Cofactor Matrices

Scott J. Beslin
Department of Mathematics
Nicholls State University
Thibodaux, Louisiana 70310

Submitted by Robert Hartwig

ABSTRACT

The function which maps a square matrix A to its cofactor matrix cof(A) is examined. A characterization is given for the image of the function. Its injective properties on the general linear group of nonsingular matrices are also addressed.

1. INTRODUCTION AND PRELIMINARIES

Let $A=(a_{ij})$ be any $n\times n$ matrix over a field. The (i,j) minor of A, denoted M_{ij}, is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}=C_{ij}$ is called the (i,j) cofactor of A. By the cofactor matrix of A we mean the $n\times n$ matrix cof(A), the (i,j) entry of which is C_{ij}.

The transpose of this matrix, $[\mathrm{cof}(A)]^T$, is the classical adjoint of A, denoted adj(A). The importance of the adjoint (and hence the cofactor) to matrix invertibility is well known. From elementary linear algebra,

$$[\mathrm{adj}(A)]A=A[\mathrm{adj}(A)]=[\det(A)]I_n. \tag{1.1}$$

Here det(A) denotes the determinant of A, and I_n represents the $n\times n$ identity matrix. It follows that if A is invertible,

$$A^{-1}=\frac{1}{\det(A)}\mathrm{adj}(A). \tag{1.2}$$

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Although most linear algebra texts ask their readers to find $\mathrm{cof}(A)$ and $\mathrm{adj}(A)$ upon being given A, seldom is it asked whether, given A, there is a matrix B such that $\mathrm{cof}(B)=A$. It is the purpose of this paper to settle the question of exactly which matrices are cofactor matrices (and hence adjoint matrices). We choose to address cofactor matrices in order to avoid repeatedly taking transposes. Moreover, we will see that the *cofactor* operator yields a homomorphism of semigroups; the *adjoint* is an antihomomorphism.

More specifically, let F be an abstract field, $M_n(F)$ the collection of $n\times n$ matrices over F, and $cof: M_n(F)\to M_n(F)$ the function which maps A to $\mathrm{cof}(A)$. We investigate the elements of the image of *cof*, and also address the question of whether *cof* is injective. In what follows, let R and C be the fields of real and complex numbers respectively.

We will see that results differ according to three criteria: the action of *cof* on the nonsingular matrices, the action of *cof* on the singular matrices, and the dimension n. Many of our results will be valid in the context of the real field R; we will note the modifications for C and F in explanatory remarks.

First we note the equality

$$\mathrm{cof}(AB)=\mathrm{cof}(A)\,\mathrm{cof}(B). \tag{1.3}$$

See [1], [2], or the comments in [3, p. 20]. That is, *cof* is a semigroup homomorphism. Hence, the image of *cof* is closed under multiplication.

2. THE OPERATOR *cof* ON THE GROUP $\mathrm{GL}_n(F)$ OF NONSINGULAR $n\times n$ MATRICES

In this section we examine the cofactor matrices of invertible matrices. The building blocks of invertible matrices are the elementary matrices; every nonsingular matrix is the product of such matrices.

We wish to study the precise action of *cof* on the elementary matrices; for example, which elementary matrices are cofactor matrices? Theorems 1 and 2 will give us a scheme to find a solution to the equation $\mathrm{cof}(X)=A$ for a given invertible A. The results developed below will be used in the next section on singular matrices.

We adopt the following notation. Let J_{ij} be the type 1 elementary matrix resulting from I_n by switching rows i and j. By $U_i(r)$, $r\neq 0$, we mean the elementary type 2 matrix formed by multiplying row i of I_n by r. Finally, if $r\neq 0$, the transvection matrix $X_{ij}(r)$ is the elementary type 3 matrix obtained by replacing row i of I_n with (row i)$+r\cdot$(row j). It is well known that $\det(J_{ij}) = -1$, $\det(U_i(r)) = r$, and $\det(X_{ij}(r)) = 1$. A similar notation may be used for the corresponding column-elementary matrices.

The following theorem describes the action of *cof* on elementary matrices; the proof is straightforward and is omitted. Observe that J_{ij} is a symmetric matrix for any i and j.

THEOREM 1.

(i) $\operatorname{cof}(J_{ij}) = -J_{ij}$;
(ii) $\operatorname{cof}(U_i(r)) = \prod_{k \neq i} U_k(r)$;
(iii) $\operatorname{cof}(X_{ij}(r)) = X_{ji}(-r)$.

Similar results hold for column-elementary matrices.

We now answer the question of which nonsingular matrices are cofactor matrices. Observe from Theorem 1(iii) that each elementary type 3 is a cofactor matrix.

To classify the invertible cofactor matrices is easy when we observe the following: if $\operatorname{adj}(X) = A$, then $AX = XA = [\det(X)]I_n$; hence $[\det(X)]^{n-1} = \det(A)$. Thus it is necessary that the equation $y^{n-1} = \det(A)$ have a solution in F. Conversely, if such y exists, set $X = yA^{-1}$.

Hence, for the real field R, we have

THEOREM 2. *If n is even, every nonsingular $n \times n$ matrix A is a cofactor matrix. If n is odd, the nonsingular cofactor matrices are precisely those with positive determinant. In fact, $A = \operatorname{cof}[r \cdot (A^{-1})^T]$, where r is an $(n-1)$st root of* $\det(A)$.

COROLLARY 1. *If n is even, then $U_i(r)$, $r \neq 0$, is a cofactor matrix for every pair (i, r). If n is odd, $U_i(r)$ is a cofactor matrix if and only if r is positive.*

COROLLARY 2. *The type* 1 *matrix J_{ij} is a cofactor matrix if and only if n is even.*

REMARK 1. For an abstract field F, Theorem 2 must be modified to the following: $A \in M_n(F)$ is a cofactor matrix if and only if $\det(A)$ is an $(n-1)$st power in F. For example, $U_2(-\frac{1}{8})$ is a 4×4 rational cofactor matrix, but $U_2(\frac{1}{2})$ is not. Of course, every nonsingular complex matrix is a cofactor matrix over C.

The function cof is an injective mapping of $GL_n(R)$ into itself if n is even. Hence, from Theorem 2 and (1.3) we have

Corollary 3. *If n is even, the mapping $A \to \text{cof}(A)$ is a group isomorphism of* $\text{GL}_n(R)$ *with* $\text{GL}_n(R)$.

Letting $\text{GL}_n^+(R)$ denote the subgroup of $\text{GL}_n(R)$ consisting of matrices with positive determinant, we obtain

Corollary 4. *If $n > 2$ is odd, the mapping $A \to \text{cof}(A)$ is a group homomorphism of $GL_n(R)$ onto $GL_n^+(R)$. The kernel of the homomorphism is* $\{\pm I_n\}$.

Remark 2. We may obtain similar results for an abstract field F by introducing the sets $V = \{A \in \text{GL}_n(F) : \det(A) = y^{n-1}$ for some $y \in F\}$ and $K = \{(n-1)\text{st roots of unity in } F\}$. Then we have

Theorem 3. *The mapping $A \to \text{cof}(A)$ is a group homomorphism of* $\text{GL}_n(F)$ *onto V; the kernel of the homomorphism is $K \cdot I_n$.*

For example, if $F = Z_p$, the finite field of integers modulo a prime p, and $n = p$, then V is equal to the special linear group $\text{SL}_p(Z_p) = \{A \in \text{GL}_p(Z_p) : \det(A) = 1\}$, and $K = Z_p \setminus 0$.

3. THE OPERATOR *cof* ON THE COLLECTION OF SINGULAR $n \times n$ MATRICES

We first observe that the mapping $A \to \text{cof}(A)$ is no longer injective on the singular matrices. For example,

$$\text{cof}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \text{cof}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

To examine the multiplicative subsemigroup of the singular $n \times n$ matrices defined by the image of *cof*, we look for "primitive" elements, much as the identity I_n is a primitive element for the nonsingular matrices. These can be found in the matrix units E_{ij}. The $n \times n$ matrix E_{ij} has 1 in the (i, j) position and 0 elsewhere. Hence, if e_k denotes the $n \times 1$ column matrix with 1 in the kth position and 0 elsewhere, it is true that $E_{ij} = e_i e_j^T$.

Theorem 4. *For every i and j, and for every scalar α, αE_{ij} is a cofactor matrix.*

Proof. Any matrix B whose (i,j) cofactor is α, and whose ith row and jth column are zero, satisfies $\operatorname{cof}(B)=\alpha E_{ij}$. ■

For example, let σ be the one-to-one mapping of $S=\{1,\dots,i-1, i+1,\dots,n\}$ onto $\{1,\dots,j-1,j+1,\dots,n\}$ which preserves increasing order. Choose some fixed $m\in S$. Define the $n\times n$ matrix T_{ij} to be

$$\pm\alpha E_{m,\sigma(m)}+\sum_{\substack{k\in S\\ k\neq m}}E_{k,\sigma(k)}=\pm\alpha e_m e_{\sigma(m)}^T+\sum_{\substack{k\in S\\ k\neq m}}e_k e_{\sigma(k)}^T,$$

so that the ith row and jth column of T_{ij} are zero, and the submatrix formed by deleting row i and column j is the $(n-1)\times(n-1)$ matrix $\operatorname{diag}(1,\dots,\alpha,\dots,1)$ if $i+j$ is even, or $\operatorname{diag}(1,\dots,-\alpha,\dots,1)$ if $i+j$ is odd. Then $\operatorname{cof}(T_{ij})=\alpha E_{ij}$. For an easier formulation when $i\neq j$, let $B=I_n-\alpha E_{ji}-E_{ii}-E_{jj}$; again, $\operatorname{cof}(B)=\alpha E_{ij}$.

In fact, we have the following proposition.

Proposition 1. *If $\alpha\neq 0$, then $\operatorname{cof}(B)=\alpha E_{ij}$ if and only if the (i,j) cofactor of B is α, row i of B is zero, and column j of B is zero.*

Proof. Sufficiency is straightforward. To prove necessity, suppose $\operatorname{cof}(B)=\alpha E_{ij}$. Obviously the (i,j) cofactor of B must be α. Furthermore, $B^T\operatorname{cof}(B)=0$ implies that $e_ie_j^TB=0$. Hence $e_i^TB=0$ and $Be_j=0$. The proposition follows. ■

In reference to Proposition 1, the $(n-1)\times(n-1)$ submatrix B_{ij} of B formed by deleting the ith row and jth column must satisfy

$$\det(B_{ij})=\begin{cases}\alpha & \text{if } i+j \text{ is even}\\ -\alpha & \text{if } i+j \text{ is odd.}\end{cases}$$

Since the determinant function is a group homomorphism of $\mathrm{GL}_{n-1}(F)$ onto $F\setminus 0$ with the special linear group $\mathrm{SL}_{n-1}(F)$ as kernel, it is true that B_{ij} has the form ST, where $S\in\mathrm{SL}_{n-1}(F)$ and $\det(T)=\pm\alpha$. Letting

$$T=\begin{cases}\operatorname{diag}(\alpha,1,1,\dots,1) & \text{if } i+j \text{ is even,}\\ \operatorname{diag}(-\alpha,1,1,\dots,1) & \text{if } i+j \text{ is odd,}\end{cases}$$

we have

Corollary 5. *If* $\alpha \neq 0$, *then* $\mathrm{cof}(B) = \alpha E_{ij}$ *if and only if* $B_{ij} \in \mathrm{SL}_{n-1}(F)\cdot T$, *row* i *of* B *is zero* ($e_i^T B = 0$), *and column* j *of* B *is zero* ($Be_j = 0$).

Remark 3. For the real field, we may extend Theorem 4 by considering *polynomials* $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, with x replaced by some $n \times n$ matrix unit E_{ij}, $n \geqslant 3$. If $a_0 = 0$, then $f(E_{ij}) = \beta E_{ij}$ for some scalar β, and hence $f(E_{ij})$ is a cofactor matrix.

Now suppose $a_0 \neq 0$. If $i \neq j$, then $f(E_{ij}) = a_0 I_n + a_1 E_{ij}$ is invertible. Therefore, by Theorem 2, $f(E_{ij})$ is always a cofactor matrix when n is even; if n is odd, $f(E_{ij})$ is a cofactor matrix if and only if a_0 is positive. If $i = j$, then $f(E_{ij}) = f(E_{ii}) = a_0 I_n + \alpha E_{ii}$, where $\alpha = a_1 + a_2 + \cdots + a_m$. It is well know that *if the rank of an* $n \times n$ *matrix* Y *is* $n-1$ *or* $< n-1$, *then* $\mathrm{rank}(\mathrm{cof}(Y)) = 1$ *or* 0 *respectively*. So if $a_0 + \alpha = 0$, then $f(E_{ii})$ is a singular matrix of rank > 1, and hence not a cofactor matrix. Otherwise, $a_0 + \alpha \neq 0$ and $f(E_{ii})$ is invertible. If n is even, this is a cofactor matrix; if n is odd, $f(E_{ii})$ is a cofactor matrix if and only if $a_0 + \alpha$ is positive.

For the complex field, $f(E_{ij})$ is not a cofactor matrix if and only if $a_0 \neq 0$, $i = j$, and $a_0 + a_1 + \cdots + a_m = 0$.

The main result of this section is

Theorem 5. *A singular* $n \times n$ *matrix* A *is a cofactor matrix if and only if* $\mathrm{rank}(A) \leqslant 1$.

Proof. Necessity follows from Remark 3. To prove sufficiency, suppose A has rank 1. Type 3 row operations alone will reduce A to the form

$$B = \begin{pmatrix} & & 0 & & \\ * & * & * & * & * \\ & & 0 & & \end{pmatrix}.$$

Hence type 3 column operations on B (or type 3 row operations on the transpose of B) then yield the form αE_{ij} for some i and j. Since type 3 elementary matrices are cofactor matrices (and cofactor matrices are closed under transpose and composition), the result follows from Theorem 4. ∎

Theorems 1–5 combined afford us a scheme to find a solution to $\mathrm{cof}(X) = A$, given a rank 1 matrix A. For example,

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not a cofactor matrix, but

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}^T$$

is a real cofactor matrix, since

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_{21}\left(\tfrac{3}{2}\right)\cdot(2E_{12}).$$

4. SUMMARY AND FURTHER REMARKS

As a summary of our findings, let G denote $\mathrm{GL}_n(R)$ for n even, and $\mathrm{GL}_n^+(R)$ for n odd; let H denote the collection of $n \times n$ matrices A with $\mathrm{rank}(A) \leqslant 1$. Then $cof : M_n(R) \to G \cup H$ is a surjective mapping, injective on $\mathrm{GL}_n(R)$, when n is even. In fact, it is an epimorphism of semigroups.

For the complex number field C, G may be replaced with $\mathrm{GL}_n(C)$ (for n either even or odd), since C is algebraically closed. For an abstract field F, the notation would be modified as in Theorem 3.

REMARK 4. Let A be an $n \times n$ matrix.

If A is nonsingular, the solutions X to $\mathrm{cof}(X) = A$ depend on A^{-1} and $\det(A)$. Theorems 1 and 2 furnish a scheme for finding X when possible.

If $A = 0$, we know $\mathrm{cof}^{-1}(A)$ is the collection of singular matrices of rank $< n-1$.

If $\mathrm{rank}(A) = 1$, Theorems 1–5 along with Proposition 1 enable us to find all solutions to $\mathrm{cof}(X) = A$. From the proof of Theorem 5, A can be factored as $P(\alpha E_{ij})Q$, where P is a product of type 3 row-elementary matrices, and Q is a product of type 3 column-elementary matrices. Observe that there may be several possibilities for P, αE_{ij}, and Q. In any case, X is an element of $\mathrm{cof}^{-1}(P)\mathrm{cof}^{-1}(\alpha E_{ij})\mathrm{cof}^{-1}(Q)$ for some decomposition $P(\alpha E_{ij})Q$ of A.

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