Positive Solutions of Semilinear Differential Equations with Singularities

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1. INTRODUCTION

The existence of positive solutions of a second order differential equation of the form

\[ z'' + g(t)f(z) = 0 \]  (1.1)

with suitable boundary conditions has proved to be important in theory and applications whether \( g \) is continuous in \([0, 1]\) or \( g \) has singularities. These equations often arise in the study of positive radial solutions of a nonlinear elliptic equation of the form

\[ Au + h(|x|) f(u) = 0, \]  (1.2)

for example, see \([3, 7, 8, 18, \text{ and } 23]\). Moreover, Eq. (1.1) contains many important equations which arise from other fields. For example, the generalized Emden–Fowler equation, where \( f = z^p, \ p > 0 \) and \( g \) is continuous (see \([21]\) and \([24]\)), arises in the fields of gas dynamics, nuclear physics, and chemically reacting systems \([24]\); and the Thomas–Fermi equation, where \( f = z^{3/2} \) and \( g = r^{-3/2}, \) so \( g \) has a singularity at 0 (see \([9, 10 \text{ and } 21]\)), was developed in studies of atomic structures (see, for example, \([21]\)) and atomic calculations \([5]\).

When \( g \) is continuous, the existence of positive solutions of Eq. (1.1) with suitable boundary conditions has been studied in \([23]\) by using norm-type cone expansion and compression theorems. The key conditions on \( f \) are either \( f \) is superlinear, that is, \( \lim_{x \to 0} f(x)/x = 0 \) and \( \lim_{x \to \infty} f(x)/x = \infty \) or \( f \) is sublinear, that is, \( \lim_{x \to 0} f(x)/x = \infty \) and \( \lim_{x \to \infty} f(x)/x = 0. \) However, it is known that Eq. (1.1) with \( g \equiv 1 \) has positive solutions for
some functions which may not be superlinear. Such a result was obtained by D. Guo [11] (see Example 2.3.1, p. 96, in [13]) again using norm-type cone expansion and compression theorems, when $f$ satisfies $0 \leq \limsup_{x \to 0} f(x)/x < 8$ and $\frac{24}{\sqrt{3}} \leq \limsup_{x \to 0} f(x)/x \leq \infty$.

When $g$ has a singularity at 0 and $f$ is decreasing and satisfies other stronger conditions, Eq. (1.1) was studied by Z. Guo (see Theorem 4 in [14]) using Leray-Schauder degree. The Thomas-Fermi equation where $g = t^{-1/2}$, which has a singularity at 0, was studied by Granas, Guenther and Lee [9] and [10], using the Topological Transversality Theorem of A. Granas, and by Luning and Perry [20], using iterative techniques.

In this paper we improve on previous results in several ways. We consider Eq. (1.1) when $g \in L^1(0, 1)$ (in particular, $g$ is allowed to have a finite number of singularities) and $f$ satisfies either $0 \leq \limsup_{x \to 0} f(x)/x < a$ and $b < \liminf_{x \to \infty} f(x)/x \leq \infty$ or $0 \leq \limsup_{x \to \infty} f(x)/x < a$ and $b < \liminf_{x \to 0} f(x)/x \leq \infty$ for suitable $a$ and $b$. We shall give varieties of estimates for $a$ and $b$ and our results are often new even when $g$ is continuous.

The main idea is to change Eq. (1.1) into a Hammerstein integral equation of the form

$$z(t) = \int_{0}^{1} k(t, s) g(s) f(z(s)) \, ds = Az(t). \tag{1.3}$$

Although $g \in L^1(0, 1)$, we can still show that $A$ is compact. This allows us to apply the theory of fixed point index for compact maps. We shall employ a well-known nonzero fixed point theorem (see Lemma 2.1) and prove that $A$ has a positive solution under our weak conditions. One of the key steps is to find a function $\phi$ such that $A$ satisfies the condition $x \neq Ax + A\phi$ in the cited fixed point theorem. It seems to be difficult to utilize the norm-type expansion and compression theorems to prove our Theorem 2.2.

As a first application of our new results we consider the existence of eigenvalues for the equation $z'' + \lambda g(t) f(z) = 0$. The eigenvalue problem was recently studied in [16] when $g$ is continuous on $[0, 1]$. Our new results are well suited to treat such eigenvalue problems. Moreover, we give examples to show that our results are improvements of the earlier ones.

As a second application we consider the existence of positive radial solutions for Eq. (1.2) with the usual boundary conditions. We allow $h$ to have suitable singularities and $f$ need not be superlinear or sublinear. Previous results only treat the cases when $f$ is superlinear or sublinear, for example, see [3, 8, 18 and 23]. Equation (1.2) with $h(t) \equiv 1$ in general bounded domains has been intensively studied in recent years (see [2, 4, 19, and 22]).
2. EXISTENCE OF POSITIVE SOLUTIONS OF INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

In this section we shall consider the existence of positive solutions of the Hammerstein equation of the form

\[ z(t) = \int_0^1 k(t, s) g(s) f(z(s)) \, ds \equiv Az(t). \] (2.1)

We always assume the following conditions.

1. \( k : [0, 1] \times [0, 1] \to [0, \infty) \) is continuous.
2. \( f : [0, \infty) \to [0, \infty) \) is continuous.
3. \( g \in L^1(0, 1) \) and \( g(s) > 0 \) a.e.

It is clear that the following condition is a special case of the condition (3):

3'. For given points \( t_1, \ldots, t_m \), \( g : E = [0, 1] \setminus \{ t_i : i = 1, \ldots, m \} \to [0, \infty) \) is continuous and the integral \( \int_0^1 g(s) \, ds \) exists.

We emphasize that the condition (3') allows \( g \) to have finitely many singularities at \( t_1, \ldots, t_m \).

Let \( P = \{ x \in C[0, 1] : x(t) \geq 0 \text{ for } t \in [0, 1] \} \). Then \( P \) is a cone in \( C[0, 1] \).

It is known that if \( g \) is defined on \([0, 1]\) and is continuous in \([0, 1]\), the map \( A : P \to P \) is compact, that is, \( A \) is continuous and \( A(Q) \) is compact for each bounded subset \( Q \subset P \). A more general result can be found, for example, in Chapter 1, p.46, [17]. We shall generalize the simple result to the case when \( g \) satisfies the weak condition (3).

**Theorem 2.1.** Under the hypotheses (1)-(3) above, the map \( A \) defined in (2.1) maps \( P \) into \( P \) and is compact.

**Proof.** It is easy to verify that under the hypotheses (1)-(3) \( A \) is well defined and for every \( z \in P, Az(t) \) is non-negative and continuous on \([0, 1]\). Hence \( A \) maps \( P \) into \( P \). Now, we prove that \( A \) is compact. This is trivial if \( \int_0^1 g(s) \, ds = 0 \), so suppose that \( \int_0^1 g(s) \, ds > 0 \). We first prove that \( A \) is continuous. Let \( M = \max \{ k(t, s) : t, s \in [0, 1] \} \). Assume that \( z_n, z_0 \in P \) and \( z_n \to z_0 \). Then \( \|z_n\| \leq M' < \infty \) for every \( n \geq 0 \). Since \( f \) is continuous on \([0, M']\), it is uniformly continuous. Therefore, for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |f(z') - f(z'')| < \delta \) implies that \( |f(z') - f(z'')| < \varepsilon(M \int_0^1 g(s) \, ds)^{-1} \). Since \( z_n \to z_0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \|z_n - z_0\| < \delta \) for \( n \geq n_0 \). Thus we have

\[ |f(z_n(t)) - f(z_0(t))| < \varepsilon \left( M \int_0^1 g(s) \, ds \right)^{-1} \quad \text{for } n \geq n_0 \text{ and } t \in [0, 1]. \]
This implies that
\[
|A_z(t) - A_0(t)| \leq \int_0^1 k(t, s) g(s) |f(z_0(s)) - f(z_0(s))| \, ds < \varepsilon
\]
for \( t \in [0, 1] \) and \( n \geq n_0 \).

and therefore \( \|A_z - A_0\| \leq \varepsilon \) for \( n \geq n_0 \).

Next, let \( B \in P \) be bounded, i.e., \( \|x\| \leq m \) for all \( x \in B \) and some \( m > 0 \), and let \( b = \max \{ f(x) \colon 0 \leq x \leq m \} \int_0^1 g(s) \, ds \). Then \( A(B) \) is uniformly bounded, because we have \( \|Ax\| \leq bM \) for \( x \in B \). To prove \( A(B) \) is compact it is sufficient to prove that \( A(B) \) is equicontinuous. In fact, since \( k \) is uniformly continuous, for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |k(t_1, s) - k(t_2, s)| < \delta b^{-1} \) for \( |t_1 - t_2| < \delta \) and \( s \in [0, 1] \). This implies that \( |A_z(t_1) - A_z(t_2)| \leq \int_0^1 |k(t_1, s) - k(t_2, s)| \times g(s) f(z(s)) \, ds < \varepsilon \).

We shall need the following well-known result (see, for example, Theorem 12.3 in \[1\]).

Let \( K \) be a cone in a Banach space \( X \) and let \( K_r = \{ x \in K \colon \|x\| < r \} \), \( K_r^* = \{ x \in K \colon \rho \leq \|x\| \leq r \} \), where \( 0 < \rho < r < \infty \).

**Lemma 2.1.** Let \( K \) be a cone in a Banach space \( X \) and \( A \colon K_r \rightarrow K \) a compact map. Assume that the following conditions hold.

(i) \( \|Ax\| \leq \|x\| \) for \( x \in \partial K_r \).

(ii) There exists \( e \in \partial K_1 \) such that \( x \neq Ax + \lambda e \) for \( x \in \partial K_r \) and \( \lambda > 0 \). Then \( A \) has a fixed point in \( K_r \). The same conclusion remains valid if (i) holds on \( \partial K_r \) and (ii) holds on \( \partial K_r^* \).

**Notation.** Let \( \tilde{f}^* = \lim sup_{x \to *} f(x)/x \) and \( \tilde{f}_* = \lim inf_{x \to *} f(x)/x \), where \( * \) denotes either 0 or \( \infty \).

Now we give the main result in this section.

**Theorem 2.2.** Assume that the following conditions hold.

(G) There exist \( a, b \in [0, 1] \) with \( a < b \) such that \( \int_a^b g(s) \, ds > 0 \).

(C) There exist a continuous function \( \Phi \colon [0, 1] \rightarrow \mathbb{R}^+ \) and a number \( \gamma \in (0, 1] \) such that
\[
k(t, s) \leq \Phi(s) \quad \text{for} \quad t, s \in [0, 1] \quad \text{and} \quad \gamma \Phi(s) \leq k(t, s) \quad \text{for} \quad t \in [a, b] \quad \text{and} \quad s \in [0, 1].
\]
One of the following conditions holds.

(1) $0 < f_0 \neq M_1$, and $m_1 < f_\infty$, $\leq \infty$.

(2) $0 < f_0$ and $m_1 < f_\infty$, $\leq \infty$, where

$$M_1 = (\max_{0 < \varepsilon < 1} \int_{0}^{1} k(t, s) g(s) ds)^{-1}$$

and

$$m_1 = (\min_{0 < \varepsilon < 1} \int_{0}^{1} k(t, s) g(s) ds)^{-1}.$$

Then Eq. (2.1) has a solution $z \in P$ with $z(t) \neq 0$ for $t \in [0, 1]$.

Proof. From Theorem 2.1, $A : P \to P$ is compact. Let

$$P^{(1)} = \{ x \in P : \min \{ x(t) ; a \leq t \leq b \} \geq \gamma \| x \| \}.$$

Then $P^{(1)}$ is a cone in $C[0, 1]$. We prove that $AP \subset P^{(1)}$. Indeed, by (C) we have for each $z \in P$, $\| Az \| \leq \int_{0}^{1} \Phi(s) g(s) f(z(s)) ds$ and

$$\min \{ Az(t) ; a \leq t \leq b \} \geq \gamma \int_{0}^{1} \Phi(s) g(s) f(z(s)) ds.$$

Hence, $\min \{ Az(t) ; a \leq t \leq b \} \geq \gamma \| Az \|$ and $Az \in P^{(1)}$ for all $z \in P$.

(1) Assume that (h1) holds. By the first part of (h1), there exists $\rho > 0$ such that $f(x) \leq M_1 \rho$ for $0 \leq x \leq \rho$. For every $z \in \partial P^{(1)}$, we have

$$Az(t) = \int_{0}^{1} k(t, s) g(s) f(z(s)) ds \leq M_1 \rho \int_{0}^{1} k(t, s) g(s) ds \leq \rho = \| z \|.$$

This implies $\| Az \| \leq \| z \|$ for every $z \in \partial P^{(1)}$.

By the second part of (h1), there exists $\eta > \gamma \rho$ such that $f(x) \geq m_1 x$ for $x \geq \eta$. Let $r = \gamma^{-1} \eta$. Then we have

$$\min \{ z(t) ; a \leq t \leq b \} \geq \gamma \| z \| = \eta \quad \text{for} \quad z \in \partial P^{(1)}.$$

Let $\phi(t) \equiv 1$ for $t \in [0, 1]$. Then $\phi \in \partial P^{(1)}$. We prove that

$$z \neq Az + \lambda \phi \quad \text{for} \quad z \in \partial P^{(1)} \quad \text{and} \quad \lambda > 0.$$

In fact, if not, there exist $z_0 \in \partial P^{(1)}$ and $\lambda_0 > 0$ such that $z_0 = Az_0 + \lambda_0 \phi$. Let

$$\mu = \min \{ z_0(t) ; a \leq t \leq b \} \geq \eta. \quad \text{Then we have, for} \quad a \leq t \leq b,$$

$$z_0(t) = \int_{0}^{1} k(t, s) g(s) f(z_0(s)) ds + \lambda_0 \phi(t) \geq m_1 \int_{a}^{b} k(t, s) g(s) z_0(s) ds + \lambda_0$$

$$\geq m_1 \mu \int_{a}^{b} k(t, s) g(s) ds + \lambda_0 \geq \mu + \lambda_0.$$

This implies $\mu \geq \mu + \lambda_0 > \mu$, a contradiction. It follows from Lemma 2.1 that $A$ has a fixed point $z$ in $P^{(1)}_{\rho, \gamma}$. 

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(II) Assume that \((h_2)\) holds. Choose some \(\beta \in (f^\infty, M_1)\). By the first part of \((h_2)\), there exists \(r_1 > 0\) such that \(f(x) \leq \beta x\) for \(x \geq r_1\). Since \(f\) is continuous, we have \(c = \max \{ f(x) : 0 \leq x \leq r_1\} < \infty\). Hence \(0 \leq f(x) \leq c + \beta x\) for \(0 \leq x < \infty\). Let \(r = c(M_1 - \beta)^{-1}\). Then we have for \(z \in \partial P_r\),

\[
|Az(t)| \leq (c + \beta|z|) \int_0^1 k(t, s) g(s) ds \leq (c + \beta|z|)/M_1 = r = |z|.
\]

This implies \(|Az| \leq |z|\) for every \(z \in \partial P_r\). On the other hand, by the last part of \((h_2)\), there exists \(\rho \in (0, r)\) such that \(f(x) \geq m_1 x\) for \(0 \leq x \leq \rho\). By a similar argument to that used in (I) we have \(z \neq Az + \lambda \phi\) for \(z \in \partial P_\rho\) and \(\lambda > 0\). The result follows from Lemma 2.1.

**Remark 2.1.** In the proof of Theorem 2.2 one of the key steps is to find the function \(\phi\). We note that it seems to be difficult to prove the theorem by using norm-type cone expansion and compression theorem.

### 3. Existence of Positive Solutions of \(z'' + g(t)f(z) = 0\)

In this section we consider the existence of positive solutions for the equation of the form

\[
z'' + g(t)f(z) = 0, \quad \text{a.e. on } [0, 1], \quad (3.1)
\]

subject to one of the following sets of boundary conditions:

\[
z(0) = z(1) = 0, \quad (3.2)_1
\]

\[
z(0) = z'(1) = 0, \quad (3.2)_2
\]

\[
z'(0) = z(1) = 0. \quad (3.2)_3
\]

By a solution to (3.1)-(3.2), we mean a function \(z \in C^1[0, 1], z' \in AC[0, 1]\) and \(z\) satisfies (3.1)-(3.2), \(i = 1, 2, 3\), where \(AC[0, 1]\) denotes the space of absolutely continuous functions defined on \([0, 1]\).

**Remark 3.1.** It is easy to verify that \(z(t)\) is a solution of Eq. (3.1) with (3.2)_2 if and only if \(z(1 - t)\) is a solution of (3.1) with (3.2)_3. Hence we only consider Eq. (3.1) with (3.2)_1 and with (3.2)_2 in the following leaving the statement of results for Eq. (3.1) with (3.2)_3 to the reader.

For each \(i = 1, 2\), we define \(k_i : [0, 1] \times [0, 1] \to [0, \infty)\) as follows.

\[
k_1(t, s) = \begin{cases} t(1-s) & \text{if } 0 \leq t \leq s \leq 1, \\ s(1-t) & \text{if } 0 \leq s < t \leq 1, \end{cases}
\]

\[
k_2(t, s) = \begin{cases} t & \text{if } 0 \leq t \leq s \leq 1, \\ s & \text{if } 0 \leq s < t \leq 1. \end{cases}
\]
The following result shows that existence of solutions of Eq. (3.1) can be derived from existence of solutions of Eq. (2.1). Its proof is straightforward and we omit it.

LEMMA 3.1. Assume that \( g \in L^1(0, 1) \). Let \( k = k_i, i = 1, 2 \) in Eq. (2.1), respectively. Then for \( z \in P \),

(i) \( Az \in C^1[0, 1] \)
(ii) \( (Az)' \in AC[0, 1] \),
(iii) \( (Az)'(t) = -g(t) f(z(t)) \) a.e. on \([0, 1]\) and
(iv) if \( z \in P \) is a solution of Eq. (2.1) with \( k = k_i, i = 1, 2 \), respectively, then \( z \) is also a solution of (3.1)-(3.2), \( i = 1, 2 \), respectively.

Remark 3.2. In Lemma 3.1 if \( g \) satisfies the condition (3'), then we have, for \( z \in P \),

(R1) \( (Az)'(t) = -g(t) f(z(t)) \) for \( t \in E \);
(R2) \( Az(t) \in C^2(E) \cap C^1[0, 1] \);
(R3) if \( z \) is a solution of Eq. (2.1) with \( k = k_i, i = 1, 2 \), respectively, then \( z \) is also a solution of (3.1)-(3.2), \( i = 1, 2 \), respectively, and \( z \in C^2(E) \cap C^1[0, 1] \).

We first give a general result on the existence of positive solutions of the equation (3.1)-(3.2), by using Theorem 2.2.

THEOREM 3.1. Assume that there exist \( a, b \in (0, 1) \) with \( a < b \) such that \( \int_a^b g(s) \, ds > 0 \) and \( f \) satisfies \((h_1)\) or \((h_2)\) of Theorem 2.2 with \( k = k_1 \). Then Eq. (3.1)-(3.2) has a solution \( z \in P \) with \( z(t) \neq 0 \) for \( t \in [0, 1] \).

Proof. Let \( \Phi(s) = s(1 - s) \) for \( s \in [0, 1] \) and \( \gamma = \min\{a, 1 - b\} \). Then (C) of Theorem 2.2 with \( k = k_1 \) holds. It follows from Theorem 2.2 and (iii) of Lemma 3.1 that the Equation (3.1)-(3.2) has a positive solution.

As special cases of Theorem 3.1 we give several Corollaries. The conditions used do not depend explicitly on \( k_1 \) and so are easily verified in applications.

COROLLARY 3.1. Assume that there exist \( a, b \in (0, 1) \) with \( a < b \) such that \( \int_a^b g(s) \, ds > 0 \) and \( f \) satisfies one of the following conditions.

(i) \( 0 \leq f^a < 4(\int_a^b g(s) \, ds)^{-1} \) and \( (a(1 - b)) \int_a^b g(s) \, ds)^{-1} < f_\infty \leq \infty \).
(ii) \( 0 \leq f^\infty < 4(\int_a^b g(s) \, ds)^{-1} \) and \( (a(1 - b)) \int_a^b g(s) \, ds)^{-1} < f_0 \leq \infty \).

Then the result of Theorem 3.1 holds.
Proof. Since \( k_1(t, s) \leq s(1 - s) \leq 1/4 \) for \( t, s \in [0, 1] \) and \( k_1(t, s) \geq a(1 - b) \) for \( t, s \in [a, b] \), (i) and (ii) imply \((h_1)\) and \((h_2)\) in Theorem 3.1, respectively.

When \( g \) is continuous in \([0, 1]\), the integral in (i) or (ii) can be replaced by other values. Let \( L_1 = \max\{g(s): 0 \leq s \leq 1\}, \ w = \frac{1}{2} \min\{(a(b - a)(2a - b), (1 - b)(b^2 - a^2))\} \) and \( d_1 = \min\{g(t): a \leq t \leq b\}. \) Then we have

**Corollary 3.2.** Assume that \( g: [0, 1] \to \mathbb{R}^+ \) is continuous and there exist \( a, b \in (0, 1) \) with \( a < b \) such that \( d_1 > 0 \). Assume that \( f \) satisfies one of the following conditions.

(i) \( 0 \leq f^0 < 8(L_1)^{-1} \) and \( (d_1w)^{-1} < f_{\infty} \leq \infty. \)

(ii) \( 0 \leq f^0 < 8(L_4)^{-1} \) and \( (d_1w)^{-1} < f_0 \leq \infty. \)

Then Eq. (3.1) has a solution \( z \in C^2[0, 1] \) such that \( z(t) > 0 \) for \( 0 < t < 1. \)

Proof. Since \( \int_0^1 k_1(t, s) \, ds \leq 1/8 \) for \( t \in [0, 1] \), it is easy to verify that \( 8L_2^{-1} \leq M_3. \) Let \( F(t) = \int_a^b k_1(t, s) \, ds \) for \( t \in [a, b] \). By calculation we have for \( a \leq t \leq b, \)

\[
F(t) = \frac{1}{2} \left[ \frac{(a + 2b - b^2)^2}{4} - a^2 - \left( t - \frac{a^2 + 2b - b^2}{2} \right)^2 \right].
\]

Noting that \( 2a \leq a^2 + 2b - b^2 \leq 2b, \) we obtain that \( F(t) \geq \min\{F(a), F(b)\} = w \) for \( t \in [a, b]. \) Hence \( m_t \leq (d_1w)^{-1}. \) The result follows from Theorem 3.1.

As a special case of Corollary 3.2 we have the following result.

**Corollary 3.3.** Let \( g: [0, 1] \to \mathbb{R}^+ \) be continuous such that \( g \) is not identically zero in \([0, 1]\). Assume that \( f \) satisfies either \( 0 \leq f^0 < 8(L_1)^{-1} \) and \( f_{\infty} = \infty \) or \( 0 \leq f^0 < 8(L_1)^{-1} \) and \( f_0 = \infty. \) Then the conclusion of Corollary 3.2 holds.

**Remark 3.3.** Let \( f(x) = x^p. \) Then if \( p > 1, f^0 = 0 \) and \( f_{\infty} = \infty, \) and if \( 0 < p < 1, f^0 = 0 \) and \( f_0 = \infty. \) Hence, Corollary 3.3 generalizes Corollary 4.2 in [15].

In Corollary 3.2 when \( g(t) \equiv 1 \) for \( t \in [0, 1], \) the best estimates are obtained by making \( w \) as large as possible. It is easily checked by calculus that the maximum of \( w \) on the region \( 0 \leq a < b \leq 1 \) occurs on the line \( a + b = 1 \) and the two formulae coincide on that line. Therefore, \( w \) takes its maximum \( 1/16 \) when \( a = 1/4 \) and \( b = 3/4. \) Hence, when \( g(t) \equiv 1, a = 1/4 \) and \( b = 3/4, \) we obtain the following interesting result.
COROLLARY 3.4. Assume that $f$ satisfies either $0 \leq f_0 < 8$ and $16 < f_\infty \leq 0$ or $0 < f_\infty < 8$ and $16 < f_0 \leq \infty$. Then the equation $z''(t) + f(z(t)) = 0$ with (3.2) has a solution $z \in C^2[0, 1]$ such that $z(t) > 0$ for $0 < t < 1$.

Remark 3.4. A special case of Corollary 3.4 was first obtained by Guo [11], where the number 16 in the first condition of Corollary 3.4 is replaced by the larger number $24 - \sqrt{3}$ (also see Example 2.3.1, p. 96 in [13]). Therefore, Corollary 3.4 not only generalizes Guo’s result but also applies when the second condition in Corollary 3.4 holds.

As an application of Theorem 3.1 we consider the following eigenvalue problem.

$$z'' + g(t) f(z) = 0$$

with the boundary condition (3.2). We list the following conditions:

(P) $\int_{1/4}^{3/4} g(s) \, ds > 0$;

(P') $f_\infty > 0$, $f_0 \neq \infty$ and $m_1/f_\infty < M_1/f_0$;

(P'') $f_\infty \neq \infty$, $f_0 > 0$ and $m_1/f_0 < M_1/f_\infty$.

We write $m_1/f_0 = 0$ if $f_\infty = \infty$ and $M_1/f_\infty = \infty$ if $f_0 = 0$, where $\alpha = 0, \infty$.

We have the following new result on existence for the eigenvalue problem (3.2).

THEOREM 3.2. Assume that (P) and (P') hold. Then for every $\lambda \in (m_1/f_0, M_1/f_\infty)$, Eq. (3.2) with (3.2) has a solution $z$ with $z(t) \geq 0$ for $t \in [0, 1]$ and $z(t) \neq 0$ if $[0, 1]$. The same result remains valid for every $\lambda \in (m_1/f_0, M_1/f_\infty)$, if (P) and (P') hold.

Proof. If (P') holds, $0 < \lambda f_0 < M_1$ and $\lambda f_\infty > m_1$ so Theorem 3.1 applies.

Remark 3.5. The first and second results in Theorem 3.2 generalize Theorems 2 and 3 in [16] respectively in the following ways: (1) $g$ may not be continuous; (2) $g$ may vanish on $[0, 1/4]$ and $[3/4, 1]$; (3) $f_\infty$ may not equal $f_0$, where $\alpha$ denotes either 0 or $\infty$ and (4) even when $f_\infty = f_0$, the interval $(m_1/f_\infty, M_1/f_\infty)$ contains the interval $(4/c_1 f_0, 1/c_2 f_\infty)$ given in Theorem 2 in [16] and the interval $(m_1/f_0, M_1/f_\infty)$ contains the interval $(4/c_1 f_0, 1/c_2 f_\infty)$ given in Theorem 3 in [16], where $c_1 = \max_{0 < c < 1} \int_{1/4}^{3/4} k(t, s) g(s) \, ds$ and $c_2 = \int_{0}^{1} (1-s) g(s) \, ds$.

The above inclusions can be easily proved by considering the following inequalities:

$$k(t, s) \leq s(1-s) \quad \text{for} \quad t, s \in [0, 1]$$

and

$$s(1-s) \leq 4k(t, s) \quad \text{for} \quad t \in [1/4, 3/4], s \in [0, 1].$$
Moreover, the inclusions may be strict. For example, taking \( g(t) \equiv 1 \), \( a = 1/4 \) and \( b = 3/4 \) in Theorem 3.2 we obtain \( m_1 = 16 \), \( M_1 = 8 \). If \( g(t) \equiv 1 \), \( c_1 = 3/32 \) and \( c_2 = 1/6 \). If we take \( f(x) = x^2/(1 + x) \) for \( x \in [0, \infty) \), then \( f^0 = 0 \) and \( f_\infty = 1 \); if we take \( f(x) = x/(1 + \sqrt{x}) \) for \( x \in [0, \infty) \), then \( f_0 = 1 \) and \( f_\infty = 0 \). In both cases, the inclusions are \((128/3, \infty) \subset (16, \infty)\).

Now we consider Eq. (3.1)-(3.2).

**Theorem 3.3.** Assume that there exist \( a \in (0, 1) \) and \( b \in (a, 1] \) such that \( \int_a^b g(s) \, ds > 0 \) and \( f \) satisfies one of the following conditions:

- (h1) \( 0 < f^0 < M_2 \) and \( m_2 < f_\infty \leq \infty \);
- (h4) \( 0 < f^\infty < M_2 \) and \( m_2 < f_0 \leq \infty \), where \( M_2 = (\max_{a \leq s \leq 1} \int_a^1 k_2(t, s) g(s) \, ds)^{-1} \), \( m_2 = (\min_{a \leq s \leq 1} \int_a^1 k_2(t, s) g(s) \, ds)^{-1} \).

Then Eq. (3.1)-(3.2) has a positive solution \( z \) with \( z(t) \neq 0 \) on \([0, 1]\).

**Proof.** Let \( \Phi(s) = s \) for \( s \in [0, 1] \). Then the condition (C) of Theorem 2.2 holds with \( k = k_2 \) and \( \gamma = 1 \). It follows from Theorem 2.2 and (iv) of Lemma 3.1 that Eq. (3.1)-(3.2) has a positive solution \( z \) with \( z(t) \neq 0 \) on \([0, 1]\).

Similar to Corollaries 3.1–3.4, we have the following results.

**Corollary 3.5.** Assume that \( g \) satisfies the condition in Theorem 3.3 and \( f \) satisfies one of the following conditions:

- (i) \( 0 < f^0 < (\int_a^b g(s) \, ds)^{-1} \) and \( (a \int_a^b g(s) \, ds)^{-1} < f_\infty \leq \infty \);
- (ii) \( 0 < f^\infty < (\int_a^b g(s) \, ds)^{-1} \) and \( (a \int_a^b g(s) \, ds)^{-1} < f_0 \leq \infty \).

Then the conclusion of Theorem 3.3 holds.

**Corollary 3.6.** Assume that \( g : [0, 1] \to \mathbb{R}^+ \) is continuous and there exist \( a \in (0, 1) \) and \( b \in (a, 1] \) such that \( d_1 > 0 \). Assume that \( f \) satisfies one of the following conditions:

- (i) \( 0 < f^0 < 2(L_1)^{-1} \) and \( (a(b-a) d_1)^{-1} < f_\infty \leq \infty \);
- (ii) \( 0 < f^\infty < 2(L_1)^{-1} \) and \( (a(b-a) d_1)^{-1} < f_0 \leq \infty \), where \( L_1 = \max \{ g(s) : 0 \leq s \leq 1 \} \) and \( d_1 = \min \{ g(t) : a \leq t \leq b \} \).

Then Eq. (3.1)-(3.2) has a solution \( z \in C^2[0, 1] \) such that \( z(t) > 0 \) for \( 0 < t < 1 \).

**Proof.** Since \( \int_a^1 k_2(t, s) \, ds \leq 1/2 \) for \( t \in [0, 1] \), then \( 2L_1^{-1} \leq M_2 \). Let \( G(t) = \int_a^b k_2(t, s) \, ds \) for \( t \in [0, 1] \). By a small calculation we have \( G(t) = \frac{(b^2-a^2)-(b-a)^2}{2} \geq a(b-a) \). Hence \( m_2 \leq (a(b-a) d_1)^{-1} \). The result follows from Theorem 3.3.
Corollary 3.7. Let \( g: [0, 1] \rightarrow \mathbb{R}^+ \) be continuous such that \( g \) is not identically zero in \([0, 1]\). Assume that \( f \) satisfies either 0 \( \leq \int_0^1 f < 2/L_1 \) and \( f_\infty = \infty \) or 0 \( \leq \int_0^1 f < 2/L_1 \) and \( f^0 = \infty \). Then the conclusion of Corollary 3.6 holds.

In Corollary 3.6, let \( g \equiv 1, a = 1/2, \) and \( b = 1 \). Then we have

Corollary 3.8. Assume that \( f \) satisfies either 0 \( \leq \int_0^1 f < 2 \) and \( 4 < f_\infty \leq \infty \) or 0 \( \leq \int_0^1 f < 2 \) and \( 4 < f_0 \leq \infty \). Then the equation \( z'' + f(z(t)) = 0 \) with \( (3.2)_2 \) has a solution \( z \in C^2([0, 1]) \) such that \( z(t) > 0 \) for \( 0 < t < 1 \).

4. ON EXISTENCE OF POSITIVE RADIAL SOLUTIONS
OF THE EQUATION \( Au + h(|x|) f(u) = 0 \)

In this section we shall apply the results obtained in the above section to the existence of positive radial solutions in an annulus for the equation

\[
Au + h(|x|) f(u) = 0, \quad \text{a.e. on } |x| \in [R_1, R_0], \quad x \in \mathbb{R}^n, \ n \geq 2. \quad (4.1)
\]

with one of the following sets of boundary conditions:

\[
u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad |x| = R_0, \quad (4.2)_1
\]

\[
u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_0, \quad (4.2)_2
\]

\[
\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_0. \quad (4.2)_3
\]

where \( r = |x| \) and \( \frac{\partial u}{\partial r} \) denotes differentiation in the radial direction.

We always assume that the following conditions hold.

1. \( 0 < R_1 < R_0 < \infty \).
2. \( f: [0, \infty) \rightarrow [0, \infty) \) is continuous.
3. \( h \in L^1(R_0, R_1) \) and \( h(r) \geq 0 \) a.e.

We note that the condition (3) allows \( h \) to have finitely many singularities.

By a solution to (4.1)-(4.2), we mean a function \( u \in C^1[R_1, R_0], \) \( u' \in AC[R_1, R_0] \) and \( u \) satisfies (4.1)-(4.2)_i, \( i = 1, 2, 3 \).

Since we consider the existence of a positive radial solution \( u = u(r) \) for (4.1)-(4.2), we can write (4.1) in the form

\[
u''(r) + \frac{n-1}{r} u'(r) + h(r) f(u(r)) = 0 \quad \text{a.e. on} \quad [R_1, R_0], \ n \geq 2. \quad (4.3)
\]
Now we change Eq. (4.3) into Eq. (3.1) by means of the following variables.

Let \( r_2(t) = R_1^{1-t} R_t^{t} \), \( r_d(t) = (a_n + (b_n - a_n) t)^{-1/(n-2)} \) for \( n \geq 3 \) and \( z(t) = u(r_d(t)) \), where \( a_n = R_0^{-(n-2)} \) and \( b_n = R_1^{-(n-2)} \). Let \( \phi_d(t) = (R_0/(1-t) \log(R_0/R_t))^t \) for \( t \in [0, 1] \) and \( \phi_d(t) = ((b_n - a_n)/(n-2))^2 (a_n + (b_n - a_n) t)^{-r_n} \), where \( v_n = 2(n-1)/(n-2) \). Then Eq. (4.3) becomes

\[
\frac{z'(t)}{g_d(t)} f(t) = 0, \quad \text{a.e. on } [0, 1]
\]

where \( g_d(t) = \phi_d(t) h(r_d(t)) \) for \( t \in [0, 1] \). Moreover, the boundary conditions (4.2), become (3.2), for \( i = 1, 2, 3 \). Hence the results in Section 3 can be applied directly to Eq. (4.5) with (3.2), and thus, the existence of positive radial solutions for (4.1)-(4.2), can be derived from Eq. (4.5) with (3.2).

Let \( p_d(r) = (\log(R_0/R_1))^{-1} \log(R_0/r), \quad p_d(r) = (1/r^{n-2} - a_n)/(b_n - a_n) \) for \( n \geq 3 \). Let \( \alpha_2 = (\log(R_0/R_1))^{-1} \) and \( \alpha_n = (n-2)/(b_n - a_n) \) for \( n \geq 3 \).

By using Corollary 3.1 we have the following:

**Theorem 4.1.** Assume that there exist \( r_1, r_2 \in (R_1, R_0) \) such that

1. \( 0 < f_0 < 4(W_n)^{-1} \) and \( (w_n)^{-1} < f_\infty \).
2. \( 0 < f_\infty < 2(W_n)^{-1} \) and \( (w_n)^{-1} < f_\infty \).

Then Eq. (4.1)-(4.2), has a positive radial solution.

**Proof.** We only prove (4.5) with the boundary condition (3.2), has a positive solution. It is sufficient to show that \( f \) and \( g_d \) satisfy all the conditions of Corollary 3.1. Noting that \( \phi_n \) is decreasing in \([0, 1]\) and \( \int_0^1 h(r_d(s)) \, ds = \alpha_n \int_0^{r_d(1)} h((u/u^{n-1}) \, du \quad \text{for } t_1, t_2 \in [0, 1] \text{ with } t_1 < t_2, \) we have

\[
\int_0^{r_d(1)} h((u/u^{n-1}) \, du = W_n.
\]

and

\[
a(1-b) \int_0^1 g_d(s) \, ds \geq a(1-b) \phi_d(b) \alpha_n \int_0^{r_d(1)} h((u/u^{n-1}) \, du = w_n.
\]

The result follows from Corollary 3.1. \( \blacksquare \)

As a special case of Theorem 4.1 we have the following:

**Corollary 4.1.** Let \( h: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\} \) be a function such that for any \( R_1, R_0 \in \mathbb{R}^+ \) with \( R_1 < R_0 \), there exist at most \( r^1, \ldots, r^m \in \{R_1, R_0\} \) such that \( h: [R_1, R_0] \setminus \{r^i: i = 1, \ldots, m\} \rightarrow \mathbb{R}^+ \) is continuous. \( \int_{R_1}^{R_0} h(r) \, dr \) exists and \( h \) is not identically zero in \([R_1, R_0]\). Assume that \( f \) satisfies either \( f^0 = 0 \) and
Proof. For \( R_1, R_0 \in (0, \infty) \) with \( R_1 < R_0 \), there exist \( r_1, r_2 \in [R_1, R_0] \) with \( r_1 < r_2 \) such that \( \int_{r_1}^{r_2} h(r) \, dr > 0 \). It is easy to see that \( W_n > 0 \) and \( w_n > 0 \). The result follows from Theorem 4.1.

Remark 4.1. Corollary 4.1 generalizes Theorem 2.7 in [18], where \( f \in C^1([0, \infty)) \) and \( h \in C^1((0, \infty)) \); and Theorem 1 in [23], where \( h \) is required to be continuous.

When \( h \) is continuous in \([R_1, R_0]\), Corollary 3.2 gives the following result:

**Theorem 4.2.** Assume that \( h: [R_1, R_0] \to \mathbb{R}^+ \) is continuous and there exist \( r_1, r_2 \in (R_1, R_0) \) with \( r_1 < r_2 \) such that \( d_2 := \min\{h(r): r_1 \leq r \leq r_2\} > 0 \). Assume that \( f \) satisfies one of the following conditions.

1. \( 0 \leq f^0 < 8(\beta_n)^{-1} \) and \((\gamma_n)^{-1} < f_\infty \leq \infty\).
2. \( 0 \leq f^\infty < 8(\beta_n)^{-1} \) and \((\gamma_n)^{-1} < f_0 \leq \infty\), where \( \beta_n = \phi_n(0) \max\{h(r): R_1 \leq r \leq R_0\} \), \( \gamma_n = \phi_n(h) \), \( b = p(r_1) \), \( a = p(r_2) \) and \( w = \frac{1}{2} \min\{a(b-a) \}

Then Eq. (4.1)-(4.2) has a positive radial solution.

Let \( h(r) \equiv 1 \) for \( r \in [R_1, R_0] \), and \( a = 1/4 \) and \( b = 3/4 \). Theorem 4.2 gives the following corollary:

**Corollary 4.2.** Assume that \( f \) satisfies either \( 0 \leq f^0 < 8(\phi_n(0))^{-1} \) and \( 16(\phi_n(\frac{1}{2}))^{-1} < f_\infty \leq \infty \) or \( 0 \leq f^\infty < 8(\phi_n(0))^{-1} \) and \( 16(\phi_n(\frac{1}{2}))^{-1} < f_0 \leq \infty \). Then the equation \( \Delta u + f(u) = 0 \) with \( (4.2)_1 \) has a positive radial solution.

Similar to the arguments in Theorem 4.1, by using Corollary 3.5 we have

**Theorem 4.3.** Let \( r_1 \in (R_1, R_0) \) and \( r_2 \in [R_1, R_2] \) be such that \( \int_{r_1}^{r_2} h(r) \, dr > 0 \). Assume that \( f \) satisfies one of the following conditions.

1. \( 0 \leq f^0 < (W_n)^{-1} \) and \((w_n)^{-1} < f_\infty \leq \infty\).
2. \( 0 \leq f^\infty < (W_n)^{-1} \) and \((w_n)^{-1} < f_0 \leq \infty\), where \( W_n \) is the same as in Theorem 4.1, \( w_n = \alpha \phi_n(h) \), \( \alpha_n \int_{r_1}^{r_2} h(r(s)) \, ds \) and \( a = p_1(r_2) \) and \( b = p_1(r_1) \). Then Eq. (4.1)-(4.2) has a positive radial solution.

**Remark 4.2.** Theorem 4.3 gives generalizations of Theorem 2.7 in [18] and Theorem 1 in [23].

When \( h \) is continuous in \([R_1, R_0]\), by using Corollary 3.6 we have
Theorem 4.4. Assume that \( h: [R_1, R_0] \rightarrow \mathbb{R}^+ \) is continuous and there exist \( r_2 \in (R_1, R_0) \) and \( r_1 \in [R_1, r_2) \) such that \( d_2 := \min \{ h(r) : r_1 \leq r \leq r_2 \} > 0 \). Assume that \( f \) satisfies one of the following conditions.

(i) \( 0 \leq f^0 < 2(\beta_n)^{-1} \) and \( (\tau_n)^{-1} < f^\infty \leq \infty \).

(ii) \( 0 \leq f^\infty < 2(\beta_n)^{-1} \) and \( (\tau_n)^{-1} < f_0 < \infty \), where \( \beta_n \) is the same as in Theorem 4.2. \( \tau_n = (b - a) \phi_n(b) d_2 \), \( a = p_n(r_2) \) and \( b = p_n(r_1) \). Then Eq. (4.1)–(4.2) has a positive radial solution.

Corollary 4.3. Assume that \( f \) satisfies either \( 0 \leq f^0 < 2(\beta_n)^{-1} \) and \( 4/\phi_n(1) < f^\infty \leq \infty \) or \( 0 \leq f^\infty < 2/\phi_n(0) \) and \( 4/\phi_n(1) < f_0 < \infty \). Then the equation \( Au + f(u) = 0 \) with (4.2) has a positive radial solution.

Remark 4.3. According to Remark 3.1 we can modify the corresponding conditions of the above results for Eq. (4.1) with (4.2) to obtain results on Eq. (4.1) with (4.2). We omit the obvious statements.

REFERENCES