Large orbits of odd-order subgroups of solvable linear groups

Yong Yang

Department of Mathematics, University of Wisconsin–Parkside, Kenosha, WI 53141, USA

Article history:
Received 22 October 2010
Available online 14 December 2011
Communicated by Michel Broué

MSC:
20C20

Keywords:
Representations of solvable groups
Solvable linear groups
Regular orbits

1. Introduction

One of the most important and natural questions about orbit structure is to establish the existence of an orbit of a certain size. For a long time, there has been a deep interest and need to examine the size of the largest possible orbits in linear group actions. Obviously the size of an orbit can never exceed the order of the group. If \( V \) is a \( G \)-module with a regular orbit, this means that there is an orbit \( \{ v^g \mid g \in G \} \) such that \( C_G(v) = 1 \) holds. However, there do not always exist regular orbits even for nilpotent groups; although in this case it is possible to find \( v_1 \) and \( v_2 \) in \( V \) such that \( C_G(v_1) \cap C_G(v_2) = 1 \). This fact easily implies the existence of a vector \( v \in V \) such that \( |G : C_G(v)|^2 \geq |G| \).

Suppose that \( G \) is a finite group and \( V \) is a finite, faithful and completely reducible \( G \)-module. The existence of regular orbits or large orbits have been studied extensively in the literature (for example [1–6,13,16]).

In [3], the following result was obtained. Let \( G \) be a solvable group of odd order and let \( V \) be a finite, faithful, irreducible and quasi-primitive \( G \)-module over a field of odd characteristic, then either \( G \leq \Gamma(V) \) or \( G \) has at least two regular orbits on \( V \). As an application of this theorem, Espuelas [3]
proved the following. Let $G$ be a finite odd-order group and let $V$ be a finite, faithful and completely reducible $G$-module over a field of odd characteristic, then $G$ has at least two regular orbits on $V \oplus V$. Dolfi [1, Theorem 3.1] extended this to the case where $V$ is over a field of characteristic 2.

In [4], Espuelas showed the following. Let $G$ be a solvable group of odd order and $V$ be a faithful and completely reducible $G$-module over a field of odd characteristic, assume that $V$ is endowed with a non-singular symplectic form fixed by $G$, then $G$ has at least two regular orbits on $V$.

Espuelas and Navarro [6] proved the following result. Let $G$ be a group of odd order and let $H$ be a Hall $\pi$-subgroup of $G$. Let $V$ be a faithful $G$-module, over possibly different finite fields of odd $\pi$-characteristic and assume that $V_{\odot \pi(G)}$ is completely reducible, then there exists $v \in V$ such that $C_H(v) \subseteq O_{\pi}(G)$.

All of the results in the previous paragraphs can be extended to the situation when $H$ is an odd-order subgroup of a solvable group $G$ where $G$ acts faithfully and completely reducibly on $V$. We prove the following results.

**Theorem 1.1.** Let $G$ be a solvable group and let $V$ be a finite, faithful and quasi-primitive $G$-module over a field of odd characteristic. Let $H$ be an odd-order subgroup of $G$. Then either $G \leq \Gamma(V)$ or $H$ has at least two regular orbits on $V$.

**Theorem 1.2.** Suppose that $G$ is a finite solvable group and $V$ is a finite, faithful and completely reducible $G$-module. Let $H$ be an odd-order subgroup of $G$, then $H$ has at least 2 regular orbits on $V \oplus V$. Suppose in addition that $|V|$ is odd, then there exists $v \in V$ in a regular orbit of $F(G) \cap H$ such that $C_H(v) \subseteq F_2(G)$.

**Theorem 1.3.** Let $G$ be a finite solvable group and $V$ be a faithful and completely reducible $G$-module over a field of odd characteristic, assume that $V$ is endowed with a non-singular symplectic form fixed by $G$. Let $H$ be an odd-order subgroup of $G$, then $H$ has at least two regular orbits on $V$.

**Theorem 1.4.** Let $G$ be a solvable group and let $H$ be an odd-order Hall $\pi$-subgroup of $G$. Let $V$ be a faithful $G$-module, over possibly different finite fields of odd $\pi$-characteristic and assume that $V_{\odot \pi(G)}$ is completely reducible, then there exists $v \in V$ such that $C_H(v) \subseteq O_{\pi}(G)$.

The orbit theorems of this kind turn out to be useful when dealing with problems about character degrees or conjugacy class sizes. We provide some applications of these orbit theorems at the end of the paper.

2. Notation and lemmas

Notation:

1. Let $G$ be a finite group, let $S$ be a subset of $G$ and let $\pi$ be a set of different primes. For each prime $p$, we denote $SP_p(S) = \{ (x) \mid o(x) = p, \; x \in S \}$ and $EP_p(S) = \{ x \mid o(x) = p, \; x \in S \}$. We denote $SP(S) = \bigcup_{p \text{ prime}} SP_p(S)$, $SP_\pi(S) = \bigcup_{p \in \pi} SP_p(S)$, $EP(S) = \bigcup_{p \text{ prime}} EP_p(S)$ and $EP_\pi(S) = \bigcup_{p \in \pi} EP_p(S)$. We denote $NEP(S) = |EP(S)|$, $NEP_p(S) = |EP_p(S)|$ and $NEP_\pi(S) = |EP_\pi(S)|$.

2. Let $n$ be an even integer, $q$ a power of a prime. Let $V$ be a symplectic vector space of dimension $n$ over $\mathbb{F}_q$. We use $SCRSp(n, q)$ or $SCRSp(V)$ to denote the set of all solvable completely reducible subgroups of $Sp(V)$. We use $SIRSp(n, q)$ or $SIRSp(V)$ to denote the set of all solvable irreducible subgroups of $Sp(V)$.

3. Let $V$ be a finite vector space and let $G \subseteq GL(V)$. We define $PC(G, V, p, i) = \{ x \mid x \in EP_p(G) \}$ and $\dim(C_V(x) = i)$ and $NPC(G, V, p, i) = |PC(G, V, p, i)|$. We will drop $V$ in the notation when it is clear in the context.

4. If $V$ is a finite vector space of dimension $n$ over $GF(q)$, where $q$ is a prime power, we denote by $\Gamma^\circ(q^n) = \Gamma^\circ(V)$ the semilinear group of $V$, i.e.,

$$
\Gamma^\circ(q^n) = \{ x \mapsto a x^\sigma \mid x \in GF(q^n), \; a \in GF(q^n)^*, \; \sigma \in Gal(GF(q^n)/GF(q)) \}.
$$
and we define
\[ I_0(q^n) = \{ x \mapsto ax \mid x \in GF(q^n), a \in GF(q^n) \}. \]

(5) We use \( F(G) \) to denote the Fitting subgroup of \( G \). Let \( F_0(G) \leq F_1(G) \leq F_2(G) \leq \cdots \leq F_n(G) = G \) denote the ascending Fitting series, i.e. \( F_0(G) = 1, F_1(G) = F(G) \) and \( F_{i+1}(G)/F_i(G) = F(G/F_i(G)) \). \( F_i(G) \) is the \( i \)th ascending Fitting subgroup of \( G \).

(6) We use \( H \cdot S \) to denote the wreath product of \( H \) with \( S \) where \( H \) is a group and \( S \) is a permutation group.

(7) Let \( G \) be a finite group and denote by \( b(G) = \max\{|\psi| \mid \psi \in \text{Irr}(G)\} \) the largest degree of an irreducible character of \( G \).

(8) Let \( G \) be a finite group and \( K \leq L \) be normal subgroups of \( G \). Let \( H \) be a subgroup of \( G \) and we denote \( \pi_{L/K}(H) \) to be the image of \( H \cap L \) on \( L/K \).

**Definition 2.1.** Suppose that a finite solvable group \( G \) acts faithfully, irreducibly and quasi-primitively on a finite vector space \( V \). Let \( F(G) \) be the Fitting subgroup of \( G \) and \( F(G) = \prod_i P_i \), \( i = 1, \ldots, m \) where \( P_i \) are normal \( p_i \)-subgroups of \( G \) for different primes \( p_i \). Let \( Z_i = \Omega_1(Z(P_i)) \). We define
\[
E_i = \begin{cases} 
\Omega_1(P_i) & \text{if } p_i \text{ is odd;} \\
[P_i, G, \ldots, G] & \text{if } p_i = 2 \text{ and } [P_i, G, \ldots, G] \neq 1; \\
Z_i & \text{otherwise.}
\end{cases}
\]

By proper reordering we may assume that \( E_i \neq Z_i \) for \( i = 1, \ldots, s \), \( 0 \leq s \leq m \) and \( E_i = Z_i \) for \( i = s + 1, \ldots, m \). We define \( E = \prod_{i=1}^s E_i, Z = \prod_{i=1}^s Z_i \) and we define \( \bar{E}_i = E_i/Z_i, \bar{E} = E/Z \). Furthermore, we define \( e_i = \sqrt{|E_i/Z_i|} \) for \( i = 1, \ldots, s \) and \( e = \sqrt{|E/Z|} \).

**Theorem 2.2.** Suppose that a finite solvable group \( G \) acts faithfully, irreducibly and quasi-primitively on an \( n \)-dimensional finite vector space \( V \) over finite field \( F \) of characteristic \( r \). We use the notation in Definition 2.1. Then every normal abelian subgroup of \( G \) is cyclic and \( G \) has normal subgroups \( Z \leq U \leq F \leq A \leq G \) such that:

1. \( F = EU \) is a central product where \( Z = E \cap U = Z(E) \) and \( C_G(F) \leq F \);
2. \( F/U \cong E/Z \) is a direct sum of completely reducible \( G/F \)-modules;
3. \( E_i \) is an extraspecial \( p_i \)-group for \( i = 1, \ldots, s \) and \( e_i = p_i^{n_i} \) for some \( n_i \geq 1 \). Furthermore, \((e_1, e_j) = 1\) when \( i \neq j \) and \( e = e_1 \cdots e_s \) divides \( n \), also gcd\((r, e) = 1\);
4. \( A = C_G(U) \) and \( G/A \leq \text{Aut}(U) \) and \( G \) acts faithfully on \( E/Z \);
5. \( A/C_A(E_i/Z_i) \leq \text{Sp}(2n_i, p_i) \);
6. \( U \) is cyclic and acts fixed point freely on \( W \) where \( W \) is an irreducible submodule of \( V_U \);
7. \(|V| = |W|^{n}\) for some integer \( b \) and \(|G:A| \cdot \dim(W)\);
8. \( G/A \) is cyclic.

**Proof.** This follows from [15, Theorem 2.2]. The fact that \( G/A \) is cyclic is proved there although it is not explicitly stated. \( \square \)

**Lemma 2.3.** Suppose that a finite solvable group \( G \) acts faithfully, irreducibly and quasi-primitively on a finite vector space \( V \). Using the notation in Theorem 2.2, we have \(|G| \cdot |\dim(W)\cdot |A/F|\cdot e^2 \cdot (|W| - 1)|\).

**Proof.** By Theorem 2.2, \(|G| = |G/A||A/F||F|\) and \(|F| = |E/Z||U|\). Since \(|G/A| \cdot \dim(W), |E/Z| = e^2\) and \(|U| (|W| - 1)\), we have \(|G| \cdot \dim(W)\cdot |A/F|\cdot e^2 \cdot (|W| - 1)|. \( \square \)

**Lemma 2.4.** Suppose that a finite solvable group \( G \) acts faithfully and quasi-primitively on a finite vector space \( V \) over the field \( \mathbb{F} \). Let \( g \in \text{EP}_s(G) \) where \( s \) is a prime and we use the notation in Theorem 2.2.
(1) If \( g \in F \) then \( |C_V(g)| \leq |W|^{1/e} \).
(2) If \( g \in A \setminus F \) then \( |C_V(g)| \leq |W|^{1/e} \).
(3) If \( g \in A \setminus F \), \( s \geq 3 \) and \( s \nmid |E| \), then \( |C_V(g)| \leq |W|^{1/e} \).
(4) If \( g \in G \setminus A \) then \( |C_V(g)| \leq |W|^{1/eb} \).

**Proof.** (1) is a slightly improvement of [14, Lemma 2.4(1)], the proof is similar to [14, Lemma 2.4(3)].
(2) and (3) follow from [14, Lemma 2.4(2) and (3)] respectively. Let \( K \) be the algebraic closure of \( F \), then \( W \otimes_K K = W_1 \oplus W_2 \oplus \cdots \oplus W_m \) where the \( W_i \) are Galois conjugate, non-isomorphic irreducible \( U \)-modules. In particular, each \( W_i \) is faithful, \( \dim_K W_i = 1 \). Clearly \( N_G(W_i) \supseteq C_G(U) \) for each \( i \). Furthermore, \( [N_G(W_i), U] \leq C_U(W_i) \) since \( U \) is normal. Thus \( N_G(W_i) = C_G(U) = A \). It follows that \( G/A \) permutes the set \( \{W_1, \ldots, W_m\} \) in orbits of length \( |G : A| \) and thus \( |G/A| | \dim(W) \). Since \( G/A \) permutes the \( W_i \) fixed point freely, for all \( g \in G \setminus A \) of order \( s \) where \( s \) is a prime, \( |C_V(g)| = |W|^{1/eb} \). This proves (4). \( \square \)

**Lemma 2.5.** Let \( G \) be a finite solvable group and \( V \) be a finite, faithful irreducible \( FG \)-module with dimension \( \prod p_i^{n_i} \) where \( p_i \) are different primes. \( F \) is algebraically closed and \( \text{char}(F) = s \) where \( s \prod p_i = 1 \). \( E \) is a direct product of normal extraspecial subgroups \( E_i \)'s of \( G \) and \( |E_i| = p_i^{2n_i+1} \). Define \( Z_i = Z(E_i) \) and \( Z = \prod Z_i \). Consider \( x \in G \), \( x \) is of prime order different than the characteristic of \( V \) and \( x \) acts trivially on \( Z \). In [9] Isaacs defined good element. Let \( C/Z = C_{E/Z}(x) \), in our situation, \( x \) is a good element if \( [x, C] = 1 \). We call an element bad if it is not good. We have the following:

(1) If \( x \) is a good element, then we have that the Brauer character of \( x \) on \( V \), say \( \chi(x) \) is then such that 
\[ |\chi(x)|^2 = |C_{E/Z}(x)|. \]
(2) If \( x \) is a bad element, then \( \chi(x) = 0 \).

**Proof.** By [9, Theorem 3.5]. \( \square \)

**Lemma 2.6.** Suppose that a finite solvable group \( G \) acts faithfully, irreducibly and quasi-primitively on a finite vector space \( V \) over a field \( F \). Using the notation in Theorem 2.2, let \( x \in E_{p_i}(A \setminus F) \) where \( s \) is a prime and \( s, \text{char}(F) = 1 \). Let \( C/Z = C_{E/Z}(x) \), following Isaacs [9], we say \( x \) is a good element if \( [x, C] = 1 \), we say \( x \) is a bad element if it is not good.

(1) Assume \( x \) is a bad element and let \( \beta = e/s \), then \( |C_V(x)| \leq |W|^{\beta} \).
(2) Assume \( x \) is a good element and \( |C_{E/Z}(x)| \leq a \), let

\[ \beta = \left[ \frac{1}{s} (e + (s - 1)a^{1/2}) \right] \]

then \( |C_V(x)| \leq |W|^{\beta} \).
(3) Assume \( b = 1 \) and \( |W| \) is a prime, then \( \dim_F(C_V(x)) = \frac{1}{s} \sum_{y \in \langle x \rangle} \chi(y) \).

**Proof.** Note that \( \text{char}(F) \nmid |\langle x \rangle| \). Let \( K \) be a splitting field for \( \langle x \rangle \) which is a finite extension of \( F \) and set \( V_K = V \otimes_K K \). Since \( \dim_K(C_{V_K}(x)) = \dim_F(C_V(x)) \), we may consider \( V_K \) instead of \( V \). Let \( 0 = V_0 \subset V_1 \subset \cdots \subset V_l = V_K \) be an \( \langle x \rangle \)-composition series for \( V_K \) with quotient \( V_j = V_j/V_{j-1} \) for \( j = 1, \ldots, l \). Thus each \( V_j \) is an absolutely irreducible \( \langle x \rangle \)-module. Since \( V_K \) is obtained by tensoring a quasi-primitive module up to a splitting field, \( V_{K/Z} \) is a direct sum of Galois conjugate irreducible modules, \( Z \) is faithful on every irreducible summand of \( V_{K/Z} \). By the Jordan–Holder Theorem, these are the only irreducibles that can occur in \( V_{j/Z} \) and thus \( Z \) acts faithfully on \( V_j \). Since all nontrivial normal subgroups of \( \langle x \rangle \) have nontrivial intersections with \( Z \), \( \langle x \rangle \) is faithful on \( V_j \). Since \( x \) centralizes \( Z \), \( V_j \) is also an irreducible \( E \)-module. Thus we know that \( \dim V_j = \prod p_{i}^{n_i} = e \) and Lemma 2.5 can be applied on \( \langle x \rangle \) and \( V_j \). We let \( \chi_j \) be the Brauer character of \( V_j \). If \( x \)
is a bad element, then all the nontrivial elements \( y \in (x) \) are bad elements and \( \chi_j(y) = 0 \) and \( \dim_{\mathbb{K}}(\mathcal{C}_{Y_j}(x)) = \frac{1}{2} \sum_{y \in (x)} \chi_j(y) = \frac{1}{2} \prod p_i^{n_i} = \frac{6}{7} \). If \( x \) is a good element, then all the nontrivial elements \( y \in (x) \) are good elements and \( |\chi_j(y)|^2 = |\mathcal{C}_{E/z}(y)| \leq a \) and thus \( |\chi_j(y)| \leq a^{1/2} \). \( \dim_{\mathbb{K}}(\mathcal{C}_{Y_j}(x)) = \frac{1}{2} \sum_{y \in (x)} |\chi_j(y)| \leq \left( \frac{1}{2} + (s - 1)a^{1/2} \right) \). Since \( \dim_{\mathbb{K}}(\mathcal{C}_{Y}(x)) \leq \sum_j \dim_{\mathbb{K}}(\mathcal{C}_{Y_j}(x)) \), (1) and (2) hold.

If \( b = 1 \) and \( |W| \) is a prime, \( \dim_{\mathbb{F}} V = \dim_{\mathbb{S}} V_{\mathbb{K}} = e \). Thus \( V_{\mathbb{K}} = V_0 \) and (3) is clear. □

**Lemma 2.7.** Suppose that \( G \leq (V) \) and let \( G \bigcap H_0(V) = U \). Assume \( x \in G \setminus U \) and \( o(x) = r \) where \( r \) is a prime, then \( |C_V(x)| \leq |V|^{1/r} \).

**Proof.** We may write \( x = \sigma a \) where \( \sigma \in \text{Gal}(GF(p^n) : GF(p)) \), \( o(\sigma) = r \) and \( a \in U \). Suppose that \( \nu_0 \) is a nonzero fixed point of \( x \), then \( v_0^\nu = v_0 \) and this implies that \( v_0^p = a = v_0 \). If \( v^x = v \) then \( v^p = a = v \) and we have that \( (v_0^{-1})^{p^{n/r} - 1} = 1 \). Thus \( |C_V(x)| \leq |V|^{1/r} \). □

**Lemma 2.8.** Assume \( G \) satisfies Theorem 2.2 and we adopt the notation in it. Let \( p \) be a prime and \( x \in E_{P_F}(\alpha ; F) \) and assume \( |E_{F/z}(x)| = \prod p_i^{M_i} \). Define \( |U|_p = \text{gcd}(|U|, p) \). We have the following:

1. \( \text{NEP}_p(A \setminus F) \leq \text{NEP}_p(A/F) / \{|U|_p| \}
2. \( \text{NEP}_p(A \setminus F) \leq \prod_{p_i \neq p} |U|_p / \prod p_i^{M_i} \)
3. \( \text{NEP}_p(xF) \leq \prod p_i^{2M_i} / \prod_i |U|_p \)

**Proof.** (1) and (2) follow from [14, Lemma 2.7]. By the proof of [14, Lemma 2.7(3)], we know that \( \text{NEP}_p(xF/U) \leq M_i \). Let \( a \in A \) and \( o(a) = p \), we consider \( \text{NEP}_p(\alpha U) \). Since \( U \leq Z(A) \), \( \text{NEP}_p(\alpha U) \leq |U|_p \) and (3) follows. □

**Lemma 2.9.** Let \( V \) be a symplectic vector space of dimension \( n \) with base field \( \mathbb{F} \) and \( G \in \text{SIRSp}(n, \mathbb{F}) \). Assume that the action is not quasi-primitive and \( N = G \) is maximal such that \( V_N \) is not homogeneous. Let \( V_N = V_1 \oplus \cdots \oplus V_t \), where \( V_i \)'s are the homogeneous \( N \)-modules and clearly \( t \geq 2 \). Then either all \( V_i \) are non-singular or all are totally isotropic. In the first case, \( \dim(V_i) \) is even, \( G \leq H : S \) as linear groups where \( H \in \text{SIRSp}(V_1) \). In the second case \( t = 2, V_2 \) is isomorphic to \( V_1^* \) as an \( N \)-module, and we say that \( V_N \) is a pair.

**Proof.** This is [14, Lemma 2.9]. □

**Lemma 2.10.** Let \( V \) be a symplectic vector space of dimension \( 2n \) with base field \( \mathbb{F} \) and \( G \in \text{SIRSp}(2n, \mathbb{F}) \), \( |\mathbb{F}| = p \) where \( p \) is a prime. Suppose \( G \) acts irreducibly and quasi-primitively on \( V \) and we adopt the notation in Theorem 2.2. Assume \( e = 1 \), then we have the following:

1. \( G \leq (p^{2n}) \), \( G/U \) is cyclic and \( |G/U| \mid 2n \).
2. \( U \leq G_0(p^{2n}) \) and \( |U| \mid p^{n} + 1 \).

**Proof.** By [13, Proposition 3.1(1)], \( G \) may be identified with a subgroup of the semidirect product of \( GF(p^{2n}) \times \text{Gal}(GF(p^{2n}) : GF(p)) \) acting in a natural manner on \( GF(p^{2n}) \). Also \( G \cap GF(p^{2n}) = U \) and \( |G \cap GF(p^{2n})| = p^{n} + 1 \). Clearly \( G/U \) is cyclic of order dividing \( 2n \). Now (1) and (2) hold. □

**Lemma 2.11.** Let \( V \) be a finite, faithful irreducible \( G \)-module and \( G \) is solvable. Suppose that \( V \) is a vector space of dimension \( n \) over the field \( \mathbb{F} \). Let \( (n, \mathbb{F}) = (4, \mathbb{F}_2) \), then \( |G|_2 \leq 15 \).
Proof. Assume $V$ is irreducible, then $G$ satisfies one of the following:

1. $G \cong S_3 \times S_2$, $|G| = 72$ and thus $|G|_2^r \leq 9$.
2. $V$ is quasi-primitive and $e = 1$. Thus $G \cong \Gamma(2^4)$, $|G|$ $| 60$ and $|G|_2^r \leq 15$.

Assume $V$ is reducible, then $G \cong S_3 \times S_3$ and thus $|G|_2^r \leq 9$. 

Lemma 2.12. Let $n$ be an even integer and $V$ be a symplectic vector space of dimension $n$ of field $\mathbb{F}$. Let $G \in \text{SCRSp}(n, \mathbb{F})$.

1. Let $(n, \mathbb{F}) = (2, \mathbb{F}_2)$, then $G \cong S_3$, $|G| \leq 6$, $\text{NEP}_2(G) \leq 3$ and $\text{NEP}_3(G) \leq 2$.
2. Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$, then $|G|_2^r \leq 9$.
3. Let $(n, \mathbb{F}) = (6, \mathbb{F}_2)$, then $|G|_2^r \leq 81$.
4. Let $(n, \mathbb{F}) = (8, \mathbb{F}_2)$, then $|G|_2^r \leq 243$.
5. Let $(n, \mathbb{F}) = (2, \mathbb{F}_3)$, then $G \cong \text{SL}(2, 3)$.
6. Let $(n, \mathbb{F}) = (4, \mathbb{F}_3)$ and let $H$ be an odd-order subgroup of $G$, then $|H| \leq 9$, $\text{NEP}_3(H) \leq 8$, $\text{NEP}_5(H) \leq 4$, $\text{NEP}_{(3, 5)}(H) = 0$, $\text{NPC}(H, 3, 2) \leq 4$ and $\text{NPC}(H, 3, 3) \leq 4$.

Proof. If $V$ is not irreducible, we may choose an irreducible submodule $W$ of $V$ of smallest possible dimension and set $\dim(W) = m$. Since the form $( , )$ is $G$-invariant, the subspace $\{v \in W \mid (v, v') = 0$ for all $v' \in W\}$ is a submodule of $W$ and the form $( , )$ is either totally isotropic or non-singular on $W$. If the form $( , )$ is totally isotropic on $W$, then set $W^\perp = \{v \in V \mid (v, w) = 0$ for all $w \in W\}$. For $v \in V$, we consider the map $f_v \in W^* := \text{Hom}(W, \mathbb{F})$, defined by $f_v(w) = (v, w)$, $w \in W$. Then $v \mapsto f_v$, $v \in V$, induces a $G$-isomorphism between $V/W^\perp$ and the dual space $W^*$. Since $V$ is completely reducible, we may find an irreducible $G$-submodule $U \cong W^*$ such that the form is non-singular on $X = W \oplus U$. If the form $( , )$ is totally isotropic on $W$ and $n > 2m$, then $V = X \oplus X^\perp$ where $X^\perp$ is not trivial. If the form $( , )$ is non-singular on $W$, then $V = W \oplus W^\perp$ where $W^\perp$ is not trivial. In both cases we may view $G \cong \text{SCRSp}(V_1) \times \text{SCRSp}(V_2)$ as linear groups where $\dim(V_1), \dim(V_2) = n$. If the form $( , )$ is totally isotropic on $W$ and $n = 2m$, then $V = W \oplus U$ and the action of $G$ on $V$ is a pair. We use this classification repeatedly in the following arguments.

We prove these different cases one by one.

1. Let $(n, \mathbb{F}) = (2, \mathbb{F}_2)$ and the result is clear.
2. Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$ Assume $V$ is irreducible, then $G$ satisfies one of the following:
   a. $G \cong S_3 \times S_2$, $|G| \leq 72$ and $|G|_2^r \leq 9$.
   b. $V$ is quasi-primitive and $e = 1$. $|G| \leq 54$ by Lemma 2.10 and $|G|_2^r \leq 5$.
   Assume $V$ is reducible, then $G \cong S_3 \times S_3$ and thus $|G|_2^r \leq 9$. Hence the result holds in all cases.
3. Let $(n, \mathbb{F}) = (6, \mathbb{F}_2)$. Assume $V$ is irreducible, then $G$ satisfies one of the following:
   a. $t = 2$ and $\dim(V_1) = 3$, the action of $N$ on $V$ must be a pair by Lemma 2.9. Thus $|G|_2^r \leq 21$.
   b. $G \cong S_3 \times S_3$ and $|G|_2^r \leq 81$.
   c. $V$ is quasi-primitive and $e = 1$. $|G| \leq 15$ by Lemma 2.10 and $|G|_2^r \leq 27$.
   d. $V$ is quasi-primitive and $e = 3$, then $A/F \cong \text{SL}(2, 3)$ and $|A/F| \leq 24$, $|W| \leq 2^2 \cdot 3^3$ by Lemma 2.3. Thus $|G|_2^r \leq 81$.
   Assume $V$ is reducible, then $G$ satisfies one of the following:
   a. $G \cong \text{GL}(3, 2) \times \text{GL}(3, 2)$ and the action of $G$ on $V$ is a pair. Thus $|G|_2^r \leq 21$.
   b. $G \in \text{SCRSp}(2, 2) \times \text{SCRSp}(4, 2)$ and $|G|_2^r \geq 27$ by (1) and (2).
   Hence the result holds in all cases.
4. Let $(n, \mathbb{F}) = (8, \mathbb{F}_2)$. Assume $V$ is irreducible and not quasi-primitive, then $G$ satisfies one of the following:
   a. $G \cong H \cong S_2$ where $H$ is an irreducible subgroup of $\text{GL}(4, 2)$, $|G|_2^r \leq 15^2$ by Lemma 2.11.
   b. $G \cong S_3 \times S_4$ and thus $|G|_2^r \leq 3^5$.

Assume $V$ is irreducible and quasi-primitive. Since $2 \nmid e \mid 8$, $e = 1$. By Lemma 2.10, $|G| \leq (2^4 + 1) \cdot 8 = 17 \cdot 8$ and $|G|_2^r \leq 17$. 

Assume $V$ is reducible, then $G$ satisfies one of the following:

(a) $G \leq \text{GL}(4, 2) \times \text{GL}(4, 2)$ and the action of $G$ on $V$ is a pair. Thus $|G|_{2'} \leq 15$ by Lemma 2.11.

(b) $G \in \text{SCRSp}(4, 2) \times \text{SCRSp}(4, 2)$ and thus $|G|_{2'} \leq 3^4$ by (2).

(c) $G \in \text{SCRSp}(2, 2) \times \text{SCRSp}(6, 2)$ and thus $|G|_{2'} \leq 3^5$ by (1) and (3).

Hence the result holds in all cases.

(5) Let $(n, F) = (2, F_3)$ and the result is clear.

(6) Let $(n, F) = (4, F_3)$. It is well known that a maximal subgroup of $\text{Sp}(4, 3)$ is isomorphic to one of the five groups $M_1$, $M_2$, $M_3$, $M_4$ and $M_5$, where $M_1 = \text{SL}(2, 3) \times S_2$, $|M_2| = |M_3| = 2^4 \cdot 3^4$, $|O_3(M_2)| = |O_3(M_3)| = 3^3$, $M_4 = 2.5_6$ and $M_5 = (D_8 \rtimes Q_8).A_5$. If $G \leq M_1$, then $H \leq M_2$ and the result is clear. If $G$ is a subgroup of $M_2$ or $M_3$, then clearly $|G| \leq 48$ and $|G|_{2'} \leq 3^4$. If $G$ is a subgroup of $M_4$, it is not hard to show [2, Lemma 3.2] that $|G| \leq 96$ or $|G| \leq 40$, thus $|G|_{2'} = 1$, $|G|_{2'} = 3$ or $|G|_{2'} = 5$. If $G$ is a subgroup of $M_5$, then $G \leq (D_8 \rtimes Q_8)L$ where $L = S_3$, $A_4$ or $F_{10}$, thus $|G|_{2'} \leq 3$ or $|G|_{2'} \leq 5$. Hence the result holds in all cases. □

**Proposition 2.13.** Let $G$ be a solvable primitive subgroup of $\text{GL}(n, p)$, $p$ a prime number, $n$ a positive integer, and let $V$ be the natural module for $G$. Then $G$ has at least $p$ regular orbits on $V \oplus V$, unless $G$ is one of the following groups:

1. $\text{GL}(2, 2)$;
2. $\text{SL}(2, 3)$ or $\text{GL}(2, 3)$;
3. $3^{1+2} \cdot \text{SL}(2, 3)$ or $3^{1+2} \cdot \text{GL}(2, 3)$;
4. $(Q_8 \rtimes Q_8).K \leq \text{GL}(4, 3)$ where $K$ is isomorphic to a subgroup of index 1, 2 or 4 of $O^+(4, 2)$.

**Proof.** This is [2, Theorem 3.4]. □

**Proposition 2.14.** Assume that $V$ is not quasi-primitive, then there exists a normal subgroup $N$ of $G$ such that $VN = V_1 \oplus \cdots \oplus V_m$ for $m > 1$ homogeneous components $V_i$ of $VN$. If $N$ is maximal with this property, then $S = G/N$ primitively permutes the $\mathcal{O} = \{V_1, \ldots, V_m\}$. Also $V = V_1^G$ is induced from $N_\mathcal{O}(V_1)$. Let $L_1 = \text{NC}(V_1)/\text{C}(V_1)$, then $L_1$ acts faithfully and irreducibly on $V_1$ and $G$ is isomorphic to a subgroup of $L_1 \wr S$. Let $H$ be an odd-order subgroup of $G$ and let $H_1 = N_H(V_1)/\text{C}_H(V_1)$. Suppose that $H_1$ has at least two orbits of elements with representatives $v_1$, $u_1 \in V_1$ such that $C_{H_1}(v_1)$, $C_{H_1}(u_1) \subseteq F_n(L_1)$ for some $n > 0$. Then $H$ has at least two orbits of elements with representatives $v$, $u \in V$ such that $C_{H_1}(v)$, $C_H(u) \subseteq F_n(G)$.

**Proof.** $S = G/N$ is a primitive permutation group and $KN/N$ is an odd-order subgroup of $S$. It follows from [10, Theorem 5.6] that $KN/N$ has a strongly regular orbit on $\mathcal{P}(\mathcal{O})$ where $\mathcal{P}(\mathcal{O})$ is the power set of $\mathcal{O}$ (strongly regular orbit means the orbit is regular and for each element $\beta$ in the orbit, $|\beta| = |\mathcal{O}|/2$). The rest of the argument is routine (see the proof of [12, Theorem 4.2]). □

### 3. Main theorems

The orbit structure of quasi-primitive solvable linear groups is useful in the study of solvable linear groups. In [3], the following result was obtained. Let $G$ be a solvable group of odd order and let $V$ be a finite, faithful, irreducible quasi-primitive $G$-module over a field of odd characteristic, assume $F(G)$ is nonabelian, then $G$ has at least two regular orbits on $V$. In this paper we generalize this result.

**Theorem 3.1.** Let $G$ be a solvable group and let $V$ be a finite, faithful and quasi-primitive $G$-module over a field of odd characteristic. Let $H$ be an odd-order subgroup of $G$. Then either $G \leq \Gamma(V)$ or $H$ has at least two regular orbits on $V$.

**Proof.** We adopt the notation in Theorem 2.2. Assume $G \not\leq \Gamma(V)$, then $e \neq 1$ by [10, Corollary 2.3]. By [14, Theorem 3.1] and [15, Theorem 3.1], we may assume $e = 2, 3, 4, 8, 9, 16$.

Assume $v$ generates a regular orbit of $H$. Since $|H|$ is odd and $|V|$ is odd, we know that $-v$ will generate a different regular orbit of $H$. Thus it suffices to show that $H$ has at least one regular orbit
on $V$. In order to show that $H$ has at least one regular orbit on $V$ it suffices to check that

$$\left| \bigcup_{P \in \text{SP}(H)} C_V(P) \right| < |V|^\#.$$

We will divide the set $\text{SP}(H)$ into a union of sets $A_i$, it is clear that $|\bigcup_{P \in \text{SP}(H)} C_V(P)\#| \leq \sum_{i} |\bigcup_{P \in A_i} C_V(P)\#|$. We will find $\beta_i$ such that $|C_V(P)| \leq |W|^{\beta_i}$ for all $P \in A_i$ and find $a_i$ such that $|A_i| \leq a_i$. Now it suffices to check that

$$\sum_{i} a_i \cdot (|W|^{\beta_i} - 1) < |W|^{\beta_i} - 1 < 1.$$

We call this inequality $\ast$. Since for all $P \in \text{SP}(U)$ we have $|C_V(P)\#| = 0$, we do not need to count the elements in $H \cap U$.

Let $e = 16$. Define $A_1 = \{(x) \mid x \in \text{EP}_3(H \cap A) \text{ for all primes } s \geq 3\}$. Thus for all $P \in A_1$, $|C_V(P)| \leq |W|^{6b}$ by Lemma 2.4(1) and we set $\beta_1 = 8$. $|A_1| \leq |\pi_{A/F}(H)\cap H/F|/2 \leq 243 \cdot ((|W|-1)/2)/2 = a_1$ by Lemma 2.12(4). Define $A_2 = \{(x) \mid x \in \text{EP}_3(H \cap (G \setminus A)) \text{ for all primes } s \geq 3\}$. Thus for all $P \in A_2$, $|C_V(P)| \leq |W|^{6b}$ by Lemma 2.4(4) and we set $\beta_2 = 6$. $|A_2| \leq 123 \cdot 3^4 \cdot ((|W|-1)/2)/2 = a_2$ by Lemma 2.12(6) and 2.8(2). Define $A_3 = \{(x) \mid x \in \text{EP}_3(H \cap (G \setminus A)) \text{ for all primes } s \geq 3\}$. Thus for all $P \in A_3$, $|C_V(P)| \leq |W|^{6b}$ by Lemma 2.4(4) and we set $\beta_3 = 3$. $|A_3| \leq |H|/2 \leq \text{dim}(W) \cdot 243 \cdot ((|W|-1)/2)/2 = a_3$ by Lemma 2.12(4). Thus for all $P \in A_3$, $|C_V(P)| \leq |W|^{6b}$ by Lemma 2.4(4) and we set $\beta_3 = 3$. $|A_3| \leq |H|/2 \leq \text{dim}(W) \cdot 243 \cdot ((|W|-1)/2)/2 = a_3$ by Lemma 2.12(4). Suppose $b > 1$ or $|W| \geq 19$, it is routine to check that $\ast$ is satisfied.

Thus we assume $b = 1$ and $|W| = 13$ or $|W| = 7$. Since $\text{dim}(W) = 1, |G/A| = 1$.

Define $A_1 = \{(x) \mid x \in \text{EP}_3(H \cap (F \setminus U))\}$. Thus for all $P \in A_1$, $|C_V(P)| \leq |W|^{4.5b}$ by Lemma 2.4(1) and we set $\beta_1 = 4.5$. $|A_1| \leq |H \cap F|/2 \leq 3^3 \cdot ((|W|-1)/2)/2 = a_1$. By Lemma 2.12(6), all the elements of prime order in $H \cap (A/F)$ are of order 3 or 5. Define $A_2 = \{(x) \mid x \in \text{EP}_3(H \cap (A/F)) \text{ and } |C_V(x)| \leq |W|^{\beta_2}\}$. Thus we set $\beta_2 = 5$ and $|A_2| \leq |\text{EP}_3(H \cap (A/F))| \leq 2 \leq 3^3 \cdot 3/2 = a_2$ by Lemma 2.12(6), Define $A_3 = \{(x) \mid x \in \text{EP}_3(H \cap (A/F)) \text{ and } |C_V(x)| > |W|^{\beta_3}\}$. We set $\beta_3 = 3$. $|A_3| \leq |H|/2 \leq \text{dim}(W) \cdot 243 \cdot ((|W|-1)/2)/2 = a_3$ by Lemma 2.12(6). Suppose $b > 1$ or $|W| \geq 19$, it is routine to check that $\ast$ is satisfied.

Thus we assume $b = 1$ and $|W| = 13$ or $|W| = 7$. Since $\text{dim}(W) = 1, |G/A| = 1$.

Define $A_1 = \{(x) \mid x \in \text{EP}_3(H \cap (F \setminus U))\}$.
Lemma 2.12(5). Suppose \(|32| \neq |W|^3 b\) by Lemma 2.4(4) and we set \(\beta_2 = 4/3\). \(|A_2| \leq |H|/2 \leq \dim(W) \cdot 9 \cdot (|W| - 1)/2/2 = a_2\) by Lemma 2.12(2). It is routine to check that \(*\) is satisfied.

Let \(e = 3\). Since \(3 \mid |W| - 1\) and \(|W|\) is odd, \(|W| \geq 7\).

Define \(A_1 = \{ (x) \mid x \in EP_3(H \cap (F \cup)) \}\). Thus for all \(P \in A_1\), \(|\mathbf{C}_V(P)| \leq |W|^3 b\) by Lemma 2.4(1) and we set \(\beta_1 = 1\). \(|A_1| \leq 3^2 \cdot 3/2 = a_1\). Define \(A_2 = \{ (x) \mid x \in EP_3(H \cap (A \setminus F)) \}\). Thus for all \(P \in A_2\), \(|\mathbf{C}_V(P)| \leq |W|^3 b\) by Lemma 2.4(2) and we set \(\beta_2 = 2\). \(|A_2| \leq 2 \cdot 3^2 \cdot 3/2 = a_2\) by Lemmas 2.12(5) and 2.8(3). Define \(A_3 = \{ (x) \mid x \in EP_3(H \cap (G \setminus A)) \}\) for all primes \(s \geq 3\). Thus for all \(P \in A_3\), \(|\mathbf{C}_V(P)| \leq |W|^3 b\) by Lemma 2.4(4) and we set \(\beta_3 = 1\). \(|A_3| \leq |H|/2 \leq \dim(W) \cdot 3 \cdot 3^2 \cdot (|W| - 1)/2/2 = a_3\) by Lemma 2.12(5). Suppose \(b > 1\) or \(|W| \geq 31\), it is routine to check that \(*\) is satisfied.

Assume \(b = 1\) and \(|W| = 13\) or \(19\). Since \(\dim(W) = 1\), \(|G/A| = 1\). Define \(A_1 = \{ (x) \mid x \in EP_3(H \cap (F \cup)) \}\). Thus for all \(P \in A_1\), \(|\mathbf{C}_V(P)| \leq |W|\) by Lemma 2.4(1) and we set \(\beta_1 = 1\). \(|A_1| \leq 3^2 \cdot 3/2 = a_1\). Define \(A_2 = \{ (x) \mid x \in EP_3(H \cap A) \cdot F \} \) and \(|\mathbf{C}_V(x)| \leq |W|\). Thus we set \(\beta_2 = 1\) and \(|A_2| \leq 2 \cdot 3^2 \cdot 3/2 = a_2\) by Lemmas 2.12(5) and 2.8(3). Define \(A_3 = \{ (x) \mid x \in EP_3(H \cap (A \setminus F)) \}\). Let \(x \in EP_3(H \cap (A \setminus F))\). If \(x\) is a bad element, then \(|\mathbf{C}_V(x)| \leq |W|\) by Lemma 2.6(1). Thus for all \(1 \neq x \in P \in A_3\), \(x\) is a good element, \(|\mathbf{C}_E(x)| = |x|^2 = 3\) and \(|W|^2 \leq |\mathbf{C}_V(x)| = |W|^2 \cdot |x + \overline{x}| \cdot |x - \overline{x}| |\chi(x)| = |W|^2 \cdot |x|^2 |\chi(x)| / 2 \leq 2 \cdot 3^2 \cdot 3/2 = a_3\) by Lemmas 2.12(5) and 2.8(3).

It is routine to check that \(*\) is satisfied.

Assume \(b = 1\) and \(|W| = 7\). Since \(|G| \leq 24 \cdot 9 \cdot 6\) by Lemma 2.3, \(|H| \leq 81\). Since \(|V| = 7^2\), \(|(H|, |V|)| = 1\) and \(H\) acts completely reducibly on \(V\). Thus we have one of the following:

(1) The action is not irreducible, then it is not hard to check that \(H\) has at least two regular orbits on \(V\).

(2) The action is irreducible and \(H \leq \Gamma(7^3)\). It is not hard to show that a 3-subgroup of \(\Gamma(7^3)\) will have at least two regular orbits on the \(V\).

(3) The action is irreducible and \(V = W^H\) for a primitive module \(W\) of \(L\) where \(L \leq \Gamma(7^1)\). Since a 3-subgroup of \(\Gamma(7^1)\) will have two regular orbits on \(W\), \(H\) has at least two regular orbits on \(V\).

Assume \(b = 1\) and \(|W| = 25\). Since \(|G/2 \cdot 24 \cdot 9 \cdot 24\) by Lemma 2.3, \(|H| \leq 81\). Since \(|V| = 5^6\), \(|(H|, |V|)| = 1\) and \(H\) acts completely reducibly on \(V\). Thus we have one of the following:

(1) The action is not irreducible, then it is not hard to check that \(H\) has at least two regular orbits on \(V\).

(2) The action is irreducible and \(H \leq \Gamma(5^6)\). It is not hard to show that a 3-subgroup of \(\Gamma(5^6)\) will have at least two regular orbits on the \(V\).

(3) The action is irreducible and \(V = W^H\) for a primitive module \(W\) of \(L\) where \(L \leq \Gamma(5^2)\). Since a 3-subgroup of \(\Gamma(5^2)\) will have two regular orbits on \(W\), \(H\) has at least two regular orbits on \(V\).

Let \(e = 2\). Clearly \(A/F \leq S_3\). Define \(A_1 = \{ (x) \mid x \in EP_3(H \cap (A \setminus F)) \}\). Thus for all \(P \in A_1\), \(|\mathbf{C}_V(P)| \leq |W|^3 b\) by Lemma 2.4(3) and we set \(\beta_1 = 1\). \(|A_1| \leq |NEP_3(\pi_A/F)(H)\cdot |3/2 \leq 2 \cdot 3/2 = a_1\) by Lemmas 2.12(1) and 2.8(3). Define \(A_2 = \{ (x) \mid x \in EP_3(H \cap (G \setminus A)) \}\) for all primes \(s \geq 3\). Thus for all \(P \in A_2\), \(|\mathbf{C}_V(P)| \leq |W|^3 b\) by Lemma 2.4(4) and we set \(\beta_2 = 2/3\). \(|A_2| \leq |H|/2 \leq \dim(W) \cdot 3 \cdot (|W| - 1)/2/2 = a_2\) by Lemma 2.12(1). We set \(a_2 = 0\) if \(\dim(W) > 1\) and \(a_2 = \dim(W) \cdot 3 \cdot (|W| - 1)/2/2\) if \(\dim(W) > 1\). It is routine to check that \(*\) is satisfied.

The bound of Theorem 3.1 is tight. Let \(G = GL(2, 3)\) act on \(V = F_3^2\). Then \(G\) and \(V\) satisfy the condition of Theorem 3.1. Let \(H\) be an odd-order subgroup of \(G\), then \(|H| = 3\) and it is clear that \(H\) has exactly two regular orbits on \(V\).
Theorem 3.2. Suppose that $V$ is a finite faithful irreducible $G$-module over a field of odd characteristic. Then $V = W^G$ for an irreducible primitive module $W$ of $K$ for some $K \leq G$ (possibly $K = G$). Let $H$ be an odd-order subgroup of $G$ and assume that $K/C_K(W) \not\leq \Gamma(W)$, then $H$ will have at least two regular orbits on $V$.

Proof. This follows from Theorem 3.1 and Proposition 2.14. \qed

Moretó and Wolf [12, Theorem 4.2] proved the following result. Suppose that $V$ is a finite, faithful and completely reducible $G$-module with $|G||V|$ odd. Then there exists $v \in V$ in a regular orbit of $F(G)$ such that $C_G(v) \subseteq F_2(G)$. The following is a generalization of their theorem.

Theorem 3.3. Suppose that $G$ is a finite solvable group and $V$ is a finite, faithful and completely reducible $G$-module over a field of odd characteristic. Let $H$ be an odd-order subgroup of $G$, then there exists $v \in V$ in a regular orbit of $F(G) \cap H$ such that $C_H(v) \subseteq F_2(G)$.

Proof. Assume false and consider a counterexample with $|G| + \dim(V)$ as small as possible.

Suppose that $V$ is not irreducible then $V = V_1 + V_2$ where $V_1$, $V_2$ are $G$-submodules. Let $C_i = C_G(V_i)$ and let $K_i = F_2(G/C_i)$. We can find $x_i, y_i \in V_i$ such that $x_i$, $y_i$ are in different $HC_i/C_i$ orbits, $C_i(x_i) \cap H \cap K_i$ and $C_i(y_i) \cap H \cap K_i$. Let $x = x_1 + x_2$, $y = y_1 + y_2$ and we have $C_H(x), C_H(y) \subseteq H \cap K_1 \cap K_2 = H \cap F_2(G)$. Clearly $x$ and $y$ are not $H$-conjugate.

We may assume $V$ is irreducible. First assume $V$ is a quasi-primitive $G$-module. By Theorem 3.1, either $H$ has two regular orbits on $V$ or $G \not\leq \Gamma(V)$. The result is clear if $H$ has two regular orbits on $V$. If $G \not\leq \Gamma(V)$, then $G = F_2(G)$ and $H \cap F(G)$ acts fixed point freely on $V$. Since $|V|$ is odd, there exist two $H$-orbits with representatives $v_a, v_b \in V$ such that $v_a$, $v_b$ are in a regular orbit of $F(G) \cap H$ and $C_H(v_a), C_H(v_b) \subseteq F_2(G)$. This is a contradiction.

Thus $V$ is not quasi-primitive and there exists a normal subgroup $N$ of $G$ that $V_N = V_1 \oplus \cdots \oplus V_m$ for $m > 1$ homogeneous components $V_i$ of $V_N$. If $N$ is maximal with this property, then $S = G/N$ primitively permutes the $V_i$. Also $V = V_1^G$, induced from $N_G(V_1)$. Let $L_1 = N_G(V_1)/C_G(V_1)$, then $L_1$ acts faithfully and irreducibly on $V_1$ and $G$ is isomorphic to a subgroup of $L_1 \wr S$. By minimality, $H_1 := N_H(V_1)C_G(V_1)/C_G(V_1)$ has at least two orbits of elements with representatives $v_1, u_1 \in V_1$ such that $v_1, u_1$ are in a regular orbit of $F(L_1) \cap H_1$ and $C_{H_1}(\{v_1\}), C_{H_1}(\{u_1\}) \subseteq F_2(L_1)$. By Proposition 2.14, there exist two $H$-orbits with representatives $v_a, v_b \in V$ such that $v_a$, $v_b$ are in a regular orbit of $F(G) \cap H$ and $C_H(v_a), C_H(v_b) \subseteq F_2(G)$. Final contradiction. \qed

Dolfi [1, Theorem 1.3] proved the following result. Let $G$ be a solvable group of odd order and $V$ a finite, faithful and completely reducible $G$-module. Then there exist $v, w \in V$ such that $C_G(v) \cap C_G(w) = 1$. The following is a generalization of this theorem.

Theorem 3.4. Suppose that $G$ is a finite solvable group and $V$ is a finite, faithful and completely reducible $G$-module. Let $H$ be an odd-order subgroup of $G$, then $H$ has at least two regular orbits on $V \oplus V$.

Proof. Assume false and consider a counterexample with $|G| + \dim(V)$ as small as possible.

Suppose that $V$ is not irreducible then $V = V_1 \oplus V_2$ where $V_1$, $V_2$ are $G$-submodules. Let $C_i = C_G(V_i)$ where $i = 1, 2$. $HC_i/C_i$ has two regular orbits on $V_i \oplus V_i$ by minimality and clearly $H$ has two regular orbits on $V \oplus V$. This is a contradiction.

Assume $V$ is not primitive. We hence assume that there exists a proper subgroup $L_1$ of $G$ and an irreducible $L_1$-submodule $V_1$ of $V$ such that $V = V_1^G$. Then, clearly, $V \oplus V = (V_1 \oplus V_1)^G$. We may choose $L_1$ to be a maximal subgroup of $G$. In particular, $S \cong G/N$ is a primitive permutation group. A right transversal $\Omega$ of $L_1 \cap G$, where $N$ is the normal core of $L_1$ in $G$. Let $V_N = V_1 \oplus \cdots \oplus V_t$, where the $V_i$ are irreducible $L_1$-modules where $L_1 = N_G(V_i)$ and $t > 1$. We know $G/N$ primitively permutes the $\Omega = \{V_1, \ldots, V_t\}$. Since $HN/N$ permutes the $\Omega = \{V_1, \ldots, V_t\}$, we can take the representatives of each orbit as $\Omega_1 = \{V_{1t}, \ldots, V_{st}\}$. Let $H_{V_{kt}} = N_{H_{V_{kt}}}(\{V_{1t}\})C_{H_{V_{kt}}}(\{V_{1t}\})/C_{H_{V_{kt}}}(\{V_{1t}\})$ for all $1 \leq k \leq s$. By minimality, $H_{V_{kt}}$ has at least two regular orbits on $V_{kt} \oplus V_{kt}$. Thus we know that $H$ has at least two regular orbits on $V \oplus V$ by Proposition 2.14.
Thus we may assume $V$ is a primitive $G$-module. By Proposition 2.13, we may assume that $G$ is one of the following:

1. $\text{GL}(2, 2)$, then the odd-order subgroup of $G$ will have at least 5 regular orbits on $V \oplus V$ by direct calculation.
2. $\text{SL}(2, 3)$ or $\text{GL}(2, 3)$. Thus $e = 2$ and $p = 3$. By Theorem 3.1, the odd-order subgroup of $G$ will have at least two regular orbits on $V$ and at least $2|V| = 18$ regular orbits on $V \oplus V$.
3. $3^{1+2} \cdot \text{SL}(2, 3)$ or $3^{1+2} \cdot \text{GL}(2, 3) \leq \text{GL}(6, 2)$. Let $H$ be an odd-order subgroup of $G$, then since $(|H|, |V|) = 1$, $H$ acts completely reducible on $V$ and $H$ has at least two regular orbits on $V \oplus V$ by [1, Theorem 3.1].
4. $(Q_8 \rtimes Q_8).K \leq \text{GL}(4, 3)$ where $K$ is isomorphic to a subgroup of index 1, 2 or 4 of $O^+(4, 2)$. Thus $e = 4$ and $p = 3$. By Theorem 3.1, the odd-order subgroup of $G$ will have at least two regular orbits on $V$ and at least $2|V| = 162$ regular orbits on $V \oplus V$.

Final contradiction. □

The bound of Theorem 3.4 is tight. Let $G = \Gamma(2^3)$ act on $V = \mathbb{F}_2^3$. $|G| = 21$ and $G$ has exactly two regular orbits on $V \oplus V$.

**Corollary 3.5.** Suppose that $G$ is a finite solvable group and $V$ is a faithful and completely reducible $G$-module (possibly of mixed characteristic). Let $H$ be an odd-order subgroup of $G$, then there exists $v \in V$ such that $|C_H(v)| \leq \sqrt{|H|}$.

**Proof.** By Theorem 3.4, there is an element $(v, u) \in V \oplus V$ such that $C_H((v, u)) = C_H(v) \cap C_H(u) = 1$. Since $|C_H(v)||C_H(u)| = |C_H((v, u))| = |C_H(v)||C_H(u)| \leq |H|$, it follows that, either $|C_H(v)| \leq \sqrt{|H|}$ or $|C_H(u)| \leq \sqrt{|H|}$. □

In [4], Espuelas shows the following. Let $G$ be a solvable group of odd order and $V$ be a faithful and completely reducible $G$-module with odd characteristic, assume that $V$ is endowed with a non-singular symplectic form fixed by $G$, then $V$ contains at least two regular orbits of $G$. The following is a generalization of this theorem.

**Theorem 3.6.** Let $G$ be a finite solvable group and $V$ be a faithful and completely reducible $G$-module over a finite field $\mathbb{F}$ of odd characteristic, assume that $V$ is endowed with a non-singular symplectic form fixed by $G$. Let $H$ be an odd-order subgroup of $G$, then $H$ has at least two regular orbits on $V$.

**Proof.** Assume false and consider a counterexample with $|G| + \dim(V)$ as small as possible.

Let $W$ be an irreducible $G$-submodule of $V$. Since the form $(,)$ is $G$-invariant, the subspace \{ $v \in W \mid (v, v') = 0$ for all $v' \in W$ \} is a submodule of $W$ and the form $(,)$ is either totally isotropic or non-singular on $W$. Define $W^\perp = \{ v \in V \mid (v, w) = 0 \text{ for all } w \in W \}$. If the form $(,)$ is non-singular on $W$, then $V = W \oplus W^\perp$. If the form $(,)$ is totally isotropic on $W$, then for $v \in V$, we consider the map $f_v \in W^* := \text{Hom}_F(W, \mathbb{F})$, defined by $f_v(w) = (v, w)$, $w \in W$. Then $v \mapsto f_v$, $v \in V$, induces a $G$-isomorphism between $V/W^\perp$ and the dual space $W^*$. Since $V$ is completely reducible, we may find an irreducible $G$-submodule $U \cong W^*$ such that the form is non-singular on $X = W \oplus U$. Now $X^\perp \cap X = 0$ and therefore $X = X \oplus X^\perp$. Repeating this argument we arrive at $V = W_1 \perp \cdots \perp W_t$, where each $W_i$ is either $G$-irreducible or the sum of an irreducible $G$-module and its dual. If $t > 1$, then $H$ has at least two regular orbits on $V$ by minimality. Suppose that $V$ is the sum of an irreducible $G$-module and its dual, i.e. $V = W \oplus W^*$. The action of $G$ on $W$ is faithful by [2, Lemma 2.3] and the action of $G$ on $W \oplus W^*$ is orbit isomorphic to the action of $G$ on $W \oplus W$ by [8, Lemma 3.33]. By Theorem 3.4, $H$ will have at least two regular orbits on $V = W \oplus W^*$. Thus we may assume $V$ is $G$-irreducible.

Suppose the $V$ is not quasi-primitive and that $N \triangleleft G$ is maximal such that $V_N$ is not homogeneous. By [10, Proposition 0.2], $S = G/N$ faithfully and primitively permutes the homogeneous components
of $V_N$. Set $I = N_G(V_1)$, by Clifford’s Theorem, $V_1$ is an irreducible $I$-module. Since the form $(\ , \ )$ is $G$-invariant, the subspace $\{v \in V_1 \mid (v, v') = 0 \text{ for all } v' \in V_1\}$ is an $I$-submodule of $V_1$ and the form $(\ , \ )$ is either totally isotropic or non-singular on $V_1$. Since $G$ transitively permutes the $V_i$, the $G$-invariant form $(\ , \ )$ is simultaneously totally isotropic or non-singular on all the $V_i$. If the form is non-singular on each $V_j$ then let $K = N_G(V_1)/C_G(V_1)$. $K$ acts faithfully and irreducibly on $V_1$ and preserves the form on $V_1$. Thus $H$ has at least two regular orbits on $V$ by minimality and Proposition 2.14. Hence, we assume that each $V_i$ is totally isotropic. Let $j \in \{1, \ldots, t\}$ and set $V_j = \{v \in V \mid (v, v_j) = 0 \text{ for all } v_j \in V_j\}$. For $v \in V$, we consider the map $f_v \in V_j^* := \text{Hom}_F(V_j, F)$, defined by $f_v(v_j) = (v, v_j)$.

**Proof.** If $\exists$ counterexample minimizing $\dim(H \cap U)$, since $p$ is odd, we have $|H \cap U| \leq m$. Thus the number of primes dividing $|H \cap U|$ is at most $\log_2 m$. Hence

$$B_1 := \bigcup_{x \in H^*} C_V(x) \leq (\log_2 m)|H \cap U|p^{2m/s}.$$

We need to prove that $(|V| - B_1)/|G| > 1$ or sufficiently, that

$$p^{2m} - (\log_2 m)|H \cap U|p^{2m/s} - m|H \cap U| > 0.$$ 

Since $p$ and $|H \cap U|$ are odd and $|H \cap U| \leq 2$ by [4, Lemma 1.1], $|H \cap U| \leq (p^m + 1)/2$. It suffices to show that $2p^{2m}/(p^m + 1) - \log_2 m \cdot p^{2m/3} - m > 0$ and this inequality is satisfied for all $p \geq 3$ and $m \geq 1$. □

The bound of Theorem 3.6 is tight. Let $G = SL(2, 3)$ act on $V = \mathbb{F}_2^2$. Then $G$ and $V$ satisfy the condition of Theorem 3.6. Let $H$ be an odd-order subgroup of $G$, then $|H| = 3$ and it is clear that $H$ has exactly two regular orbits on $V$.

Espuelas and Navarro [6] proved the following result. Let $G$ be a group of odd order and let $H$ be a Hall $\pi$-subgroup of $G$. Then $V$ is a faithful $G$-module, over possibly different finite fields of odd $\pi$-characteristic and assume that $V_{O_\pi(G)}$ is completely reducible, then there exists $v \in V$ such that $C_H(v) \subseteq O_\pi(G)$. The following is a generalization of this theorem.

**Theorem 3.7.** Let $G$ be a solvable group and let $H$ be an odd-order Hall $\pi$-subgroup of $G$. Let $V$ be a faithful $G$-module, over possibly different finite fields of odd $\pi$-characteristic. Assume that $V_{O_\pi(G)}$ is completely reducible, then there exists $v \in V$ such that $C_H(v) \subseteq O_\pi(G)$.

**Proof.** If $H \subseteq O_\pi(G)$ there is nothing to prove. Thus we may assume that $H \not\subseteq O_\pi(G)$. Let $G$ be a counterexample minimizing $\dim(V)$.

Step 1. $V$ is a completely reducible $G$-module.

Let $R$ be a Hall $\pi'$-subgroup of $O_\pi\pi'(G)$. If $h \in H - O_\pi(G)$, let $1 \neq Y(h)$ be a Hall $\pi'$-subgroup of $[h, R]$.

We claim that there exists an irreducible $G$-submodule $V(h)$ of $V$ such that $Y(h)$ acts nontrivially on $V(h)$. Since $V_{O_\pi(G)}$ is completely reducible and the fields have $\pi$-characteristic, we know that $V_{O_\pi\pi'(G)}$ is completely reducible.
Write $V_{O_{\pi}(G)} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where the $V_i$'s are the homogeneous components.

Since $Y(h) > 1$, suppose for instance that $Y(h)$ acts nontrivially on $V_1$. Now consider the $G$-module $\sum_{x \in G} V_1 x$ and choose an irreducible $G$-submodule $W$ of it. Let $X$ be an irreducible $O_{\pi}(G)$-submodule of $W$. Since for every $x \in G$, the $V_1 x$'s are homogeneous components, it follows that $X \subseteq V_1 x$, for some $x \in G$. Since $W = V$, we will have that $V_1 \cap W > 0$.

Suppose now that $Y(h)$ acts trivially on $W \cap V_1$ and let $Y$ be an irreducible $O_{\pi}(G)$-submodule of $W \cap V_1$. Therefore, since $V_1$ is a direct sum of modules isomorphic to $Y$, it follows that $Y(h)$ acts trivially on $V_1$. This shows that $Y(h)$ does not act nontrivially on $V(h = W)$, as claimed.

Let $U = \sum_{b \in H - O_{\pi}(G)} V(h)$, it is a completely reducible $G$-module of $V$. If $U \not\subseteq V$, by minimality, there exists $u \in U$ such that $C_{\pi}(u) \subseteq O_{\pi}(G)$, where $G = G_{\pi}(C)$.

Let $C = C_{\pi}(U)$ and let $K/C = O_{\pi}(G)$. Observe that $[K/C O_{\pi}(G), O_{\pi}(G)/O_{\pi}(G)] = 1$. If $h \in H \cap K - O_{\pi}(G)$, then $[h, R] \subseteq [K, O_{\pi}(G)] \subseteq O_{\pi}(G) C$. Since $C$ contains the $\pi'$-subgroups of $O_{\pi}(G)/C$, it follows that $Y(h) \subseteq C$, which is a contradiction. This proves that $C_H(u) \subseteq H \cap K \subseteq O_{\pi}(G)$ and we may assume $U = V$. Hence $V$ is a completely reducible $G$ module.

Step 2. $V$ is an irreducible $G$-module. Assume not, we have $V = V_1 \oplus V_2$ and each $V_i$ is a nontrivial $G$-module. Let $K_i = C_{\pi}(V_i)$ and $V_i$ is a faithful irreducible $G/K_i$-module. By minimality, let $v_i \in V_i$ such that $C_{K_i/K_i}(v_i) \subseteq O_{\pi}(G/K_i)$ and consider $v = v_1 + v_2$. Then $C_H(v) \subseteq O_{\pi}(G)$.

Step 3. $V$ is a quasi-primitive $G$-module. Assume not and let $N$ be a normal subgroup of $G$ such that $V_N = V_1 \oplus \cdots \oplus V_r$, where the $V_i$'s are homogeneous $N$-modules and $r > 1$. If $N$ is maximal with this property, then $G/N$ primitively permutes the $\Omega = \{V_1, \ldots, V_r\}$. Moreover, $V_1$ is an irreducible $C_{\pi}(V_1)$-module with $V^G = V$. Since $H$ permutes the $\Omega = \{V_1, \ldots, V_r\}$, we can take the representatives of each orbit as $\Omega_1 = \{V_1, \ldots, V_\ell\}$. Let $H_{\ell}$ be a Hall $\pi$-subgroup of $C_{\pi}(V_{\ell})$ containing $C_{V_{\ell}}(K_{\ell})$ for all $1 \leq k \leq s$. By minimality, there exist $v_{k_1}, v_{k_2} \in V_{\ell}$ such that $C_{H_{\ell}}(C_{V_{\ell}}(K_{\ell}))(V_{k_1}) \subseteq O_{\pi}(C_{V_{\ell}}(K_{\ell})) = C_{H_{\ell}}(C_{V_{\ell}}(K_{\ell}))(V_{k_2})$. By Gluck's Theorem [10, Corollary 5.7 (b)], $HN/N$ has a regular orbit on the power set of $\Omega$. Let $\Lambda \subseteq \Omega$ be a representative of such an orbit. Let $x_{\Lambda} \in H$ with $V_{\ell} x_{\Lambda} = V_{k_1}$. Let $w_1 = x_{\Lambda} + \cdots + x_{\Lambda} + \cdots + x_{\Lambda} + \cdots + x_{\Lambda} \in V$ be defined as follows: $x_{\Lambda} = v_{k_1} h_{x_{\Lambda}}$ if $V_{\Lambda} \subseteq \Lambda$ and $x_{\Lambda} = v_{k_2} h_{x_{\Lambda}}$ if $V_{\Lambda} \not\subseteq \Lambda$. Let $w_2 = x_{\Lambda} + \cdots + x_{\Lambda} + \cdots + x_{\Lambda} + \cdots + x_{\Lambda} \in V$ be defined as follows: $x_{\Lambda} = v_{k_2} h_{x_{\Lambda}}$ if $V_{\Lambda} \not\subseteq \Lambda$ and $x_{\Lambda} = v_{k_1} h_{x_{\Lambda}}$ if $V_{\Lambda} \subseteq \Lambda$. Thus $C_H(w_i) \subseteq N$ and

$$C_H(w_1) \subseteq \bigcap_{k=1}^{s} \bigcap_{j=1}^{n_k} h_{k_1}(V_{k_1}) \subseteq O_{\pi}(G).$$

Suppose that $w_1 = w_2$ for some $h \in H$, then $h$ permutes $\Lambda$ and $\Omega = \Lambda$ and $2 | o(h)$, a contradiction.

Step 4. We may assume that $G \not\subseteq \pi(V)$ since otherwise $H$ has at least two regular orbits on $V$ by Theorem 3.1. We know that $V = GF(q^h)^+ \subseteq G^* = C_n \rtimes M$ where $M = GF(q^h)^+$ and $C_n = \text{Gal}(GF(q^h) : GF(q))$. Observe that if the theorem is true for $G^*$ it is also true for $G$. Hence we may assume that $G = G^*$. Let $H = \langle (\pi) \rangle$ and $K_\pi$ are Hall $\pi$-subgroups of $C_n$ and $M$, respectively. Assume $H \not\subseteq O_{\pi}(G)$, it suffices to show that $[\bigcup_{K \not\subseteq \pi(G)} C_V(h)] < |V|$. Write $H - O_{\pi}(G) = \bigcup_{j=1}^T \langle x_j, K \rangle$, where $x_j \in \pi$. Suppose that $\bigcup_{K \not\subseteq \pi(G)} C_V(h) = V$, then $V^* = \bigcup_{j=1}^T \langle x_j(K) \rangle$.

Since for each $j$, $\bigcup_{K \not\subseteq \pi(G)} C_V(x_j) = \{v \in V^* \mid V^* \not\subseteq K \}$ is a multiplicative subgroup of $V^*$, it follows that $V^* = \bigcup_{K \not\subseteq \pi(G)} C_V(x_j)$ for some $j$. Following notation and [13, Proposition 1.3], define $N = \{x \in V^* \mid N_{x_j}(x) = 1\}$, where $N_{x_j}$ is the norm map (i.e., $N_{x_j}(y) = y_{x_j}(y) \cdots y_{x_j(x_j)}^{-1}(y)$). Then $|N| = \frac{q^n - 1}{q^{n_j} - 1}$, where $s = o(x_j)$. By [13, Proposition 1.3], we will have that $N \subseteq K_\pi$.

4. Applications

We provide some applications of the orbit theorems in the previous section.
Theorem 4.1. Suppose that $G$ is a finite $\pi$-solvable group with $O_\pi(G) = 1$ where $\pi$ is a set of odd primes and let $H$ be a Hall $\pi$-subgroup of $G$. Let $V$ be a faithful $G$-module, over possibly different finite fields of $\pi$-characteristic. Then $H$ has at least one regular orbit on $V$.

Proof. If $G$ is solvable, then this is done by Theorem 3.7. Now let $N = O_\pi(G)$ and note that $C_H(N) = 1$ since $C_G(N) \leq N$. By [11, Theorem 1.2], there exists a nilpotent $H$ invariant subgroup $K$ of $N$ such that $C_H(K) = 1$. Thus we have $O_\pi(KH) = 1$ and since $V_{KH}$ is faithful, we may assume $G = KH$. Then $G$ is solvable and we are done. \( \square \)

Remark. Theorem 4.1 extends [11, Theorem 3.1].

Theorem 4.2. Let $G$ be a finite $\pi$-solvable group where $\pi$ is a set of odd primes and $H$ be a Hall $\pi$-subgroup of $G$. Then there exists $\alpha \in \text{Irr}(H)$ such that $|G : O_{\pi'}(G)|_{\pi}$ divides $\alpha(1)$. In particular, $|G : O_{\pi'}(G)|_{\pi} \leq b(H)$.

Proof. We may assume that $O_{\pi'}(G) = 1$. Let $N = O_{\pi'}(G)$. Then, fairly standard arguments show that $C = C_G(F(N)/\Phi(N)) \leq N$. Write $V = \text{Irr}(F(N)/\Phi(N))$ and $\tilde{G} = G/C$. Thus $V$ is a faithful $\tilde{G}$-module and $O_{\pi}(\tilde{G}) = N/C$. Since $|O_{\pi}(\tilde{G})|$ is odd, $O_{\pi}(\tilde{G})$ is solvable and $V_{O_{\pi}(\tilde{G})}$ is completely reducible. By Theorem 4.1, there exists $\lambda \in \text{Irr}(V)$ such that $C_H(\lambda) \leq N$. Let $\xi \in \text{Irr}(C_H(\lambda)|\lambda)$ and $\alpha = \xi^H \in \text{Irr}(H)$. Thus $|H : N|$ divides $\alpha(1)$, as wanted. \( \square \)

Remark. Theorem 4.2 extends [6, Corollary] and [11, Theorem 3.2(1)].

Theorem 4.3. Let $G$ be a $\pi$-solvable group where $\pi$ is a set of odd primes and $H$ be a Hall $\pi$-subgroup of $G$. Let $H_0 = H \cap O_{\pi'}(G)$. If $A$ is an abelian subgroup of $H$, then $|A| \leq |H_0|$.

Proof. Since $A$ is abelian, we have $b(H) \leq |H : A|$. By Theorem 4.2, $|H : H_0| \leq b(H)$ and the result follows. \( \square \)

Remark. Theorem 4.3 partially extends [5, Proposition 2.3].

Theorem 4.4. Let $G$ be a finite $\pi$-solvable group, then $|G : F(G)|_{\bar{\pi}} \leq b(G)^2$.

Proof. Let $U = F(G)/\Phi(G)$ and $\tilde{G} = G/F(G)$. $U$ is a faithful and completely reducible $\tilde{G}$-module by Gaschütz’s Theorem [10, Theorem 1.12]. Let $V = \text{Irr}(F(G)/\Phi(G))$ and $V$ is a faithful and completely reducible $\tilde{G}$-module by [10, Proposition 12.1].

Let $H$ be a 2'-Hall subgroup of $G$ and let $\tilde{H} = HF(G)/F(G)$. By Corollary 3.5, there exists $\lambda \in V$ such that $|C_{\tilde{H}}(\lambda)| \leq |\tilde{H}|_{\bar{\pi}}^{1/2}$. Let $\xi \in \text{Irr}(C_{HF(G)}(\lambda)|\lambda)$ and $\alpha = \xi^{HF(G)} \in \text{Irr}(HF(G))$. Thus $|G : F(G)|_{\bar{\pi}} = |\tilde{H}| \leq \alpha(1)^2 \leq b(G)^2$. \( \square \)

Remark. In [7], Gluck conjectured that for finite solvable group $G$, $|G : F(G)| \leq b(G)^2$. This has been verified by Espuelas [3] for $G$ of odd order. Theorem 4.4 extends [3, Theorem 3.2].

Acknowledgment

I am greatly indebted to the referee for his or her valuable suggestions.

References