $\{1, \dots, m\}$;*
(3) *$Q \cong A^{(\alpha)} \oplus \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)}$, where $\omega \leqslant \alpha < \alpha_i$ and Λ is a proper nonempty subset of $\{1, \dots, m\}$;*
(4) *$Q \cong \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j \in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)}$, where $k_j < \omega$, $\omega \leqslant \alpha_i < \beta$, and Λ, M are disjoint subsets of $\{1, \dots, m\}$ and Λ is proper and nonempty;*
(5) *$Q \cong \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j \in M} P_{\lambda_j}^{k_j}$, where $k_j < \omega$, $\omega \leqslant \alpha_i$, and Λ, M are disjoint subsets of $\{1, \dots, m\}$ and Λ is proper and nonempty;*
(6) *$Q \cong \bigoplus_{j \in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)}$, where $k_j < \omega$, $\beta \geqslant \omega$, and M is a subset of $\{1, \dots, m\}$.*

Furthermore all the exponents $\alpha, \beta, \dots$ in the above representations are uniquely determined by Q.

Proof. By Theorem 7.5, every infinitely generated projective A-module Q is isomorphic to a direct sum of copies of A, $P_{\lambda_1}, \ldots, P_{\lambda_m}$ and $P_{\min}$ (clearly there is no harm in adding A!). Separating finite and infinite exponents of the P_{λ_i}, we obtain that

$$Q \cong A^{(\alpha)} \oplus \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j \in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)},$$

where each $\alpha_i \geqslant \omega$, $k_j < \omega$, and Λ, M are disjoint subsets of $\{1, \ldots, m\}$; and choose a representation of Q with a maximal possible α.

Suppose first that $\alpha \geqslant \omega$. Because $A = \mathrm{Tr}(A) \supset \mathrm{Tr}(P_{\lambda_i}) = I_i \supset \mathrm{Tr}(P_{\min}) = I_{\min}$, therefore, by Lemma 3.3, we can absorb projectives $P_{\mu_j}^{k_j}$ into $A^{(\alpha)}$, therefore assume that $M = \emptyset$. Similarly, if $\alpha \geqslant \alpha_i$ for some $i \in \Lambda$ then $A^{(\alpha)} \oplus P_{\lambda_i}^{(\alpha_i)} \cong A^{(\alpha)}$ (so we can drop $P_{\lambda_i}^{(\alpha_i)}$); and $A^{(\alpha)} \oplus P_{\min}^{(\beta)} \cong A^{(\alpha)}$ if $\alpha \geqslant \beta$. Furthermore, again by Lemma 3.3, $P_{\lambda_i}^{(\alpha_i)} \oplus P_{\min}^{(\beta)} \cong P_{\lambda_i}^{(\alpha_i)}$ if $\alpha_i \geqslant \beta$.

Thus either Q is free or we may assume either that $\alpha < \alpha_i < \beta$ for each $i \in \Lambda$ (or just $\alpha < \beta$ if $\Lambda = \emptyset$) or $\beta = 0$, $\Lambda \neq \emptyset$ and $\alpha < \alpha_i$ for each $i \in \Lambda$.

Suppose that $\Lambda = \{1, \ldots, m\}$ and $\alpha_j = \min_{i \in \Lambda} \alpha_i$. Since $\mathrm{Tr}(P_{\lambda_1} \oplus \cdots \oplus P_{\lambda_m}) = A$ it follows that $\bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)}$ splits off $A^{(\alpha_j)}$ as a direct summand, which can be transferred to $A^{(\alpha)}$. Since $\alpha + \alpha_j = \alpha_j > \alpha$, this contradicts our choice of α. As a result Λ is a proper subset of $\{1, \ldots, m\}$, thus we have obtained (2) and (3) of the proposition.

It remains to consider the case when $\alpha = s$ if finite. If $\Lambda \neq \emptyset$ and $j \in \Lambda$ then using Proposition 4.2 it is easily seen that $A^s \oplus P_{\lambda_j}^{(\alpha_j)}$ is isomorphic to $P_{\lambda_j}^{(\alpha_j)} \oplus \bigoplus_{i=1}^m P_{\lambda_i}^s$, therefore Q is isomorphic to a module of the form (4) or (5).

Similarly if $\Lambda = \emptyset$ and Q is not finitely generated, we obtain (6).

Arguing as in Proposition 4.3 it is easily seen that exponents $\alpha, \beta, \ldots$ are uniquely determined by Q. For instance, in (4), α_i is equal to the uniform dimension of Q/KQ, where K is the annihilator of the simple module $S_i = P_{\lambda_i}/JP_{\lambda_i}$. $\square$

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