Hermite-Birkhoff Interpolation on Roots of Unity and Walsh Equiconvergence

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ABSTRACT

Cavaretta et al. [1] had shown that in some special cases of polynomials of lacunary interpolation in the roots of unity, the analogue of a theorem of Walsh is valid. They offered a conjecture in the case of $(0, m_1, \dots, m_{s-1})$ interpolation in the n roots of unity. The objective here is to prove the conjecture and to prove a similar result in the case of 2-periodic lacunary interpolation on $2n$ roots of unity.

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1. INTRODUCTION

Let A_ρ denote the class of functions analytic in $|z| < \rho$ $(\rho > 1)$. For a given $f \in A_\rho$, let $p_{n-1}(z; f)$ denote the Lagrange interpolation to f in the n roots of unity, and let $P_{n-1}(z; f)$ denote the Taylor polynomial of degree $n-1$ for f about the origin. A beautiful theorem of Walsh [6] asserts that $\lim_{n\to\infty}[p_{n-1}(z; f) - P_{n-1}(z; f)] = 0$ for $|z| < \rho^2$ and that this convergence is uniform and geometric on compact subsets of $|z| < \rho^2$. Recently there has

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been considerable interest in this theorem and its various extensions [1, 5]. Here we are interested in its relation to the H-B problem of $(0, m_1, \ldots, m_{s-1})$ interpolation in the n roots of unity where $\{m_i\}_0^{s-1}$ are any nonnegative integers with $0 = m_0 < m_1 < \cdots < m_{s-1}$. It was shown [2, 4] that if $m_k \leqslant kn$ $(k = 0, 1, \ldots, s-1)$, then the problem is regular (i.e., has a unique solution). If $f(z) = \sum_0^\infty a_\nu z^\nu$, let $B_{ns-1,0}(z; f) = \sum_0^{ns-1} a_\nu z^\nu$ and let $b_{ns-1}(z; f)$ denote the unique polynomial of $(0, m_1, \ldots, m_{s-1})$ interpolation in n roots of unity. If $g_{\nu,j}(z) = z^{\nu+(j+s-1)n}$ $(j = 1, 2, \ldots)$, set

$$B_{sn-1,j}(z; f) = \sum_{\nu=0}^{n-1} a_{\nu+(j+s-1)n} b_{sn-1}(z; g_{\nu,j}). \tag{1.1}$$

Cavaretta et al. [1] offered the following conjecture:

CONJECTURE. For any f in A_ρ and for each positive integer l,

$$\lim_{n\to\infty} \left[b_{sn-1}(z; f) - \sum_{j=0}^{l-1} B_{sn-1,j}(z; f) \right] = 0 \tag{1.2}$$

for $|z| < \rho^{1+(l/s)}$, the convergence being uniform and geometric for all z with $|z| \leqslant Z < \rho^{1+(l/s)}$. Moreover the result (1.2) is best possible.

This conjecture was based on some special cases treated, viz., the H-B problem of $(0, 1, \ldots, r-1)$, $(0, m)$, $(0, 2, 3)$, and $(0, 1, \ldots, r-2, r+m-2)$ interpolation. The main difficulty in the proof of the conjecture in the general case derives from the difficulty of writing the polynomial $b_{sn-1}(z; f)$ in the form $(1/2\pi i)\int_\Gamma f(t) K(t, z)\, dt$ and of the finding the explicit form of the kernel. In the special cases treated, this difficulty was bypassed by adopting a different approach.

The existence and uniqueness question for a more general H-B interpolation on roots of unity was recently solved in [3]. If ω is a primitive $2n$-root of unity, let $1, \omega^2, \ldots, \omega^{2n-2}$ be the n roots of unity and let $\omega, \omega^3, \ldots, \omega^{2n-1}$ be the n roots of -1. Let $0 = m_0 < m_1 < \cdots < m_{s_1-1}$ and $0 \leqslant \tilde{m}_0 < \cdots < \tilde{m}_{s_2-1}$ be two sets of integers, with $s = s_1 + s_2$. It was shown in [3] that if $\{m_i\}$ and $\{\tilde{m}_i\}$ satisfy a suitable Polya condition [see (5.1) below], there is a unique polynomial of degree $sn-1$ which interpolates at the n roots of unity in the $(0, m_1, \ldots, m_{s_1-1})$ sense and at the n roots of -1 in the $(\tilde{m}_0, \tilde{m}_1, \ldots, \tilde{m}_{s_2-1})$ sense.

Following the terminology in [3], we shall call this the 2-periodic case of lacunary interpolation on the $2n$ roots of unity. We observe that the conditions for the n-periodic case of lacunary interpolation in (kn)th roots of unity, $k > 2$, to be regular are not known.

The objective of this note is twofold: (1) to give a proof of the conjecture of Cavaretta et al., and (2) to prove a similar result in the case of the 2-periodic case of lacunary interpolation. Section 2 deals with a few preliminaries and a lemma. In Section 3 we derive the explicit form of the kernel $K(t, z)$ such that

$$b_{sn-1}(z; f) = \frac{1}{2\pi i} \int_{\Gamma} f(t) K(t, z)\, dt \tag{1.3}$$

Here and elsewhere Γ denotes the circle $|z| = R < \rho$. In Section 4 we find a suitable form for $B_{sn-1, j}(z; f)$ and use it to give a proof of the conjecture.

Section 5 deals with the equiconvergence problem for the 2-periodic lacunary interpolation. We state the result and sketch the proof. In Section 6 we state the corresponding result for the general Hermite problem and offer a conjecture.

2. PRELIMINARIES

The following determinant will play a leading role in the sequel:

$$M(\nu) = M_n(\nu) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (\nu)_{m_1} & (\nu + n)_{m_1} & \cdots & (\nu + (s-1)n)_{m_1} \\ \vdots & \vdots & & \vdots \\ (\nu)_{m_{s-1}} & (\nu + n)_{m_{s-1}} & \cdots & (\nu + (s-1)n)_{m_{s-1}} \end{vmatrix}, \tag{2.1}$$

where $(\nu)_m = \nu(\nu - 1) \cdots (\nu - m + 1)$. We shall sometimes denote the elements of $M(\nu)$ by $a_{ij}(\nu)$ and its cofactors by $A_{ij}(\nu)$. We shall need the following lemmas:

LEMMA 1 [4]. *For fixed integers* $\{m\}_{i=1}^{q}$ *such that* $0 < m_1 < \cdots < m_q$, $m_j \leqslant jn$ $(j = 1, 2, \ldots, q)$ *and for any nonnegative integer*, ν, *the determinant*

$M(\nu)$ is positive. Also

$$\frac{1}{M(\nu)} = n^{-(m_1+\cdots+m_q)}\left[g(\nu)+O\left(\frac{1}{n}\right)\right] \quad \text{as} \quad n\to\infty, \tag{2.2}$$

where $g(\nu)$ is bounded and decreasing for $0 \leqslant \nu \leqslant n$.

Set

$$\Delta_\lambda(\nu, k) = \frac{1}{M(\nu)}\left\{A_{1,\lambda+1}(\nu) + \sum_{j=1}^{s-1}(\nu+kn)_{m_j}A_{j+1,\lambda+1}(\nu)\right\}. \tag{2.3}$$

It is easy to see that $\Delta_\lambda(\nu, k)$ is the ratio of two determinants. If $0 \leqslant k \leqslant s-1$, then $\Delta_\lambda(\nu, k) = 0$ or 1 according as $k \neq \lambda$ or $k = \lambda$. For $k \geqslant s$, the determinant in the numerator does not vanish, and, using a modified version of Lemma 1, we see that it is of the same order as the denominator. Thus, $\Delta_\lambda(\nu, k)$, $k \geqslant s$, does not vanish and is bounded by a constant as $n \to \infty$.

LEMMA 2. *The unique polynomial $q_m(t, z)$ of degree $n-1$ such that*

$$q_m(t, \omega^k) = m!\left.\frac{z^m}{(t-z)^{m+1}}\right]_{z=\omega^k}, \qquad k = 0, 1, \ldots, n-1,$$

is given explicitly by

$$q_m(t, z) = (-1)^m \sum_{\nu=0}^{n-1} \frac{d^m}{dt^m}\left(\frac{t^{n+m-\nu-1}}{t^n-1}\right)z^\nu. \tag{2.4}$$

Proof. A little manipulation shows that

$$q_m(t, z) = (-1)^m \frac{d^m}{dt^m}\left[\frac{t^m - z^m}{(t^n-1)(t-z)} + \frac{t^m z^m(t^{n-m} - z^{n-m})}{(t^n-1)(t-z)}\right],$$

which is easily seen to be a polynomial of degree $n-1$. From this we get (2.4) by differentiation. ∎

We note, for future use, the following identity which is easy to verify:

$$(-1)^m \frac{d^m}{dt^m}\left(\frac{t^{n+m-\nu-1}}{t^n-1}\right) = t^{n-1-\nu} \sum_{\lambda=0}^{\infty} \frac{(\nu+\lambda n)_m}{t^{(\lambda+1)n}}. \tag{2.5}$$

3. THE KERNEL $K(t, z)$

We shall determine the kernel $K(t, z)$ which is a polynomial of degree $ns-1$ and satisfies the conditions

$$\frac{\partial^\nu}{\partial z^\nu} K(t, z)\Big]_{z=\omega^k} = \frac{\partial^\nu}{\partial z^\nu} \frac{1}{t-z}\Big]_{z=\omega^k}$$

$$(k = 0, 1, \ldots, n-1, \quad \nu = 0, m_1, \ldots, m_{s-1}). \tag{3.1}$$

We shall prove

PROPOSITION 1. *The kernel $K(t, z)$ is given by*

$$K(t, z) = \sum_{\lambda=0}^{s-1} z^{\lambda n} P_\lambda(z), \qquad P_\lambda(z) \in \pi_{n-1}, \tag{3.2}$$

where

$$P_\lambda(z) = \frac{t^n - z^n}{(t-z)t^{(\lambda+1)n}} + \sum_{\nu=0}^{n-1} t^{n-1-\nu} z^\nu \sum_{k=s}^{\infty} \frac{\Delta_\lambda(\nu, k)}{t^{(k+1)n}} \tag{3.3}$$

and $\Delta_\lambda(\nu, k)$ is given by (2.3).

Proof. It is easy to verify that

$$\left(z^{\lambda n} P_\lambda(z)\right)^{(m_j)}_{z=\omega^k} = \omega^{-km_j} G_{j,\lambda n}(D) P_\lambda(z)\Big]_{z=\omega^k}, \tag{3.4}$$

where $D = d/dz$ and

$$G_{j,\lambda n}(D) = \sum_{p=0}^{m_j} \binom{m_j}{p} (\lambda n)_p z^{m_j-p} D^{m_j-p}, \qquad 0 \leqslant j, \lambda \leqslant q. \tag{3.5}$$

From (3.1), (3.2), (3.4), and (3.5) we obtain the following system of differential equations determining $P_\lambda(z)$:

$$\begin{aligned}\sum_{\lambda=0}^{s-1} P_\lambda(z) &= \sum_{\nu=0}^{n-1} \frac{t^{n-1-\nu}z^\nu}{t^n-1}\\ \sum_{\lambda=0}^{s-1} G_{j,\lambda n}(D)P_\lambda(z) &= (-1)^{m_j}\sum_{\nu=0}^{n-1} \frac{d^{m_j}}{dt^{m_j}}\left(\frac{t^{n+m_j-\nu-1}}{t^n-1}\right)z^\nu.\end{aligned} \tag{3.6}$$

Noting that an application of the Leibnitz rule yields $G_{j,\lambda n}(D)z^\nu = (\nu + \lambda n)_{m_j} z^\nu$, and appealing to the elementary theory of differential equations, we can derive (3.3) from (3.6) and the identity (2.5). ■

Remark 1. Recalling the fact that

$$\sum_{i=1}^{s} a_{ij}(\nu)A_{ij'}(\nu) = \begin{cases} M(\nu) & \text{if } j'=j,\\ 0 & \text{if } j'\neq j,\end{cases}$$

the relations (3.2) and (3.3) imply that

$$K(t,z) = \left\{1-\left(\frac{z^n}{t^n}\right)^s\right\}\frac{1}{t-z} + K_1(t,z), \tag{3.7}$$

where

$$K_1(t,z) = \sum_{\lambda=0}^{s-1} z^{\lambda n}R_\lambda(t,z), \tag{3.8}$$

with

$$R_\lambda(t,z) = \sum_{\nu=0}^{n-1} t^{n-1-\nu}z^\nu \sum_{k=s}^{\infty} \frac{\Delta_\lambda(\nu,k)}{t^{(k+1)n}}. \tag{3.9}$$

4. PROOF OF THE CONJECTURE

Let $g_{\nu,j}(z) = z^{\nu+(j+s-1)n}$ $(\nu = 0,1,\ldots,n;\ j = 1,2,\ldots)$. The polynomial $b_{sn-1}(z; g_{\nu,j})$ is easily determined by the method used above. We set

$$b_{sn-1}(z;g_{\nu,j}) = \sum_{\lambda=0}^{s-1} z^{\lambda n}Q_\lambda(z;\nu,j), \tag{4.1}$$

where the polynomials $Q_\lambda(z;\nu,j)\in\pi_{n-1}$ and are governed by the following set of differential equations:

$$\begin{gathered}\sum_{\lambda=0}^{s-1} Q_\lambda(z;\nu,j)=z^\nu,\\ \sum_{\lambda=0}^{s-1} G_{p,\lambda n}(D)Q_\lambda(z;\nu,j)=(\nu+(j+s-1)n)_{m_p}z^\nu \qquad (p=1,2,\ldots,s-1).\end{gathered} \tag{4.2}$$

Hence we have

$$Q_\lambda(z;\nu,j)=z^\nu\Delta_\lambda(\nu,j+s-1),$$

so that

$$b_{sn-1}(z;g_{\nu,j})=z^\nu\sum_{\lambda=0}^{s-1} z^{\lambda n}\Delta_\lambda(\nu;j+s-1). \tag{4.3}$$

We can now find the polynomials $B_{sn-1,j}(z;f)$. For $j=0$, we have

$$B_{sn-1,0}(z;f)=\sum_{0}^{sn-1} a_\nu z^\nu=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\left(1-\frac{z^{sn}}{t^{sn}}\right)dt. \tag{4.4}$$

For $j\geqslant 1$, we have

$$\begin{aligned}B_{sn-1,j}(z;f)&=\sum_{\nu=0}^{n-1} a_{\nu+(j+s-1)n}b_{sn-1}(z;g_{\nu,j})\\ &=\frac{1}{2\pi i}\int_\Gamma f(t)K_j^*(t,z)\,dt,\end{aligned} \tag{4.5}$$

where

$$K_j^*(t,z)=\frac{1}{t^{(s+j)n}}\sum_{\nu=0}^{n-1}t^{n-\nu-1}z^\nu\sum_{\lambda=0}^{s-1} z^{\lambda n}\Delta_\lambda(\nu,j+s-1). \tag{4.6}$$

From (3.7), (4.4), and (4.5), we see that

$$b_{sn-1}(z;f)+B_{sn-1,0}(z;f)-\sum_{j=1}^{l-1}B_{sn-1,j}(z;f)$$

$$=\frac{1}{2\pi i}\int_\Gamma f(t)\left[K_1(t,z)-\sum_{j=1}^{l-1}K_j^*(t,z)\right]dt$$

$$=\frac{1}{2\pi i}\int_\Gamma f(t)\sum_{\nu=0}^{n-1}t^{n-1-\nu}z^\nu\left[\sum_{k=s}^{\infty}\frac{\sum_{\lambda=0}^{s-1}z^{\lambda n}\Delta_\lambda(\nu,k)}{t^{(k+1)n}}\right.$$

$$\left.-\sum_{j=1}^{l-1}\frac{\sum_{\lambda=0}^{s-1}z^{\lambda n}\Delta_\lambda(\nu;j+s-1)}{t^{(s+j)n}}\right]dt$$

$$=\frac{1}{2\pi i}\int_\Gamma f(t)\sum_{\nu=0}^{n-1}t^{n-1-\nu}z^\nu\left[\sum_{k=s+l}^{\infty}\frac{\sum_{\lambda=0}^{s-1}z^{\lambda n}\Delta_\lambda(\nu,k)}{t^{(k+1)n}}\right]dt.$$

Since $\Delta_\lambda(\nu,k)$ is bounded as $n\to\infty$ for each fixed λ,ν,k, and $\Delta_\lambda(\nu,s+l)\neq 0$, it follows that the above difference tends to zero when $|z|<\rho^{1+(l/s)}$. This follows by considering the highest power of z in the numerator and the lowest power of t in the denominator and applying the methods used in [1].

REMARK. If $f(z)=1/\rho-z$, it follows from (1.1), (3.2), (3.3), and (4.3) that we have

$$\left[b_{ns-1}(z;f)-\sum_{j=0}^{k-1}B_{ns-1,j}(z;f)\right]_{z=\rho^{1+l/s}}$$

$$=\sum_{\lambda=0}^{s-1}\rho^{\lambda n(1+l/s)}\sum_{\nu=0}^{n-1}\rho^{n-1-\nu}\rho^{\nu(1+l/s)}\sum_{k=s+l-1}^{\infty}\frac{\Delta_\lambda(\nu,k)}{\rho^{(k+1)n}}$$

$$=\sum_{\nu=0}^{s-1}\rho^{\lambda n(1+l/s)}\sum_{\nu=0}^{n-1}\rho^{n-1-\nu}\rho^{\nu(1+l/s)}\sum_{k=s+l-1}^{\infty}\frac{\Delta_\lambda(\nu,k)}{\rho^{(k+1)n}}$$

$$=\frac{1}{\rho^{1+l/s}}\sum_{\nu=0}^{n-1}\frac{\Delta_{s-1}(\nu,s+l-1)}{\rho^{(n-1-\nu)l/s}}+O\left(\frac{1}{\rho^n}\right).$$

Since $\Delta_\lambda(v,k)$, $k \geqslant s$, does not vanish and is bounded by a constant as $n \to \infty$, it follows that in this case (1.2) does not hold when $z = \rho^{1+l/s}$. This shows that (1.2) is best possible.

5. 2-PERIODIC LACUNARY INTERPOLATION

We now extend the equiconvergence results to the case of 2-periodic lacunary interpolation where two different sequences of derivatives are prescribed at odd and at even powers of a $2n$th root of unity.

Let $0 = m_0 < m_1 < \cdots < m_{s_1-1}$ and $0 \leqslant \tilde{m}_0 < \cdots < \tilde{m}_{s_2-1}$ be two sequences of integers. We consider the problem of $(0, m_1, \ldots, m_{s_1-1})$ interpolation at ω^{2k} $(k = 0, 1, \ldots, n-1)$ and of $(\tilde{m}_0, \tilde{m}_1, \ldots, \tilde{m}_{s_2-1})$ interpolation at ω^{2k+1} $(k = 0, 1, \ldots, n-1)$, where ω is a primitive $2n$th root of unity. Thus the total number of conditions is $(s_1 + s_2)n$, and the interpolating polynomial is of degree $ns - 1$, where we set $s = s_1 + s_2$. Following [3], we require that

$$m'_k \leqslant kn, \qquad k = 0, 1, \ldots, s-1, \tag{5.1}$$

where $0 = m'_0 \leqslant m'_1 < \cdots < m'_{s-1}$ is the set of $\{m_i\} \cup \{\tilde{m}_i\}$ arranged in increasing order. This ensures the existence and uniqueness of the interpolatory polynomial.

In this case we need the determinant $\tilde{M}(\nu)$ of order s given by

$$\tilde{M}(\nu) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (\nu)_{m_1} & (\nu+n)_{m_1} & \cdots & (\nu+(s-1)n)_{m_1} \\ \vdots & \vdots & & \vdots \\ (\nu)_{m_{s_1-1}} & (\nu+n)_{m_{s_1-1}} & \cdots & (\nu+(s-1)n)_{m_{s_1-1}} \\ (\nu)_{\tilde{m}_0} & (-1)(\nu+n)_{\tilde{m}_0} & \cdots & (-1)^{s-1}(\nu+(s-1)n)_{\tilde{m}_0} \\ \vdots & \vdots & & \vdots \\ (\nu)_{\tilde{m}_{s_2-1}} & (-1)(\nu+n)_{\tilde{m}_{s_2-1}} & \cdots & (-1)^{s-1}(\nu+(s-1)n)_{\tilde{m}_{s_2-1}} \end{vmatrix}. \tag{5.2}$$

It has been shown [3, p. 672] that when (5.1) is satisfied, $\tilde{M}(\nu) \neq 0$.

Let $\tilde{b}_{ns-1}(z; f)$ be the polynomial of 2-periodic lacunary interpolation described above, and let

$$\tilde{B}_{ns-1,0}(z; f) = \sum_{\nu=0}^{ns-1} a_\nu z^\nu. \tag{5.3}$$

Further, let

$$\tilde{B}_{ns-1,j}(z;f) = \sum_{\nu=0}^{n-1} a_{\nu+(j+s-1)n}\tilde{b}_{ns-1}(z;g_{\nu,j}). \tag{5.4}$$

We shall prove

THEOREM 1. *For any f in A_ρ and for each positive integer l,*

$$\lim_{n\to\infty}\left[\tilde{b}_{ns-1}(z;f) - \sum_{j=0}^{l-1}\tilde{B}_{ns-1,j}(z;f)\right] = 0 \tag{5.5}$$

for $|z| < \rho^{1+(l/s)}$, the convergence being uniform and geometric for all z with $|z| \leqslant Z < \rho^{1+(l/s)}$. Moreover the result (5.5) *is best possible.*

Proof. The proof follows the same lines as that of the conjecture in Section 4. As in Section 3, we shall find the kernel $\tilde{K}(t,z)$ which is a polynomial in z of degree $ns-1$ and which satisfies the following conditions:

$$\left.\frac{\partial^{m_j}}{\partial z^{m_j}}\tilde{K}(t,z)\right]_{z=\omega^{2k}} = \frac{m_j!}{(t-\omega^{2k})^{m_j+1}},\ j=0,1,\ldots,s_1-1, \quad k=0,1,\ldots,n-1,$$

$$\left.\frac{\partial^{\tilde{m}_j}}{\partial z^{\tilde{m}_j}}\tilde{K}(t,z)\right]_{z=\omega^{2k+1}} = \frac{\tilde{m}_j!}{(t-\omega^{2k+1})^{\tilde{m}_j+1}},\ j=0,1,\ldots,s_2-1, \quad k=0,1,\ldots,n-1. \tag{5.6}$$

We set $\tilde{K}(t,z) = \sum_{\lambda=0}^{s-1} z^{\lambda n}\tilde{P}_\lambda(z)$ and let $G_{j,\lambda n}(D)$ and $\tilde{G}_{j,\lambda n}(D)$ denote differential operators given by

$$G_{j,\lambda n}(D) = \sum_{p=0}^{m_j}\binom{m_j}{p}(\lambda n)_p z^{m_j-p}D^{m_j-p},$$
$$\tilde{G}_{j,\lambda n}(D) = \sum_{p=0}^{\tilde{m}_j}\binom{\tilde{m}_j}{p}(\lambda n)_p z^{\tilde{m}_j-p}D^{\tilde{m}_j-p}. \tag{5.7}$$

As in Section 3, the following differential equations determine $\tilde{P}_\lambda(z)$:

$$\begin{aligned} \sum_{\lambda=0}^{s-1} G_{j,\lambda n}(D)\tilde{P}_\lambda(z) &= q_{m_j}(t,z), \qquad j=0,1,\dots,s_1-1, \\ \sum_{\lambda=0}^{s-1}(-1)^\lambda \tilde{G}_{j,\lambda n}(D)\tilde{P}_\lambda(z) &= \tilde{q}_{\tilde{m}_j}(t,z), \qquad j=0,1,\dots,s_2-1, \end{aligned} \tag{5.8}$$

where $q_m(t,z)$ is given by (2.4) and $\tilde{q}_m(t,z)$ is defined by

$$\tilde{q}_m(t,z) = (-1)^m \sum_{\nu=0}^{n-1} \frac{d^m}{dt^m}\left(\frac{t^{n+m-\nu-1}}{t^n+1}\right) z^\nu. \tag{5.9}$$

Observe that $\tilde{q}_{\tilde{m}_j}$ satisfies the condition at odd powers of ω, stated in (5.6). We note the analogue of the identity (2.5), viz.

$$(-1)^m \frac{d^m}{dt^m}\left(\frac{t^{n+m-\nu-1}}{t^n+1}\right) = t^{n-1-\nu} \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda(\nu+\lambda n)_m}{t^{(\lambda+1)n}}. \tag{5.10}$$

If $\tilde{A}_{ij}(\nu)$ denotes the cofactor of the (i,j) element in (5.2), we get from (5.8) an explicit form for $\tilde{P}_\lambda(z)$. Thus we have

$$\tilde{P}_\lambda(z) = \sum_{\nu=0}^{n-1} t^{n-\nu-1} z^\nu \sum_{j=0}^{\infty} \frac{\tilde{\Delta}_\lambda(\nu,j)}{t^{(j+1)n}} \tag{5.11}$$

where

$$\begin{aligned} \tilde{\Delta}_\lambda(\nu,j) = \Bigg\{ & \sum_{i=1}^{s_1} \tilde{A}_{i\lambda}(\nu)\cdot(\nu+jn)_{i-1} \\ & + \sum_{i=1}^{s_2} \tilde{A}_{i+s_1,\lambda}(\nu)(-1)^j(\nu+jn)_{i-1}\Bigg\}[\tilde{M}(\nu)]^{-1}. \end{aligned} \tag{5.12}$$

Observe that for $0 \leqslant j \leqslant s-1$, $j \neq \lambda$, the sum on the right in (5.12) vanishes, whereas for $j=\lambda$, $\tilde{\Delta}_\lambda(\nu,\lambda)=1$, in view of the definition of $\tilde{A}_{ij}(\nu)$ as cofactors of $\tilde{M}(\nu)$. Moreover, for $j \geqslant s$, the determinant in the numerator does not vanish, in view of the method of proof used in [3, p. 672]. Hence we have

$$\tilde{P}_\lambda(z) = \frac{t^n - z^n}{(t-z)t^{(\lambda+1)n}} + \sum_{\nu=0}^{n-1} t^{n-\nu-1} z^\nu \sum_{j=s}^{\infty} \frac{\tilde{\Delta}_\lambda(\nu, j)}{t^{(j+1)n}}$$

$$(\lambda = 0, 1, \ldots, s-1). \quad (5.13)$$

This implies

$$\tilde{K}(t,z) = \frac{t^{sn} - z^{sn}}{(t-z)t^{sn}} + \sum_{\nu=0}^{n-1} t^{n-\nu-1} z^\nu \sum_{j=s}^{\infty} \frac{\sum_{\lambda=0}^{s-1} z^{\lambda n} \tilde{\Delta}_\lambda(\nu, j)}{t^{(j+1)n}},$$

$$\tilde{b}_{sn-1,j}(z;f) - \sum_{j=0}^{l-1} \tilde{B}_{sn-1,j}(z;f) = \sum_{\nu=0}^{n-1} t^{n-\nu-1} z^\nu \sum_{j=s+l}^{\infty} \frac{\sum_{\lambda=0}^{s-1} z^{\lambda n} \tilde{\Delta}_\lambda(\nu, j)}{t^{(j+1)n}}.$$

By using the same reasoning as in the proof of Lemma 1 (cf. [4]), we can see that $\tilde{\Delta}_\lambda(\nu, j)$ is bounded for fixed λ, ν, j as $n \to \infty$. Proceeding as in Section 4, we thus obtain (5.5).

If we take $f(z) = 1/(\rho - z)$ and argue as in the Remark at the end of Section 4, we see that (5.5) is best possible. ∎

6. EXTENSIONS

An extension to general mixed Hermite-Birkhoff conditions is not straightforward, in view of the unavailability of existence and uniqueness theorems in the general case. However, the Hermite problem is always solvable, and the equiconvergence can be analyzed using our methods.

For $f \in A_\rho$ consider the Hermite interpolation polynomial $h_{ns-1}(z; f)$ at $3n$th roots of unity according to the scheme

$$\begin{array}{lll} (0,1,\ldots,s_1-1) & \text{at } \omega^{3k}, & k = 0,1,\ldots,n-1, \\ (0,1,\ldots,s_2-1) & \text{at } \omega^{3k+1}, & k = 0,1,\ldots,n-1, \\ (0,1,\ldots,s_3-1) & \text{at } \omega^{3k+2}, & k = 0,1,\ldots,n-1, \end{array} \quad (6.1)$$

where ω is a primitive root of $\omega^{3n}=1$ and $s=s_1+s_2+s_3$. Let $h_{sn-1}(z;f)$ denote the Hermite interpolant to f satisfying (6.1). Let $H_{sn-1}(z;f)=\sum_{\nu=0}^{sn-1}a_\nu z^\nu$, and set

$$H_{sn-1,j}(z;f)=\sum_{\nu=0}^{n-1}a_{\nu+(j+s-1)n}h_{sn-1}(z;g_{\nu,j}),$$

where $g_{\nu,j}$ is defined as at the beginning of Section 4. Then following the above analysis we can prove

THEOREM 2. *For any f in A_ρ and for each positive integer l,*

$$\lim_{n\to\infty}\left[h_{sn-1}(z;f)-\sum_{j=0}^{l-1}H_{sn-1,j}(z;f)\right]=0 \tag{6.2}$$

for $|z|<\rho^{1+(l/s)}$, where the convergence is uniform and geometric in $|z|\leqslant Z<\rho^{1+(l/s)}$. Moreover, the result (6.2) *is best possible.*

We conjecture that whenever the mixed Hermite-Birkhoff problem on (nq)th roots of unity has a unique solution, an analogue of Theorem 1 is true.

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