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The influence of a power law drift on the exit time of Brownian motion from a half-line

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Abstract

The addition of a Bessel drift $\frac{1}{x}$ to a Brownian motion affects the lifetime of the process in the interval $(0, \infty)$ in a well-understood way. We study the corresponding effect of a power $-\frac{\beta}{x^p}$ ($\beta \neq 0$, $p > 0$) of the Bessel drift. The most interesting case occurs when $\beta > 0$. If $p > 1$ then the effect of the drift is not too great in the sense that the exit time has the same critical value q_0 for the existence of qth moments ($q > 0$) as the exit time of Brownian motion. When $p < 1$, the influence is much greater: the exit time has exponential moments.

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1. Introduction

For any real-valued process Z_t, let $\tau_R(Z)$ be the first hitting time of R by Z:

$$\tau_R(Z) = \inf\{t > 0\colon\ Z_t = R\}.$$

It is well-known that Brownian motion W_t will exit the interval $(0, \infty)$ in finite time and in fact,

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$$P_x(\tau_0(W) > t) = \frac{2}{\sqrt{2\pi}} \int_0^{x/\sqrt{t}} \mathrm{e}^{-u^2/2}\,\mathrm{d}u \tag{1.1}$$

(Feller [10]). The addition of the drift $-\frac{\beta}{x}$ yields the Bessel process B_t with dimension $\delta = 1 - 2\beta$:

$$\mathrm{d}B_t = \mathrm{d}W_t - \beta B_t^{-1}\,\mathrm{d}t, \quad t < \tau_0(B).$$

It is known (Göing-Jaeschke and Yor [11]) that for $\delta < 2$,

$$P_x(\tau_0(B) > t) = \frac{2^{\delta/2}}{\Gamma(1-\delta/2)} \int_0^{x/\sqrt{t}} u^{1-\delta}\mathrm{e}^{-u^2/2}\,\mathrm{d}u \tag{1.2}$$

and for $\delta \geq 2$,

$$P_x(\tau_0(B) = \infty) = 1.$$

Also see DeBlassie [9] and Bañuelos and Smits [4] for other derivations.

In this article we will study the effect of changing the drift to $-\frac{\beta}{x^p}$ for $\beta \neq 0$ and $p > 0$:

$$\mathrm{d}X_t = \mathrm{d}W_t - \beta X_t^{-p}\,\mathrm{d}t, \quad t < \tau_0(X). \tag{1.3}$$

If $\beta < 0$, it is easy to show

$$P_x(\tau_0(X) = \infty) = 1 \quad \text{for } p > 1$$
$$1 > P_x(\tau_0(X) = \infty) > 0 \quad \text{for } p < 1.$$

Consequently, we will concentrate on the case $\beta > 0$. Our main results are the following theorems.

Theorem 1.1. *For $\beta > 0$ and $p > 1$,*

$$E_x[\tau_0(X)^q] < \infty \quad \text{if } q < \frac{1}{2}$$
$$E_x[\tau_0(X)^q] = \infty \quad \text{if } q > \frac{1}{2}.$$

Let

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t$$

denote the Beta function.

Theorem 1.2. *For $\beta > 0$ and $p < 1$,*

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_x(\tau_0(X) > t) = -\gamma(p,\beta)$$

where:

$$\gamma(p,\beta) = \frac{1}{2} p^{-\frac{2p}{1+p}} \beta^{\frac{2}{1+p}} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B\left(\frac{3}{2}, \frac{1-p}{2p}\right)\right] \Big/ B\left(\frac{1}{2}, \frac{1+p}{2p}\right)^{\frac{1-p}{1+p}}.$$

Here is some intuition concerning our results. Clearly, since $\beta > 0$, $\tau_0(X) \le \tau_0(W)$. Since we are computing the chance that X takes a long time to exit $(0, \infty)$, it is plausible that for this event to occur, the process must spend a lot of time away from 0. Thus the behaviour of the drift far away from 0 ought to have the most influence. Now for $\beta > 0$ and $p < 1$, for any $\alpha < 1$,

$$-\frac{\beta}{x^p} \le \frac{\alpha - 1}{2x}, \quad x \text{ large}$$

and we expect X will exit sooner than a Bessel process B of any dimension $\alpha < 1$. By (1.2), $P_x(\tau_0(B) > t) \approx ct^{(\alpha-2)/2}$, so by varying $\alpha < 1$, we see $\tau_0(X)$ ought to have all qth moments finite ($q > 0$). Theorem 1.2 tells us $\tau_0(X)$ even has exponential moments.

Notice:

$$\lim_{p\to 0^+} \gamma(p, \beta) = \frac{\beta^2}{2} \quad \text{and} \tag{1.4}$$

$$\lim_{p\to 1^-} \gamma(p, \beta) = \infty. \tag{1.5}$$

When $p = 1$ we get a Bessel process and so for some $q > 0$, $P_x(\tau_0(X) > t) \approx t^{-q}$. Thus for p close to 1 and t large, we expect:

$$\gamma \approx -t^{-\frac{1-p}{1+p}} \log P_x(\tau_0(X) > t) \approx -1 \cdot \log t^{-q},$$

which is huge, consistent with (1.5). On the other hand, when $p = 0$, we get a Brownian motion with drift ($X_t = W_t - \beta t$) and it is known that $\log P_x(\tau_0(X) > t) \approx -\frac{\beta^2}{2} t$ (Borodin and Salminen [5]). Thus for p close to 0 and t large, we expect:

$$-\gamma \approx t^{-\frac{1-p}{1+p}} \log P_x(\tau_0(X) > t) \approx t^{-1}\left(-\frac{\beta^2}{2} t\right) = -\frac{\beta^2}{2},$$

consistent with (1.4).

As for $\beta > 0$ and $p > 1$, if $\alpha < 1$ then

$$-\frac{\beta}{x^p} \ge \frac{\alpha - 1}{2x}, \quad x \text{ large}.$$

Thus we expect X to exit later than a Bessel process of dimension α close to 1. On the other hand, $\tau_0(X) \le \tau_0(W)$, and X exits sooner than W. Since W is a Bessel process with dimension 1 (at least up to time $\tau_0(W)$), we see $\tau_0(X)$ and $\tau_0(W)$ are about the same in the sense that they have the same qth moments. Theorem 1.1 shows this intuition is correct.

Our results and techniques have implications for some other problems. For example, the family of square root diffusions:

$$\mathrm{d}X_t = c\sqrt{|X_t|}\,\mathrm{d}W_t + (a + bX_t)\,\mathrm{d}t, \tag{1.6}$$

which includes the squared Bessel processes, has been the subject of much research. Aside from applications in various areas of probability and the intrinsic interest it has generated, the family of these diffusions has found an important niche in mathematical finance and economics. See the survey article of Göing-Jaeschke and Yor [11]. For example, in the seminal paper of Cox et al. [6], the family of square root diffusions was used to model short-term interest rates:

$$\mathrm{d}r_t = \sigma\sqrt{r_t}\,\mathrm{d}W_t + \kappa(\theta - r_t)\,\mathrm{d}t. \tag{1.7}$$

Justification for the use of this model includes the following empirically relevant facts:

1. interest rates are always nonnegative;
2. if the rate reaches 0, it can become positive later;
3. the infinitesimal variance of the interest rate increases as the interest rate increases;
4. there is a steady-state distribution for the interest rate.

Göing-Jaeschke and Yor [11] point out the following relationship between squared Bessel processes and square root diffusions (1.6). If Y_t is a squared Bessel process with dimension $\delta = \frac{4a}{c^2}$, that is:

$$\mathrm{d}Y_t = 2\sqrt{|Y_t|}\,\mathrm{d}W_t + \delta Y_t\,\mathrm{d}t,$$

then the solution to (1.6) can be represented by

$$X_t = \mathrm{e}^{bt}\, Y\left(\frac{c^2}{4b}(1-\mathrm{e}^{-bt})\right).$$

Thus in the context of (1.7) we can write

$$r_t = \mathrm{e}^{-\kappa t}\, Y\left(\frac{\sigma^2}{4\kappa}(\mathrm{e}^{\kappa t}-1)\right), \tag{1.8}$$

where Y_t is a squared Bessel process with dimension $\frac{4\kappa\theta}{\sigma^2}$. Instead of using the Bessel process:

$$\mathrm{d}B_t = \mathrm{d}W_t - \beta B_t^{-1}\,\mathrm{d}t,$$

(where β is chosen so that $Y_t = B_t^2$) one could use the perturbed Brownian motion

$$\mathrm{d}B_t = \mathrm{d}W_t - cB_t^{-p}\,\mathrm{d}t, \tag{1.9}$$

for some $p \neq 1$.

In order to ensure properties 1 and 2 above carry through, we use B_t^2 in place of Y_t when $p < 1$ and B_t^q in place of Y_t when $p > 1$, where $q > p + 1$ is chosen so large that $\frac{2p(p+1)}{q^2} < 1$. Moreover, under these circumstances, properties 3 and 4 will also hold.

To help decide whether or not this is a reasonable change, one could ask how long it takes the interest rate to reach 0. Then empirical data could be used to decide which process gives a better model. In light of the identity (1.8), we see the finding when the process reaches 0 comes down to seeing how long the perturbed Brownian motion (1.9) takes to hit 0—this is precisely the content of our results.

To describe another situation in which our work has implications, we review some known results. Let $W \subseteq \mathbb{R}^2$ be a wedge with angle θ. If τ_W is the exit time of Brownian motion from W, then Bañuelos and Smits [4] have shown that:

$$P_x(\tau_W > t) \sim C_1(x)t^{-\pi/2\theta} \quad \text{as } t \to \infty.$$

If τ_W^z is the lifetime of Brownian motion in W conditioned to converge to $x \in \partial W$, then Davis and Zhang [7] have shown that:

$$P_x(\tau_W^z > t) \sim C_2(x,z)t^{-\pi/\theta} \quad \text{as } t \to \infty.$$

Thus the chance of τ_W^z being large is much smaller than the chance of τ_W being large, and so the conditioning makes the process die out faster.

On the other hand, if $K \subseteq \mathbb{R}^2$ is bounded and open, and if τ_K is the exit time of Brownian motion from K, then it is known that:

$$P_x(\tau_K > t) \sim C_4(x)\mathrm{e}^{-\lambda_K t} \quad \text{as } t \to \infty,$$

where λ_K is the first Dirichlet eigenvalue of $\frac{1}{2}\Delta$ on K (Port and Stone [17]). It is also known that:

$$P_x(\tau_K^z > t) \sim C_3(x)\mathrm{e}^{-\lambda_K t} \quad \text{as } t \to \infty,$$

(DeBlassie [8], Kenig and Pipher [13] and Bañuelos and Davis [2]). In this case, the conditioning does not make the process die out any faster.

Now consider the 'parabolic-type' region:

$$D = \{(x, y) : x > 0, |y| < x^p\},$$

where $0 < p < 1$. Bañuelos et al. [3], Li [14] and Lifshits and Shi [15] have shown that

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_x(\tau_D > t) = C_5,$$

where C_5 is explicitly known. This situation is intermediate between the case of the wedge W and the bounded open set K. It would be very interesting to see if conditioning preserves this ordering. In other words, how does the conditioning affect the lifetime of the process? We do not know the answer, but our results provide the following insight.

Write

$$S = \left\{(x, y) : |y| < \frac{\pi}{2}\right\}$$

and let $F : D \to S$ be conformal with

$$\lim_{x\to\infty} F(x, y) = \infty.$$

Let $z \in \partial D$ correspond to the limiting value of $F(x, y)$ as $x \to -\infty$ in S. The Martin kernel of D with pole at z can be written as

$$h(x, y) = \mathrm{e}^{-\mathrm{Re}\, F(x,y)} \cos(\mathrm{Im}\, F(x, y))$$

and the differential operator associated with Brownian motion in D conditioned to converge to z is

$$\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla.$$

Analogous to the intuition described above concerning our results, in order for τ_D^z to be large, we expect the process to spend much time away from z. Also, the conditioning keeps the process away from $\partial D \setminus \{z\}$ and so the greatest influence should come from the horizontal component of the process. Thus:

$$h(x, y) \approx \mathrm{e}^{-\mathrm{Re}\, F(x,y)}$$

and by results of Warschawski [22],

$$F(x, y) \approx \frac{\pi}{2(1-p)} x^{1-p} \quad \text{as } x \to \infty.$$

We conclude:

$$\frac{\nabla h}{h} \cdot \nabla \approx -\frac{\pi}{2} x^{-p} \frac{\partial}{\partial x}$$

and the operator associated with the conditioned Brownian motion is much like:

$$\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\pi}{2} x^{-p} \frac{\partial}{\partial x}.$$

This is the operator studied in this article and Theorem 1.2 suggests:

$$\lim_{t\to\infty} t^{-\frac{1-p}{1+p}} \log P_x(\tau_D > t) = -\gamma\left(p, \frac{\pi}{2}\right).$$

Hence we conjecture the conditioning does not affect the lifetime very much, just as in the case of K bounded.

Here is the organization of the article. In Section 2 we use a comparison argument much like that described above to prove Theorem 1.1. The method is not precise enough to decide the critical case $q = \frac{1}{2}$. In Section 3 we make an h-transform to eliminate the first order term in the differential operator $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \beta x^{-p}\frac{\mathrm{d}}{\mathrm{d}x}$ corresponding to X. The resulting operator takes the form $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, where the potential V is singular. This transformation lets us reduce consideration to study of a Feynman-Kac type functional to which techniques of large deviations apply. In Sections 4 and 5 we use the techniques to obtain lower and upper bounds and we show the bounds coincide. The common value is given by the solution of a singular variational problem which we solve in Section 6.

2. Proof of Theorem 1.1

Throughout this section we will assume $\beta > 0$ and $p > 1$. Let $\alpha < 1$ be very close to 1 and suppose B_t is a Bessel process with dimension α:

$$\mathrm{d}B_t = \mathrm{d}W_t + \frac{1-\alpha}{2} B_t^{-1}\,\mathrm{d}t, \quad t < \tau_0(B).$$

Choose $M > 0$ so large that:

$$-\frac{\beta}{x^p} \geq \frac{\alpha - 1}{2x}, \quad x \geq M/2.$$

Then by the Comparison Theorem for Itô processes (Ikeda and Watanabe [12]),

$$P_M(\tau_{M/2}(X) > t) \geq P_M(\tau_{M/2}(B) > t). \tag{2.1}$$

The Laplace transform of $\tau_{M/2}(B)$ is known (Göing-Jaeschke and Yor [11]) to be

$$E_M[\exp(-\lambda\tau_{M/2}(B))] = 2^{-\nu} \frac{K_{-\nu}(M\sqrt{2\lambda})}{K_{-\nu}(M\sqrt{\lambda/2})}$$

where

$$\nu = \frac{\alpha}{2} - 1 < -\frac{1}{2} \tag{2.2}$$

is very close to $-\frac{1}{2}$ and K_ν is the modified Bessel function. Using this identity, after an integration by parts, the measure

$$\mu(A) = \int_A P_M(\tau_{M/2}(B) > t)\mathrm{d}t$$

has Laplace transform:

$$\omega(\lambda) = \frac{1}{\lambda}\left[1 - 2^{-\nu}\frac{K_{-\nu}(M\sqrt{2\lambda})}{K_{-\nu}(M\sqrt{\lambda/2})}\right].$$

Using the identity:

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}$$

and the series expansion

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k}}{k!\Gamma(\nu+k+1)}$$

(Abramowitz and Stegun [1] page 375), one can show that for some $c_{\nu,M}$:

$$\omega(\lambda) \sim c_{\nu,M}\lambda^{-\nu-1} \quad \text{as } \lambda \to 0.$$

Then since the density of μ is monotone, by Feller's Tauberian Theorem (Feller [10], Theorem 4 on page 446):

$$P_M(\tau_{M/2}(B) > t) \sim \frac{c_{\nu,M}}{\Gamma(\nu+1)} t^\nu \quad \text{as } t \to \infty. \tag{2.3}$$

Now we can prove Theorem 1.1. Let $x > 0$. By making M larger if necessary, it is no loss to assume $M > x$. Then by the strong Markov property, with

$$\eta = \tau_0(X) \wedge \tau_M(X),$$

we have:

$$\begin{aligned} P_x(\tau_0(X) > t) &\geq P_x(X_\eta = M, \tau_0(X) > t, \eta < t) \\ &\geq P_x(X_\eta = M)P_M(\tau_0(X) > t) \\ &\geq P_x(X_\eta = M)P_M(\tau_{M/2}(X) > t) \\ &\geq P_x(X_\eta = M)P_M(\tau_{M/2}(B) > t) \end{aligned}$$

(by (2.1)). Since $\nu < -\frac{1}{2}$ in (2.2) can be made arbitrarily close to $-\frac{1}{2}$, we get that

$$E_x[\tau_0(X)^q] = \infty \quad \text{if } q > \frac{1}{2}.$$

Note by the trivial bound

$$P_x(\tau_0(W) > t) \geq P_x(\tau_0(X) > t)$$

and (1.1), we get

$$E_x[\tau_0(X)^q] < \infty \quad \text{if } q < \frac{1}{2}.$$

This completes the proof of Theorem 1.1.

3. Transformation of the problem

For the rest of the paper, we will assume $\beta > 0$ and $0 < p < 1$. In this section an h-transform is used to convert the problem into one involving large deviations. Denote the differential operator associated with the process X_t by:

$$L = \frac{1}{2}\frac{d^2}{dx^2} - \frac{\beta}{x^p}\frac{d}{dx}$$

and write $p(t, x, y)$ for the transition density of the process killed upon striking 0:

$$P_x(X_t \in A, \tau_0(X) > t) = \int_A p(t, x, y)dy.$$

Let:

$$V(x) = -\frac{1}{2}[\beta p x^{-1-p} + \beta^2 x^{-2p}], \quad x > 0 \tag{3.1}$$

$$L_1 = \frac{1}{2}\frac{d^2}{dx^2} + V, \quad \text{and} \tag{3.2}$$

$$h(x) = \exp\left(-\frac{\beta}{1-p}x^{1-p}\right), \quad x \geq 0. \tag{3.3}$$

Note h is L_1-harmonic: $L_1 h = 0$. The h-transform L_1^h of L_1 is defined by

$$L_1^h f = \frac{1}{h}L_1(hf).$$

Then it is a simple matter to show

$$L_1^h = L. \tag{3.4}$$

Let Y_t be the process associated with L_1, killed upon reaching 0. This process exists because the potential $V(x)$ appearing in L_1 is nonpositive. By (3.4), if $p_1(t, x, y)$ is the transition density of Y_t, then

$$p(t, x, y) = p_1(t, x, y)h(y)/h(x)$$

(Pinsky [16], Theorem 4.1.1 on page 126). By the Feynman-Kac formula, for one-dimensional Brownian motion B_t,

$$\begin{aligned} P_x(\tau_0(X) > t) &= \int_0^\infty p(t, x, y)dy \\ &= \frac{1}{h(x)}\int_0^\infty p_1(t, x, y)h(y)dy \\ &= \frac{1}{h(x)}E_x[h(Y_t)I_{\tau_0(Y)>t}] \\ &= \frac{1}{h(x)}E_x\left[\exp\left(\int_0^t V(B_s)ds\right)h(B_t)I_{\tau_0(B)>t}\right]. \end{aligned} \tag{3.5}$$

Let:

$$C_0 = \{\omega\colon\ [0, 1] \to \mathbb{R} \mid \omega \text{ is continuous}, \omega(0) = 0\}$$

and equip C_0 with the sup norm. If $\omega \in C_0$ is absolutely continuous (with respect to Lebesgue measure) denote its derivative by $\dot{\omega}$ or ω'. Then set:

$$K_0 = \left\{ \omega \in C_0 \colon \int_0^1 [\dot{\omega}_u]^2 \mathrm{d}u < \infty \right\}.$$

With

$$\begin{aligned} c_1 &= \frac{1}{2}\beta^2 \quad \text{and} \\ c_2 &= \frac{\beta}{1-p}, \end{aligned} \tag{3.6}$$

for $\omega \in K_0$ define

$$F(\omega) = c_1 \int_0^1 |\omega(u)|^{-2p} \mathrm{d}u + c_2 |\omega(1)|^{1-p} + \frac{1}{2} \int_0^1 [\dot{\omega}(u)]^2 \mathrm{d}u, \tag{3.7}$$

if the first integral is finite; otherwise, set $F(\omega) = \infty$.

Theorem 3.1. *For any $x > 0$,*

$$\lim_{t \to \infty} t^{-\frac{1-p}{1+p}} \log E_x \left[\exp\left(\int_0^t V(B_s) \mathrm{d}s \right) h(B_t) I_{\tau_0(B) > t} \right] = - \inf_{\omega \in K_0} F(\omega). \quad \square$$

Notice the infimum exists and so by (3.5) this theorem will prove the existence of the limit in Theorem 1.2.

4. Proof of Theorem 3.1: Lower bound

In this section we prove:

$$\liminf_{t \to \infty} t^{-\frac{1-p}{1+p}} \log E_x \left[\exp\left(\int_0^t V(B_s) \mathrm{d}s \right) h(B_t) I_{\tau_0(B) > t} \right] \geq - \inf_{\substack{\omega \in K_0 \\ \omega \geq 0}} F(\omega). \tag{4.1}$$

For $x < 0$ set $V_i(x) = 0$, $i = 1, 2, 3$ and for $x \geq 0$ define

$$\begin{aligned} V_1(x) &= \frac{1}{2}\beta p x^{-1-p} \\ V_2(x) &= c_1 x^{-2p} \\ V_3(x) &= c_2 x^{1-p}, \end{aligned}$$

where c_1 and c_2 are from (3.6) and we take $\frac{1}{0} = \infty$. Notice the following scaling identities hold:

$$\begin{aligned} V_1(rx) &= r^{-1-p} V_1(x) \\ V_2(rx) &= r^{-2p} V_2(x) \\ V_3(rx) &= r^{1-p} V_3(x). \end{aligned}$$

Moreover, for $\omega \geq 0$

$$F(\omega) = \int_0^1 V_2(\omega(u)) \mathrm{d}u + V_3(\omega(1)) + \frac{1}{2} \int_0^1 [\dot{\omega}(u)]^2 \mathrm{d}u \tag{4.2}$$

and for $\tau_0(B) > t$,

$$\int_0^t V(B_s)\mathrm{d}s - \frac{\beta}{1-p} B_t^{1-p} = -\int_0^t [V_1(B_s) + V_2(B_s)]\mathrm{d}s - V_3(B_t).$$

Thus:

$$\begin{aligned}
&E_x\left[\exp\left(\int_0^t V(B_s)\mathrm{d}s\right) h(B_t) I_{\tau_0(B)>t}\right] \\
&= E_x\left[\exp\left(-\int_0^t [V_1(B_s) + V_2(B_s)]\mathrm{d}s - V_3(B_t)\right) I_{\tau_0(B)>t}\right] \\
&= E_x\left[\exp\left(-\int_0^1 [V_1(B_{tu}) + V_2(B_{tu})]t\ \mathrm{d}u - V_3(B_t)\right) I_{\tau_0(B)>t}\right] \\
&= E_{x/\sqrt{t}}\left[\exp\left(-\int_0^1 [V_1(\sqrt{t}\ B_u) + V_2(\sqrt{t}\ B_u)]t\ \mathrm{d}u - V_3(\sqrt{t}\ B_1)\right) I_{\tau_0(B)>1}\right]
\end{aligned}$$

(Brownian scaling)

$$\begin{aligned}
&= E_{x/\sqrt{t}}\left[\exp\left(-t^{(1-p)/2}\int_0^t V_1(B_u)\mathrm{d}u - t^{1-p}\int_0^1 V_2(B_u)\mathrm{d}u - t^{(1-p)/2} V_3(B_1)\right)\right. \\
&\quad \left.\times I_{\tau_0(B)>1}\right] \qquad (4.3)
\end{aligned}$$

(scaling properties of V_1–V_3).

Now let $g \in K_0$ be such that $g \geq 0$ and $F(g) < \infty$. Consider any $\varepsilon > 0$ and define

$$\begin{aligned}
&Z_u = B_u - g(u)/\sqrt{\varepsilon} \\
&\eta = \tau_0(Z) = \inf\{u > 0\colon\ Z_u = 0\}.
\end{aligned}$$

Note that for $Z_0 > 0$,

$$\eta > 1 \Rightarrow \tau_0(B) > 1 \quad \text{and} \qquad (4.4)$$

$$\eta > 1 \Rightarrow V_i(g(u)/\sqrt{\varepsilon}) \geq V_i(B_u) \quad \text{for all } u \leq 1,\ i = 1, 2. \qquad (4.5)$$

Lemma 4.1. *For $\delta > 0$ and $\varepsilon = t^{-(1-p)/(1+p)}$,*

$$P_{x/\sqrt{t}}\left(\tau_0(B) > 1, t^{(1-p)/2} V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u > \frac{\delta}{\varepsilon}\right) = o(P_{x/\sqrt{t}}(\tau_0(B) > 1))$$

as $t \to \infty$.

Proof. In what follows, c will be a number whose exact value might change from line to line, but it will always be independent of t. Under $P_{x/\sqrt{t}}$, the random variable $\int_0^1 g'(u)\mathrm{d}B_u$ is Gaussian with mean 0 and variance $\int_0^1 [g'(u)]^2\mathrm{d}u$. Hence

$$E_{x/\sqrt{t}}\left[\exp\left(\int_0^1 g'(u)\mathrm{d}B_u\right)\right]$$

is independent of t. Using this together with the fact that

$$\varepsilon^{-1/(1-p)}t^{-1/2} = \varepsilon^{-1/2},$$

we have for $t > 1$,

$$\begin{aligned}
&P_{x/\sqrt{t}}\left(\tau_0(B) > 1, t^{(1-p)/2}V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u > \frac{\delta}{\varepsilon}\right)\\
&\le P_{x/\sqrt{t}}\left(\tau_0(B) > 1, t^{(1-p)/2}V_3(B_1) > \frac{\delta}{2\varepsilon}\right)\\
&\quad + P_{x/\sqrt{t}}\left(\tau_0(B) > 1, \int_0^1 g'(u)\mathrm{d}B_u > \frac{\delta}{2\sqrt{\varepsilon}}\right)\\
&\le P_{x/\sqrt{t}}(B_1 > c\varepsilon^{-1/(1-p)}t^{-1/2}) + \exp\left(-\frac{\delta}{2}\varepsilon^{-1/2}\right)E_{x/\sqrt{t}}\left[\exp\left(\int_0^1 g'(u)\mathrm{d}B_u\right)\right]\\
&\le \exp(-c\varepsilon^{-1/2})E_{x/\sqrt{t}}(\mathrm{e}^{B_1}) + c\exp\left(-\frac{\delta}{2}\varepsilon^{-1/2}\right)\\
&= c\exp(-c\varepsilon^{-1/2} + xt^{-1/2}) + c\exp\left(-\frac{\delta}{2}\varepsilon^{-1/2}\right)\\
&\le c\exp(-c\varepsilon^{-1/2}). \qquad (4.6)
\end{aligned}$$

From (1.1),

$$\begin{aligned}
P_{x/\sqrt{t}}(\tau_0(B) > 1) &= \frac{2}{\sqrt{2\pi}}\int_0^{x/\sqrt{t}} \mathrm{e}^{-u^2/2}\mathrm{d}u\\
&\sim \frac{2}{\sqrt{2\pi}}\frac{x}{\sqrt{t}} = c\varepsilon^{\frac{1}{2}(1+p)/(1-p)} \qquad (4.7)
\end{aligned}$$

as $t \to \infty$. Here $f_1(t) \sim f_2(t)$ as $t \to \infty$ means $\frac{f_1(t)}{f_2(t)} \to 1$ as $t \to \infty$. Hence for t large,

$$\frac{P_{x/\sqrt{t}}\left(\tau_0(B) > 1, t^{(1-p)/2}V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u > \frac{\delta}{\varepsilon}\right)}{P_{x/\sqrt{t}}(\tau_0(B) > 1)} \le \frac{c\exp(-c\varepsilon^{-1/2})}{(\sqrt{\varepsilon})^{(1+p)/(1-p)}}$$
$$\to 0 \quad \text{as } t \to \infty, \text{ since } \varepsilon \to 0^+ \text{ iff } t \to \infty. \quad \square$$

Corollary 4.2. *For* $\varepsilon = t^{-(1-p)/(1+p)}$,

$$\liminf_{t\to\infty} \varepsilon \log P_{x/\sqrt{t}}\left(\tau_0(B) > 1, t^{(1-p)/2}V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u \le \frac{\delta}{\varepsilon}\right) \ge 0.$$

Proof. Writing $A = t^{(1-p)/2}V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u$, we have

$$\begin{aligned}
P_{x/\sqrt{t}}\left(\tau_0(B) > 1, A \le \frac{\delta}{\varepsilon}\right) &= P_{x/\sqrt{t}}(\tau_0(B) > 1) - P_{x/\sqrt{t}}\left(\tau_0(B) > 1, A > \frac{\delta}{\varepsilon}\right)\\
&= P_{x/\sqrt{t}}(\tau_0(B) > 1)\left[1 - \frac{P_{x/\sqrt{t}}(\tau_0(B) > 1, A > \frac{\delta}{\varepsilon})}{P_{x/\sqrt{t}}(\tau_0(B) > 1)}\right].
\end{aligned}$$

Take the logarithm of both sides, multiply by ε and let $t \to \infty$. Then by Lemma 4.1, and (4.7)

$$\liminf_{t\to\infty} \varepsilon \log P_{x/\sqrt{t}}\left(\tau_0(B) > 1,\, A \le \frac{\delta}{\varepsilon}\right) \ge \liminf_{t\to\infty} \varepsilon \log P_{x/\sqrt{t}}(\tau_0(B) > 1) = 0. \quad \square$$

Lemma 4.3. *For $\varepsilon = t^{-(1-p)/(1+p)}$,*

$$\liminf_{t\to\infty} \varepsilon \log E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \ge -\frac{1}{2}\int_0^1 [g'(u)]^2 \mathrm{d}u.$$

Proof. By the Cameron–Martin–Girsanov Theorem, for any $\delta > 0$,

$$\begin{aligned} &E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \\ &\quad = E_{x/\sqrt{t}}\left[I_{\tau_0(B)>1} \exp\left(-t^{(1-p)/2} V_3(B_1) - \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u - \frac{1}{2\varepsilon}\int_0^1 [g'(u)]^2 \mathrm{d}u\right)\right] \\ &\quad \ge \exp\left(-\frac{1}{2\varepsilon}\int_0^1 [g'(u)]^2 \mathrm{d}u - \frac{\delta}{\varepsilon}\right) \\ &\qquad \times P_{x/\sqrt{t}}\left(\tau_0(B) > 1,\, t^{(1-p)/2} V_3(B_1) + \frac{1}{\sqrt{\varepsilon}}\int_0^1 g'(u)\mathrm{d}B_u \le \frac{\delta}{\varepsilon}\right). \end{aligned}$$

Hence by Corollary 4.2,

$$\liminf_{t\to\infty} \varepsilon \log E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \ge -\frac{1}{2}\int_0^1 [g'(u)]^2 \mathrm{d}u - \delta.$$

Let $\delta \to 0^+$ to finish. $\square$

Now we can prove (4.1). Since $0 < p < 1$, $V_3(a+b) \le V_3(a) + V_3(b)$. Then by definition of Z, with $\varepsilon = t^{-(1-p)/(1+p)}$,

$$\begin{aligned} E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(B_1)) I_{\eta>1}] &= E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1 + g(1)/\sqrt{\varepsilon})) I_{\eta>1}] \\ &\ge \exp(-t^{(1-p)/2} V_3(g(1)/\sqrt{\varepsilon})) E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \\ &= \exp(-t^{(1-p)/2} \varepsilon^{-(1-p)/2} V_3(g(1))) E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \end{aligned}$$

(by the scaling property of V_3)

$$= \exp(-\varepsilon^{-1} V_3(g(1))) E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2} V_3(Z_1)) I_{\eta>1}] \tag{4.8}$$

(by choice of ε).

Also, by scaling properties of V_1 and V_2, and choice of ε:

$$\exp\left(-t^{(1-p)/2}\int_0^1 V_1(g(u)/\sqrt{\varepsilon})\mathrm{d}u - t^{1-p}\int_0^1 V_2(g(u)/\sqrt{\varepsilon})\mathrm{d}u\right)$$
$$= \exp\left(t^{(1-p)/2}\varepsilon^{(1+p)/2}\int_0^1 V_1(g(u))\mathrm{d}u - t^{1-p}\varepsilon^{p}\int_0^1 V_2(g(u))\mathrm{d}u\right)$$
$$= \exp\left(-\int_0^1 V_1(g(u))\mathrm{d}u - \varepsilon^{-1}\int_0^1 V_2(g(u))\mathrm{d}u\right). \tag{4.9}$$

Then:

$$E_x\left[\exp\left(\int_0^t V(B_s)\mathrm{d}s\right)h(B_t)I_{\tau_0(B)>t}\right]$$
$$= E_{x/\sqrt{t}}\left[\exp\left(-t^{(1-p)/2}\int_0^1 V_1(B_u)\mathrm{d}u - t^{1-p}\int_0^1 V_2(B_u)\mathrm{d}u - t^{(1-p)/2}V_3(B_1)\right)\right.$$
$$\left.\times\, I_{\tau_0(B)>1}\right]$$

(by (4.3))

$$\geq \exp\left(-t^{(1-p)/2}\int_0^1 V_1(g(u)/\sqrt{\varepsilon})\mathrm{d}u - t^{1-p}\int_0^1 V_2(g(u)/\sqrt{\varepsilon})\mathrm{d}u\right)$$
$$\cdot E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2}V_3(B_1))I_{\eta>1}]$$

(by (4.4) and (4.5))

$$\geq \exp\left(-\int_0^1 V_1(g(u))\mathrm{d}u - \varepsilon^{-1}\int_0^1 V_2(g(u))\mathrm{d}u\right)$$
$$\cdot \exp(-\varepsilon^{-1}V_3(g(1)))E_{x/\sqrt{t}}[\exp(-t^{(1-p)/2})V_3(Z_1)I_{\eta>1}]$$

(by (4.9) and (4.8)). By Lemma 4.3 and our choice of ε, this yields

$$\liminf_{t\to\infty} t^{-(1-p)/(1+p)}\log E_x\left[\exp\left(\int_0^t V(B_s)\mathrm{d}s\right)h(B_t)I_{\tau_0(B)>t}\right]$$
$$\geq -\int_0^1 V_2(g(u))\mathrm{d}u - V_3(g(1)) - \frac{1}{2}\int_0^1 [g'(u)]^2\mathrm{d}u.$$

Taking the supremum over all $g \in K_0$ such that $g \geq 0$ and $F(g) < \infty$ and using (4.2) gives (4.1).

5. Proof of Theorem 3.1: Upper bound

In this section we derive an upper bound that matches the lower bound from the previous section. To this end, we first show:

$$\limsup_{t\to\infty} t^{-(1-p)/(1+p)} \log E_x\left[\exp\left(\int_0^t V(B_s)\mathrm{d}s\right)h(B_t)I_{\tau_0(B)>t}\right]$$
$$\leq -\inf_{\omega\in K_0}\left[J(\omega)+\frac{1}{2}\int_0^1[\dot\omega(u)]^2\mathrm{d}u\right] \tag{5.1}$$

where

$$J(\omega)=\int_0^1 V_2(\omega(u))\mathrm{d}u+V_3(\omega(1))$$

if the integral is finite; otherwise set $J(\omega)=\infty$.

Let Q^ε be the law on $C([0,\infty),\mathbb{R})$ of $\sqrt{\varepsilon}\,B$ under P_0, where

$$\varepsilon=t^{-(1-p)/(1+p)}$$

as usual. Then by (4.3) and scaling properties of V_1-V_3, writing $f(\varepsilon)=x\varepsilon^{1/(1-p)}$,

$$E_x\left[\exp\left(\int_0^t V(B_s)\mathrm{d}s\right)h(B_t)I_{\tau_0(B)>t}\right]$$
$$=E_0\left[\exp\left(-t^{(1-p)/2}\int_0^1 V_1\left(B_u+\frac{x}{\sqrt{t}}\right)\mathrm{d}u-t^{1-p}\int_0^1 V_2\left(B_u+\frac{x}{\sqrt{t}}\right)\mathrm{d}u\right.\right.$$
$$\left.\left.-t^{(1-p)/2}V_3\left(B_1+\frac{x}{\sqrt{t}}\right)\right)I\left(\tau_0\left(B+\frac{x}{\sqrt{t}}\right)\geq 1\right)\right]$$
$$=E_0\left[\exp\left(-\int_0^1 V_1(\sqrt{\varepsilon}\,B_u+f(\varepsilon))\mathrm{d}u-\varepsilon^{-1}\int_0^1 V_2(\sqrt{\varepsilon}B_u+f(\varepsilon))\mathrm{d}u\right.\right.$$
$$\left.\left.-\varepsilon^{-1}V_3(\sqrt{\varepsilon}\,B_1+f(\varepsilon))\right)I\left(\tau_0\left(B+\frac{x}{\sqrt{t}}\right)>1\right)\right]$$
$$\leq E^{Q^\varepsilon}\left[\exp\left(-\int_0^1 V_1(\omega_u+f(\varepsilon))\mathrm{d}u-\varepsilon^{-1}\int_0^1 V_2(\omega_u+f(\varepsilon))\mathrm{d}u\right.\right.$$
$$\left.\left.-\varepsilon^{-1}V_3(\omega_1+f(\varepsilon))\right)\right]$$
$$=E^{Q^\varepsilon}\left[\exp\left(-\frac{1}{\varepsilon}J_\varepsilon(\omega)\right)\right] \tag{5.2}$$

where

$$J_\varepsilon(\omega)=\varepsilon\int_0^1 V_1(\omega_u+f(\varepsilon))\mathrm{d}u+\int_0^1 V_2(\omega_u+f(\varepsilon))\mathrm{d}u+V_3(\omega_1+f(\varepsilon))$$

if the integrals are finite; otherwise, $J_\varepsilon(\omega)=\infty$.

Notice J is lower semicontinuous on C_0 and if $\omega_n \to \omega$ in C_0 as $\varepsilon \to 0$, then

$$\liminf_{\substack{n\to\infty \\ \varepsilon\to 0^+}} J_\varepsilon(\omega_n) \ge J(\omega).$$

Hence by Varadhan's theorem (Varadhan [21], Theorem 2.3)

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \varepsilon \log E^{Q^\varepsilon}\left[\exp\left(-\frac{1}{\varepsilon} J_\varepsilon(\omega)\right)\right] &\le -\inf_{\omega\in C_0}\left[J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u\right] \\ &= -\inf_{\omega\in K_0}\left[J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u\right].\end{aligned}$$

Using this in (5.2) gives (5.1).

Next we show the lower bound from (4.1) matches the upper bound from (5.1):

$$\inf_{\substack{\omega\in K_0 \\ \omega\ge 0}} F(\omega) = \inf_{\omega\in K_0}\left[J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u\right]. \tag{5.3}$$

If $\omega \ge 0$ then by (4.2),

$$F(\omega) = J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u.$$

Hence it is clear that

$$\inf_{\substack{\omega\in K_0 \\ \omega\ge 0}} F(\omega) \ge \inf_{\omega\in K_0}\left[J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u\right].$$

For the opposite inequality, consider any $\omega \in K_0$. Then we can write $\{u\colon\ \omega(u) > 0\} = \bigcup_n (a_n, b_n)$ as a disjoint union of a countable collection of open intervals. Notice on (a_n, b_n), $\omega = \omega^+$, where $\omega^+ = \max(0, \omega)$. Then since $\bigcup_n \{a_n, b_n\}$ is countable, ω^+ is absolutely continuous on $[0, 1]$ and is 0 almost everywhere on $\{u\colon\ \omega(u) \le 0\}$. Thus

$$\begin{aligned}J(\omega) + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u &\ge \int_0^1 V_2(\omega_u^+)\mathrm{d}u + V_3(\omega_1^+) + \int_0^1 [(\omega_u^+)']^2 \mathrm{d}u \\ &= F(\omega^+) \\ &\ge \inf_{\substack{\tilde\omega\ge 0 \\ \tilde\omega\in K_0}} F(\tilde\omega).\end{aligned}$$

Taking the infimum over $\omega \in K_0$ yields the desired inequality and the proof of (5.3) is complete.

To finish the proof of Theorem 3.1, just note an argument like that for (5.3) shows

$$\inf_{\substack{\omega\in K_0 \\ \omega\ge 0}} F(\omega) = \inf_{\omega\in K_0} F(\omega).$$

6. Solution of the variational problem

The main result of this section is the explicit solution of the variational problem arising in Theorem 3.1. Combined with Theorem 3.1, it will complete the proof of Theorem 1.2.

Theorem 6.1. *We have:*

$$\inf_{\omega\in K_0} F(\omega) = \frac{1}{2} p^{-\frac{2p}{1+p}} \beta^{\frac{2}{1+p}} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B\left(\frac{3}{2}, \frac{1-p}{2p}\right)\right] \Big/ B\left(\frac{1}{2}, \frac{1+p}{2p}\right)^{\frac{1-p}{1+p}}.$$

We break up the proof into several steps, eventually reducing the problem to elementary calculus. For $\theta \geq 0$, define:

$$F_\theta(\omega) = c_1 \int_0^1 [\omega^2(u) + \theta]^{-p}\, \mathrm{d}u + c_2|\omega(1)|^{1-p} + \frac{1}{2}\int_0^1 [\dot\omega(u)]^2 \mathrm{d}u$$

if the last integral exists and take it to be ∞ otherwise. We will often write

$$\|\dot\omega\|_2 = \sqrt{\int_0^1 [\dot\omega(u)]^2\, \mathrm{d}u}\,.$$

The following lemma is on page 75 in Riesz and Sz.-Nagy [18].

Lemma 6.2. *A function* $\omega\colon\ [0,1] \to \mathbb{R}$ *is the integral of a function* F *in* L^2 *iff*

$$\sup \sum_{k=1}^m \frac{[\omega(t_k) - \omega(t_{k-1})]^2}{t_k - t_{k-1}} < \infty,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_m = 1$ *of* $[0, 1]$. *Moreover, the supremum is exactly* $\|F\|_2^2$. □

Lemma 6.3. *For* $\theta \geq 0$, $\inf_{\omega\in K_0} F_\theta(\omega)$ *is attained.*

Proof. Choose $\omega_n \in K_0$ such that $F_\theta(\omega_n) \to \inf_{\omega\in K_0} F_\theta(\omega)$. Then $a_n = F_\theta(\omega_n)$ is a bounded sequence and so by nonnegativity of all the terms in F_θ, $\|\dot\omega_n\|_2$ is a bounded sequence. By passing to a subsequence if necessary, we can assume for some γ,

$$\|\dot\omega_n\|_2 \leq 2\gamma \quad \text{for all } n \quad \text{and} \quad \|\dot\omega_n\|_2 \to \gamma \quad \text{as } n \to \infty.$$

The set $\{\omega \in K_0\colon\ \|\dot\omega\|_2 \leq 2\gamma\}$ is a compact subset of C_0 (Strassen [20], Section 1). Then by passing to a subsequence, it is no loss to assume for some $x \in K_0$ with $\|\dot x\|_2 \leq 2\gamma$, we have $\omega_n \to x$ in C_0. In fact,

$$\|\dot x\|_2 \leq \gamma,$$

as we now show. If $0 = t_0 < t_1 < \cdots < t_m = 0$ is any partition of $[0, 1]$, then by Lemma 6.2

$$\sum_{k=1}^m \frac{[\omega_n(t_k) - \omega_n(t_{k-1})]^2}{t_k - t_{k-1}} \leq \|\dot\omega_n\|_2^2.$$

Let $n \to \infty$ to get

$$\sum_{k=1}^m \frac{[x(t_k) - x(t_{k-1})]^2}{t_k - t_{k-1}} \leq \gamma^2.$$

Apply Lemma 6.2 again to see $\|\dot x\|_2^2 \leq \gamma^2$, as claimed.

To finish, observe by Fatou's lemma that:

$$\begin{aligned}
\inf_{\omega\in K_0} F_\theta(\omega) &\le F_\theta(x) = c_1\int_0^1 [x_u^2+\theta]^{-p}\mathrm{d}u + c_2|x_1|^{1-p} + \frac{1}{2}\int_0^1 [\dot{x}_u]^2\mathrm{d}u \\
&= c_1\int_0^1 \lim_{n\to\infty}[\omega_n^2(u)+\theta]^{-p}\mathrm{d}u + c_2\lim_{n\to\infty}|\omega_n(1)|^{1-p} \\
&\quad + \frac{1}{2}\lim_{n\to\infty}\int_0^1 [\dot{\omega}_n(u)]^2\mathrm{d}u - \frac{1}{2}\lim_{n\to\infty}\int_0^1 [\dot{\omega}_n(u)]^2\mathrm{d}u + \frac{1}{2}\|\dot{x}\|_2^2 \\
&\le \lim_{n\to\infty} F_\theta(\omega_n) - \frac{1}{2}\gamma^2 + \frac{1}{2}\|\dot{x}\|_2^2 \\
&\le \lim_{n\to\infty} F_\theta(\omega_n) \\
&= \inf_{\omega\in K_0} F_\theta(\omega).
\end{aligned}$$

Thus $F_\theta(x) = \inf_{\omega\in K_0} F_\theta(\omega)$. □

Lemma 6.4. *There is a sequence $(\theta_n,\omega_n)\in(0,\infty)\times K_0$ and $x\in K_0$ such that $\theta_n\to 0$, $\omega_n\ge 0$ is a minimizer of F_{θ_n} on K_0, $\lim_{n\to\infty}\|\dot{\omega}_n\|_2$ exists, $\omega_n\to x$ in C_0 and x is a minimizer of F on K_0.*

Proof. Since $\inf_{\omega\in K_0} F_\theta(\omega)$ is decreasing in θ,

$$\sup_{\theta>0}\inf_{\omega\in K_0} F_\theta(\omega) = \lim_{\theta\to 0^+}\inf_{\omega\in K_0} F_\theta(\omega).$$

Also, for each $\theta > 0$, $F(\omega)\ge F_\theta(\omega)$ and so:

$$\inf_{\omega\in K_0} F(\omega) \ge \sup_{\theta>0}\inf_{\omega\in K_0} F_\theta(\omega). \tag{6.1}$$

Next, choose $\theta_n\to 0$ such that

$$\inf_{\omega\in K_0} F_{\theta_n}(\omega) \to \sup_{\theta>0}\inf_{\omega\in K_0} F_\theta(\omega) \quad \text{as } n\to\infty.$$

By Lemma 6.3 there exist $\omega_n\in K_0$, $n = 1, 2, \ldots$, such that

$$\inf_{\omega\in K_0} F_{\theta_n}(\omega) = F_{\theta_n}(\omega_n).$$

Notice from the form of F_θ, it is no loss to assume $\omega_n\ge 0$. Since $\inf_{\omega\in K_0} F(\omega) < \infty$, by (6.1) $F_{\theta_n}(\omega_n)$ is a bounded sequence. Then we can argue exactly as in the proof of Lemma 6.3, passing to a subsequence if necessary, to get that for some: $x\in K_0$,

$$\begin{aligned}
&\|\dot{\omega}_n\|_2 \to \gamma \\
&\omega_n \to x \quad \text{in } C_0 \\
&\|\dot{x}\|_2 \le \gamma.
\end{aligned}$$

Thus all conclusions of the lemma, except the last one, follow. Moreover:

$$\begin{aligned}
\inf_{\omega\in K_0} F(\omega) &\le F(x) \\
&= \left[c_1 \int_0^1 \lim_{n\to\infty} [\omega_n^2(u)+\theta_n]^{-p} \mathrm{d}u + c_2 \lim_{n\to\infty} |\omega_n(1)|^{1-p} + \frac{1}{2}\lim_{n\to\infty} \|\dot\omega_n\|_2^2\right] \\
&\quad - \frac{1}{2}\lim_{n\to\infty} \|\dot\omega_n\|_2^2 + \frac{1}{2}\|\dot x\|_2^2 \\
&\le \lim_{n\to\infty} F_{\theta_n}(\omega_n) \\
&= \lim_{n\to\infty} \inf_{\omega\in K_0} F_{\theta_n}(\omega) \\
&= \sup_{\theta>0} \inf_{\omega\in K_0} F_\theta(\omega) \\
&\le \inf_{\omega\in K_0} F(\omega)
\end{aligned}$$

(by (6.1)). Thus equality holds throughout and x is a minimizer of F on K_0. □

Our goal is to derive an explicit expression for the minimizer x from Lemma 6.4. This will be done by showing x solves a second order differential equation which has an explicit solution. The idea is that from the calculus of variations, ω_n from Lemma 6.4 solves a certain differential equation. Letting $n\to\infty$ gives the equation for x.

Lemma 6.5. *Let $\theta > 0$ and suppose $y \ge 0$ is a minimizer of F_θ on K_0. Then*

$$2pc_1[y^2(u)+\theta]^{-p-1} y(u) + \ddot y(u) = 0 \quad \text{on } [0,1] \tag{6.2}$$

and for any $u_0 \in [0, 1]$

$$\dot y^2(u) = 2c_1 \frac{(y^2(u_0)+\theta)^p - (y^2(u)+\theta)^p}{(y^2(u_0)+\theta)^p (y^2(u)+\theta)^p} + \dot y^2(u_0), \quad u\in[0,1]. \tag{6.3}$$

Proof. Observe that for $a = y_1$

$$\begin{aligned}
\inf_{\omega\in K_0} F_\theta(\omega) &= F_\theta(y) = c_1 \int_0^1 [y_u^2+\theta]^{-p} \mathrm{d}u + c_2 a^{1-p} + \frac{1}{2}\|\dot y\|_2^2 \\
&\ge \inf_{\omega\in K_0} \left\{ c_1 \int_0^1 [\omega_u^2+\theta]^{-p} \mathrm{d}u + \frac{1}{2}\|\dot\omega\|_2^2 \colon\ \omega_1 = a \right\} + c_2 a^{1-p} \\
&\ge \inf_{\omega\in K_0} \left\{ c_1 \int_0^1 [\omega_u^2+\theta]^{-p} \mathrm{d}u + c_2|\omega_1|^{1-p} + \frac{1}{2}\|\dot\omega\|_2^2 \colon\ \omega_1 = a \right\} \\
&= \inf_{\omega\in K_0} \{F_\theta(\omega)\colon\ \omega_1 = a\} \\
&\ge \inf_{\omega\in K_0} F_\theta(\omega).
\end{aligned}$$

In particular, equality holds throughout and so for:

$$G_\theta(\omega) = c_1 \int_0^1 [\omega_u^2+\theta]^{-p} \mathrm{d}u + \frac{1}{2}\|\dot\omega\|_2^2,$$

we see $\inf_{\omega\in K_0}\{G_\theta\colon\ \omega_1 = a\}$ is taken on at y.

Define $\|\cdot\|_{K_0}$ to be the natural norm on K_0:

$$\|\omega\|_{K_0} = \sup_{[0,1]} |\omega| + \|\dot\omega\|_2.$$

If K is a functional on K_0, write δK for the variation of K:

$$\delta K(\omega, z) = \frac{\partial}{\partial \varepsilon} K(\omega + \varepsilon z)|_{\varepsilon=0}.$$

Then by the Lagrange Multiplier Theorem (Smith [19]), for $K(\omega) = \omega_1$, there is a real λ such that

$$\delta G_\theta(y, z) = \lambda \delta K(y, z) \quad \text{for all } z \in K_0.$$

That is,

$$-2pc_1 \int_0^1 [y_u^2 + \theta]^{-p-1} y_u z_u \, \mathrm{d}u + \int_0^1 \dot y_u \dot z_u \, \mathrm{d}u - \lambda z_1 = 0 \quad \text{for all } z \in K_0.$$

Integration by parts in the first integral yields:

$$\int_0^1 \left[-2pc_1 \int_u^1 [y_s^2 + \theta]^{-p-1} y_s \, \mathrm{d}s + \dot y_u \right] \dot z_u \, \mathrm{d}u - \int_0^1 \lambda \dot z_u \, \mathrm{d}u = 0 \quad \text{for all } z \in K_0.$$

In particular:

$$-2pc_1 \int_u^1 [y_s^2 + \theta]^{-p-1} y_s \, \mathrm{d}s + \dot y_u - \lambda = 0. \tag{6.4}$$

Thus we see $\dot y_u$ is continuous and differentiable on $[0, 1]$ and $\lambda = \dot y_1$. Differentiation of (6.4) yields (6.2). Finally, multiplying (6.2) by $\dot y_u$ and then integrating from u_0 to u gives (6.3). □

Lemma 6.6. *The minimizer x from Lemma 6.4 satisfies the following properties:*

(1) $x > 0$ *on* $(0, 1)$*;*
(2) $2pc_1 x_u^{-2p-1} + \ddot x_u = 0$ *on* $(0, 1)$*;*
(3) *for any* $v_0 \in (0, 1)$*,*

$$\dot x_u^2 = 2c_1 \frac{x_{v_0}^{2p} - x_u^{2p}}{x_{v_0}^{2p} x_u^{2p}} + \dot x_{v_0}^2, \quad u \in (0, 1).$$

Proof. Let θ_n and ω_n be from Lemma 6.4. By Lemma 6.5, ω_n is concave on $(0, 1)$; that is, for any $0 < \lambda < 1$,

$$\omega_n(\lambda u + (1-\lambda)v) \ge \lambda \omega_n(u) + (1-\lambda)\omega_n(v), \qquad u, v \in (0, 1).$$

Let $n \to \infty$ to see x is also concave on $(0, 1)$. Hence if $x_u = 0$ for some $u \in (0, 1)$, then $x \equiv 0$ on $(0, 1)$ and so $F(x) = \infty$, contrary to x minimizing F. Thus $x > 0$ on $(0, 1)$.

By (1) and Lemma 6.5, for any $[a, b] \subseteq (0, 1)$, $\ddot\omega_n$ is Cauchy on $C[a, b]$, the space of continuous functions on $[a, b]$ equipped with the sup norm. Moreover,

$$\sup_n \sup_{[a,b]} |\ddot\omega_n| < \infty.$$

Thus by the Mean Value Theorem the family $\{\dot{\omega}_n : n \geq 1\}$ is equicontinuous on $[a, b]$. Once we show the family is pointwise bounded on $[a, b]$, by Ascoli's Theorem there is a subsequence of $\dot{\omega}_n$ that converges uniformly on $[a, b]$. By diagonalizing we get a subsequence of $\dot{\omega}_n$ that converges uniformly on compact subsets of $(0, 1)$. Thus by passing to a subsequence, it is no loss to assume $\dot{\omega}_n$ converges uniformly on compact subsets of $(0, 1)$. The grand conclusion is that we can replace y and θ in (6.2) and (6.3) by ω_n and θ_n, respectively, and then let $n \to \infty$ to get (2) and (3).

All that remains is to verify pointwise boundedness of $\{\dot{\omega}_n\}$ on $[a, b]$. We will show for every $u \in [a, b]$,

$$\sup_n \dot{\omega}_n^2(u) < \infty. \tag{6.5}$$

By (6.3), with $y = \omega_n$ and $\theta = \theta_n$,

$$\dot{\omega}_n^2(u) = 2c_1 \frac{(\omega_n^2(b) + \theta_n)^p - (\omega_n^2(u) + \theta_n)^p}{(\omega_n^2(b) + \theta_n)^p (\omega_n^2(u) + \theta_n)^p} + \dot{\omega}_n^2(b), \quad u \in [a, b]. \tag{6.6}$$

Since $\omega_n \to x$ in C_0 and since $x > 0$ on $(0, 1)$, once we show

$$\sup_n \dot{\omega}_n^2(b) < \infty, \tag{6.7}$$

(6.5) will follow.

To prove (6.7), assume it is false. By passing to a subsequence, it is no loss to assume: $\dot{\omega}_n^2(b) \to \infty$. Then by (6.6), $\dot{\omega}_n^2(u) \to \infty$ for all $u \in [a, b]$. By Fatou's lemma this yields:

$$\begin{aligned}
\infty &= \int_0^1 \liminf_{n\to\infty} \dot{\omega}_n^2(u) \mathrm{d}u \\
&\leq \liminf_{n\to\infty} \int_0^1 \dot{\omega}_n^2(u) \mathrm{d}u \\
&= \lim_{n\to\infty} \|\dot{\omega}_n\|_2^2 \\
&< \infty,
\end{aligned}$$

by Lemma 6.4; contradiction. Thus (6.7) must hold. □

We need one more property of the minimizer x from Lemma 6.4. By Lemma 6.6, x is strictly concave on $(0, 1)$, hence one of the following must hold:

(C) for some unique $u_0 \in (0, 1]$, $\dot{x}(u_0) = 0$ and $x(u_0) = \sup x$

OR

(D) x is strictly increasing on $[0, 1]$ and $\dot{x}(1) > 0$.

Lemma 6.7. *The case* (D) *is impossible and thus* (C) *holds for* x.

Proof. Suppose (D) holds. Let $v < 1$ be very close to 1 and define

$$g(u) = \begin{cases} x(u), & 0 \leq u \leq v \\ x(v) + \dot{x}(1)(u - v), & v < u \leq 1. \end{cases}$$

Then $g \in K_0$ and since $\dot{x} > 0$ is decreasing on $(0, 1)$ (by Lemma 6.6(2)), $|\dot{g}| \leq |\dot{x}|$. Hence

$$\begin{aligned} F(g) - F(x) &\leq c_1 \left[\int_v^1 [x_v + \dot{x}_1(u - v)]^{-2p} \mathrm{d}u - \int_v^1 x_u^{-2p} \, \mathrm{d}u \right] \\ &\quad + c_2[(x_v + \dot{x}_1(1 - v))^{1-p} - x_1^{1-p}] \\ &= A + B, \quad \text{say.} \end{aligned}$$

It is easy to show

$$\lim_{v \to 1^-} \frac{A}{(1 - v)^2} = 0$$
$$\lim_{v \to 1^-} \frac{B}{(1 - v)^2} = \frac{1 - p}{2} x_1^{1-p} \ddot{x}_1.$$

Since x is strictly increasing, $x_1 \neq 0$, and by Lemma 6.6, $\ddot{x}_1 < 0$. Thus by making $v < 1$ sufficiently close to 1, $F(g) - F(x) < 0$, contrary to x being a minimizer. □

Now we are in a position to show minimizing F on K_0 is the same as minimizing over a certain smaller set $\mathcal{S}$, which we now define.

Consider any $b > 0$ such that:

$$\frac{1}{2} \leq \frac{1}{\sqrt{2c_1}} b^p \int_0^b z^p (b^{2p} - z^{2p})^{-1/2} \, \mathrm{d}z \leq 1. \tag{6.8}$$

Then:

$$\begin{aligned} &\frac{1}{\sqrt{2c_1}} b^p \int_0^b z^p (b^{2p} - z^{2p})^{-1/2} \mathrm{d}z + \frac{1}{\sqrt{2c_1}} b^p \int_0^b z^p (b^{2p} - z^{2p}) \mathrm{d}z \geq 1 \\ &\quad \geq \frac{1}{\sqrt{2c_1}} b^p \int_0^b z^p (b^{2p} - z^{2p}) \mathrm{d}z, \end{aligned}$$

and this implies there exists a unique $a = a(b) \in [0, b]$ such that

$$\frac{1}{\sqrt{2c_1}} b^p \int_0^b z^p (b^{2p} - z^{2p})^{-1/2} \mathrm{d}z + \frac{1}{\sqrt{2c_1}} b^p \int_a^b z^p (b^{2p} - z^{2p}) \mathrm{d}z = 1. \tag{6.9}$$

Define:

$$\begin{aligned} f(u) &= \frac{1}{\sqrt{2c_1}} b^p \int_0^u z^p (b^{2p} - z^{2p})^{-1/2} \mathrm{d}z, \quad 0 \leq u \leq b \\ g(u) &= f(b) + \frac{1}{\sqrt{2c_1}} b^p \int_u^b z^p (b^{2p} - z^{2p})^{-1/2} \mathrm{d}z, \quad a \leq u \leq b. \end{aligned} \tag{6.10}$$

Then f is strictly increasing, g is strictly decreasing and

$$0 < f(b) \leq g(a) = 1,$$

by (6.9). Denote their respective inverses by:

$$f^{-1}: \ [0, f(b)] \to [0, b]$$
$$g^{-1}: \ [f(b), 1] \to [a, b],$$

and then define:

$$\omega_t = \begin{cases} f^{-1}(t), & 0 \le t \le f(b), \\ g^{-1}(t), & f(b) < t \le 1. \end{cases}$$

Let $\mathcal{S}$ be the set of all such ω. It is easy to check that ω is continuous, $\omega(0) = 0$, $\omega(1) = a$, $\omega > 0$ on $(0, 1)$ and $\int_0^1 \dot{\omega}_u^2 \, \mathrm{d}u < \infty$. In particular:

$$\mathcal{S} \subseteq K_0. \tag{6.11}$$

Lemma 6.8. *We have*

$$\inf_{\omega \in K_0} F(\omega) = \inf_{\omega \in \mathcal{S}} F(\omega).$$

Proof. With x from Lemma 6.4, it suffices to show $x \in \mathcal{S}$. Let u_0 be from case (C) just before Lemma 6.7 and set $a = x(1)$, $b = x(u_0)$. Then by Lemma 6.6(3),

$$\dot{x}_u^2 = 2c_1 \frac{b^{2p} - x_u^{2p}}{b^{2p} x_u^{2p}}, \quad 0 < u < 1.$$

Since $\dot{x}_u > 0$ on $(0, u_0)$ and $\dot{x}_u < 0$ on $(u_0, 1)$,

$$\dot{x}_u = \begin{cases} \sqrt{2c_1} \sqrt{\dfrac{b^{2p} - x_u^{2p}}{b^{2p} x_u^{2p}}}, & u \in (0, u_0), \\ -\sqrt{2c_1} \sqrt{\dfrac{b^{2p} - x_u^{2p}}{b^{2p} x_u^{2p}}}, & u \in (u_0, 1). \end{cases}$$

Separation of variables leads to the solution:

$$x_t = \begin{cases} f^{-1}(t), & 0 \le t \le f(b), \\ g^{-1}(t), & f(b) < t \le 1 \end{cases}$$

where f and g are from (6.10) with our choice of $a = x(1)$ and $b = x(u_0)$. Thus $x \in \mathcal{S}$, as desired. □

For elements $\omega \in \mathcal{S}$, $F(\omega)$ takes on a very explicit form. Let

$$B_y(p, q) = \int_0^y t^{p-1}(1 - t)^{q-1} \mathrm{d}t, \quad 0 \le y \le 1$$

be the incomplete Beta function. Notice

$$B_1(p, q) = B(p, q),$$

the Beta function.

Lemma 6.9. *For $\omega \in \mathcal{S}$ with $y = 1 - \left(\frac{a}{b}\right)^{2p}$,*

$$F(\omega) = \frac{\beta}{4p} b^{1-p} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1-p}{2p}\right) + \frac{4p}{1-p}(1 - y)^{\frac{1-p}{2p}} + B\left(\frac{3}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{3}{2}, \frac{1-p}{2p}\right) \right].$$

Proof. Recall for $\omega \geq 0$ that

$$F(\omega) = c_1 \int_0^1 \omega_u^{-2p}\, \mathrm{d}u + c_2 \omega_1^{1-p} + \frac{1}{2} \int_0^1 \dot{\omega}_u^2\, \mathrm{d}u \tag{6.12}$$

where

$$c_1 = \frac{\beta^2}{2}, \qquad c_2 = \frac{\beta}{1-p}. \tag{6.13}$$

Let $\omega \in \mathcal{S}$ and take $y = 1 - \left(\frac{a}{b}\right)^{2p}$. Now for $u_0 = f(b)$, using the change of variables $u = f(v)$ for $u \in (0, u_0)$ and $u = g(v)$ for $u \in (u_0, 1)$, we have

$$\begin{aligned}\int_0^1 \omega_u^{-2p}\, \mathrm{d}u &= \int_0^{u_0} \omega_u^{-2p}\, \mathrm{d}u + \int_{u_0}^1 \omega_u^{-2p}\, \mathrm{d}u \\ &= \frac{b^p}{\sqrt{2c_1}} \left[\int_0^b v^{-p} [b^{2p} - v^{2p}]^{-1/2} \mathrm{d}v + \int_a^b v^{-p} [b^{2p} - v^{2p}]^{-1/2} \mathrm{d}v \right].\end{aligned}$$

Changing variables $w = \left(\frac{v}{b}\right)^{2p}$ then gives

$$\int_0^1 \omega_u^{-2p}\, \mathrm{d}u = \frac{b^{1-p}}{2p\sqrt{2c_1}} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1-p}{2p}\right) \right].$$

A similar argument shows

$$\int_0^1 \dot{\omega}_u^2\, \mathrm{d}u = \sqrt{2c_1} \frac{b^{1-p}}{2p} \left[B\left(\frac{3}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{3}{2}, \frac{1-p}{2p}\right) \right].$$

To finish, substitute these into (6.12), use (6.13) and the fact that

$$\begin{aligned}c_2 \omega_1^{1-p} &= \frac{\beta}{1-p} a^{1-p} \\ &= \frac{\beta}{1-p} b^{1-p} \left(\frac{a}{b}\right)^{1-p} \\ &= \frac{\beta}{1-p} b^{1-p} (1-y)^{\frac{1-p}{2p}}. \quad \square\end{aligned}$$

Because of Lemma 6.9, computing $\inf_{\omega \in \mathcal{S}} F(\omega)$ reduces to ordinary calculus.

Theorem 6.10. *We have*

$$\inf_{\omega \in \mathcal{S}} F(\omega) = \frac{1}{2} p^{-\frac{2p}{1+p}} \beta^{\frac{2}{1+p}} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B\left(\frac{3}{2}, \frac{1-p}{2p}\right) \right] \Big/ B\left(\frac{1}{2}, \frac{1+p}{2p}\right)^{\frac{1-p}{1+p}}.$$

Proof. Let $\omega \in \mathcal{S}$ and by Lemma 6.9 write

$$F(\omega) = g(b)$$

where

$$\begin{aligned}g(b) = \frac{\beta}{4p} b^{1-p} \Bigg[& B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1-p}{2p}\right) + \frac{4p}{1-p} (1-y)^{\frac{1-p}{2p}} \\ & + B\left(\frac{3}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{3}{2}, \frac{1-p}{2p}\right) \Bigg]\end{aligned}$$

with $y = 1 - \left(\frac{a}{b}\right)^{2p}$. Let $I = [b_1, b_2]$ be the set of all b satisfying (6.8). Hence by Lemma 6.9,

$$\inf_{\omega \in \mathcal{S}} F(\omega) = \inf_{b \in I} g(b).$$

Below we will show g is increasing on I. Taking this for granted, we get

$$\inf_{\omega \in \mathcal{S}} F(\omega) = g(b_1).$$

Making the change of variables $w = \left(\frac{z}{b}\right)^{2p}$ in (6.8) converts it to

$$\frac{1}{2} \le \frac{b^{1+p}}{2p\sqrt{2c_1}} B\left(\frac{1}{2}, \frac{1+p}{2p}\right) \le 1$$

and so we see b_1 is given by

$$\frac{1}{2} = \frac{b_1^{1+p}}{2p\sqrt{2c_1}} B\left(\frac{1}{2}, \frac{1+p}{2p}\right). \tag{6.14}$$

Similarly, (6.9) gets converted to

$$\frac{b^{1+p}}{2p\sqrt{2c_1}} \left[B\left(\frac{1}{2}, \frac{1+p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1+p}{2p}\right)\right] = 1, \qquad y = 1 - \left(\frac{a}{b}\right)^{2p}. \tag{6.15}$$

Plug $b = b_1$ into (6.15) and use (6.14) to see

$$B_y\left(\frac{1}{2}, \frac{1+p}{2p}\right) = B\left(\frac{1}{2}, \frac{1+p}{2p}\right),$$

which forces $y = 1$. Thus

$$g(b_1) = \frac{\beta}{4p} b_1^{1-p} \left[2B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + 2B\left(\frac{3}{2}, \frac{1-p}{2p}\right)\right]$$

and by (6.14) we get

$$g(b_1) = \frac{1}{2} p^{-\frac{2p}{1+p}} \beta^{\frac{2}{1+p}} \left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B\left(\frac{3}{2}, \frac{1-p}{2p}\right)\right] \Big/ B\left(\frac{1}{2}, \frac{1+p}{2p}\right)^{\frac{1-p}{1+p}},$$

as desired.

It remains to show g is increasing on I. By (6.15),

$$\frac{\mathrm{d}y}{\mathrm{d}b} = -2p\sqrt{2c_1}\,(1+p) b^{-2-p} y^{1/2} (1-y)^{-\frac{1-p}{2p}}.$$

Then using (6.15)

$$g'(b) = \frac{\beta}{2}\sqrt{2c_1} \left[B\left(\frac{1}{2}, \frac{1+p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1+p}{2p}\right)\right]^{-1} b^{-1-2p} g_1(y) \tag{6.16}$$

where

$$\begin{aligned} g_1(y) = (1-p)\Bigg[&B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1-p}{2p}\right) + \frac{4p}{1-p}(1-y)^{\frac{1-p}{2p}} \\ &+ B\left(\frac{3}{2}, \frac{1-p}{2p}\right) + B_y\left(\frac{3}{2}, \frac{1-p}{2p}\right)\Bigg] \\ &- (1+p)(1-y^{1/2})/(1+y^{1/2}) \left[B\left(\frac{1}{2}, \frac{1+p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1+p}{2p}\right)\right]. \end{aligned}$$

Since $y \in [0, 1]$, to show g is increasing on I, it suffices to show $g_1 \geq 0$ on $[0, 1]$. Now

$$g_1'(y) = y^{-1/2}(1 + y^{1/2})^{-2} g_2(y)$$

where

$$g_2(y) = (1-y)^{\frac{1-p}{2p}+1}(1-p) + (1+p) \times \left[-(1-y)^{\frac{1+p}{2p}} + B\left(\frac{1}{2}, \frac{1+p}{2p}\right) + B_y\left(\frac{1}{2}, \frac{1+p}{2p}\right) \right].$$

Furthermore,

$$g_2'(y) = (1+p)(1+y^{-1/2})(1-y)^{\frac{1-p}{2p}} > 0,$$

and so g_2 is increasing on $[0, 1]$. Thus its minimum value is

$$g_2(0) = -2p + (1+p) B\left(\frac{1}{2}, \frac{1+p}{2p}\right). \tag{6.17}$$

If this is nonnegative, then we will have that $g_1' \geq 0$, and g_1 is increasing on $[0, 1]$. This yields

$$\begin{aligned} g_1(y) \geq g_1(0) &= (1-p)\left[B\left(\frac{1}{2}, \frac{1-p}{2p}\right) + \frac{4p}{1-p} + B\left(\frac{3}{2}, \frac{1-p}{2p}\right) \right] \\ &\quad - (1+p) B\left(\frac{1}{2}, \frac{1+p}{2p}\right) \\ &= 4p > 0, \end{aligned}$$

using the identities

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$
$$x\Gamma(x) = \Gamma(x+1).$$

Hence by (6.16), $g'(b) \geq 0$, and g is increasing.

To see that $g_2(0)$ in (6.17) is nonnegative, write

$$g_2(0) = 2p\left[-1 + \frac{1+p}{2p} B\left(\frac{1}{2}, \frac{1+p}{2p}\right) \right]$$

and use the identities above together with the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. □

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