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Extending partial colorings of graphs

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Abstract

Recently, several authors have considered the problem of extending a partial coloring of a graph to a complete coloring. We show how similar results can be extracted from old proofs on recursive colorings of highly recursive graphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

In the late 1970s several logicians, including Manaster and Rosenstein [15], Bean [3], Schmerl [16,17], and the author [6,7] considered the computational complexity of coloring infinite graphs. They studied recursive and highly recursive graphs and tried to determine the least k such that such graphs had a k-coloring that could be calculated by a deterministic algorithm. A graph is *recursive* if there is a deterministic algorithm for deciding whether or not a given input is a vertex and whether or not two vertices are adjacent. A recursive graph is *highly recursive* if in addition every vertex has finite degree and there is a deterministic algorithm for calculating the degree of any vertex. This extra condition is very powerful because it allows one to calculate the neighborhood of any vertex in an infinite graph. It turned out that results concerning recursive graphs had natural analogs in finite graph theory that could be expressed in terms of on-line algorithms. This led to a long series of results (some of only finitary interest) including [4,5,8–12,14]. On the other hand, there seemed to be no natural analogs in finite graph theory to the results concerning highly recursive graphs. But recently several graph theorists including Tuza [19], Albertson [1], Albertson and Moore [2], and Kostochka [13] have been studying the problem of extending a partial coloring of a graph to a total coloring. The purpose of this note is to show how old techniques for coloring highly recursive graphs can be used to prove theorems similar to the results in [1,2,13].

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Albertson answered a question of Thomassen [18] about planar graphs by proving the following theorem for all graphs.

Theorem 1 (Albertson [1]). *Suppose that $\chi(G)=r$ and $W\subset V(G)$ such that the distance between any two distinct members of W is at least 4. Then any $(r+1)$-coloring of W can be extended to an $(r+1)$-coloring of G.*

This raises the following extremal problem. Let $r\geqslant 2$ and define $\mathrm{ext}(r,d)$ to be the least k such that if $\chi(G)\leqslant r$ and $W\subset V(G)$ with $\mathrm{dist}(x,y)\geqslant d$ for any two distinct vertices in W, then any k-coloring of W can be extended to a k-coloring of G. It is trivial to check that $\mathrm{ext}(r,2)=\infty$ and $\mathrm{ext}(r,d)>r$ for all d and r. By the theorem $\mathrm{ext}(r,4)=r+1$. Albertson and Moore [2] showed that $\mathrm{ext}(r,3)=\lceil (3r+1)/2\rceil$. They then considered the problem when $W=W_1\cup W_2\cup\cdots\cup W_m$ is the union of k-cliques W_i such that $\mathrm{dist}(x,y)\geqslant d$ whenever $i\neq j$ with $x\in W_i$ and $y\in W_j$. We write $\mathrm{ext}(r,k,d)$ for the corresponding extremal function. (So $\mathrm{ext}(r,d)=\mathrm{ext}(r,1,d)$.) Albertson [1] showed that $\mathrm{ext}(r,k,6k-2)\leqslant r+1$. Kostochka [13] improved this to $\mathrm{ext}(r,k,4k)\leqslant r+1$. Finally, Albertson and Moore [2] proved $\mathrm{ext}(r,k,2k+2\lfloor r/2\rfloor)\leqslant r+1$.

We shall consider a slightly different problem. For a graph $G=(V,E)$ let

$$N[W]=\{v\in V\colon v \text{ is adjacent to some vertex in } W\}\cup W,$$

$$N(W)=N[W]\setminus W,$$

$$N^0[W]=W=N^0(W),$$

$$N^{s+1}[W]=N[N^s[W]]$$

and

$$N^{s+1}(W)=N^{s+1}[W]\setminus N^s[W].$$

Let $\mathrm{Pre}(r,d,t)$, resp. $\mathrm{PPre}(r,d,t)$, be the least k such that if $G=(V,E)$ is an r-colorable graph, resp. perfect graph, $W=W_1\cup W_2\cup\cdots\cup W_m\subset V$ where $\mathrm{dist}(W_i,W_j)\geqslant d$ whenever $i\neq j$, and f is a k-coloring of W such that $f|W_i$ can be extended to an r-coloring of $N^t[W_i]$ for all i, then f can be extended to a k-coloring of G. Note that we have placed the additional condition on the coloring f that it can be partially extended.

However when the W_i are all cliques this condition is automatically implied by the r-colorability of G. In the next section we will warm-up by showing that $\mathrm{Pre}(r,3,0)=r^2+r$, $\mathrm{Pre}(r,d,0)=2r$ for all $d\geqslant 4$, $\mathrm{Pre}(r,3,1)=r^2$, and $\mathrm{Pre}(r,d,1)\leqslant 2r$ for all $d\geqslant 4$. In Section 3 we will extract a proof from Schmerl's paper [16] on coloring highly recursive graphs to show that $\mathrm{Pre}(r,d,1)=2r-1$ for all $d\geqslant 6$ and $\mathrm{Pre}(r,d,2)=2r-1$ for all $d\geqslant 5$. When G is a perfect graph we can do even better. We shall extract a proof from the author's paper [6] to show that $\mathrm{PPre}(r,4r,2r-1)=r+1$.

We end this section with some notation. For positive integers n and k let $[n]$ denote the set $\{1,2,\ldots,n\}$ and $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$. For a function f and a subset $S\subset\mathrm{domain}(f)$, let $f|S$ denote f restricted to S. Let $G=(V,E)$ be a graph.

A function $f: V \to S$ is an S-coloring, or simply a coloring, of G if $f(x) \neq f(y)$ for all adjacent vertices x and y. A coloring is an r-coloring if its range has cardinality at most r. Thus an $[r]$-coloring is an r-coloring that uses colors from $[r]$. We may indicate that f is a coloring of G by writing $f: V \xrightarrow{G} S$. The distance dist(x, y) between two vertices x and y is the number of edges in the shortest path between them. The distance dist(X, Y) between two subsets of vertices X and Y is the number edges in the shortest path from a vertex in one to a vertex in the other.

2. Warm-up

Proposition 2. PPre$(r,3,0) =$ Pre$(r,3,0) = r^2 + r$ *and* PPre$(r,3,1) =$ Pre$(r,3,1) = r^2$.

Proof. We shall show that $r^2+r \leqslant \mathrm{PPre}(r,3,0) \leqslant \mathrm{Pre}(r,3,0) = r^2+r$ and $r^2 \leqslant \mathrm{PPre}(r,3,1) \leqslant \mathrm{Pre}(r,3,1) = r^2$. First, we show that $r^2 + r \leqslant \mathrm{PPre}(r,3,0)$. For $i \in [r]$ let $X^i = \{x^i_S \colon S \in \binom{[r^2+r-1]}{r}\}$ and for $i \in [r]$, $S \in \binom{[r^2+r-1]}{r}$ let W^i_S, be pairwise disjoint r-sets which are also disjoint from the X^i. Let $G = (V, E)$ be the graph with

$$V = \bigcup_{i\in[r]} \left(X^i \cup \bigcup_{S\in\binom{[r^2+r-1]}{r}} W^i_S \right)$$

and

$$E = \left\{ x^i_S x^j_T \colon i \neq j, \text{ and } S, T \in \binom{[r^2+r-1]}{r} \right\}$$
$$\cup \left\{ x^i_S v \colon i \in [r], S \in \binom{[r^2+r-1]}{r}, v \in W^i_S \right\}.$$

Then G is a perfect graph. Let $f: \bigcup_{S\in\binom{[2r+1]}{r}} W^i_S \xrightarrow{G} [r^2 + r - 1]$ be such that range$(f|W_S) = S$. If $g: V \xrightarrow{G} [r^2 + r - 1]$ were an extension of f then g would need to use at least $r + 1$ colors on each X^i and the sets of colors used on X^i and X^j would need to be disjoint whenever $i \neq j$. Clearly, there are not enough colors.

Next, we note in passing that PPre$(r,3,0) \leqslant$ Pre$(r,3,0)$ and show that Pre$(r,3,0) = r^2 + r$. Suppose that $G = (V, E)$ is an r-colorable graph, $W = W_1 \cup W_2 \cup \cdots \cup W_m \subset V$ where dist$(W_i, W_j) \geqslant 3$ whenever $i \neq j$, and f is a $[r^2+r]$-coloring of W such that $f|W_i$ is an r-coloring for all i. We must show that f can be extended to a $[r^2+r]$-coloring g of G. Fix an $[r]$-coloring h of G. For each $v \in N(W)$ let $m(v, j)$ be the greatest integer $c \leqslant j$ such that $c \notin \{f(w) \colon w \in N(v) \cap W\}$. Since dist$(W_i, W_j) \geqslant 3$ whenever

$i \neq j$ and $f|W_i$ is an r-coloring, $j - r \leqslant m(v,j)$. Define $g : V \to [r^2 + r]$ by

$$g(v) = \begin{cases} f(v) & \text{if } v \in W, \\ h(v)(r+1) & \text{if } v \notin N[W], \\ m(v, h(v)(r+1)) & \text{if } v \in N(W). \end{cases}$$

Using the fact that $\text{dist}(W_i, W_j) \geqslant 3$ whenever $i \neq j$, it is easy to check that g is an $[r^2 + r]$-coloring of G.

The proof that $r^2 \leqslant \text{PPre}(r,3,1) \leqslant \text{Pre}(r,3,1) = r^2$ is similar. The only difference is that now a vertex in $N(W_i)$ can only be adjacent to vertices of W_i with $r - 1$ different colors. □

Proposition 3. $\text{PPre}(r,d,0) = \text{Pre}(r,d,0) = 2r$, *for all* $d \geqslant 4$.

Proof. We shall show that $2r \leqslant \text{PPre}(r,d,0) \leqslant \text{Pre}(r,d,0) \leqslant 2r$, for all $d \geqslant 4$. First we show that $2r \leqslant \text{PPre}(r,d,0)$. Let $G = (V,E)$, $W_1 = W \subset V$, and $f : W \to [r]$ be defined by

$$V = [r] \times [r+1],$$

$$E = \{(i,j)(i,r+1)\colon i,j \in [r]\} \cup \{(i,r+1)(j,r+1)\colon i,j \in [r] \text{ and } i \neq j\},$$

$$W = [r] \times [r],$$

$$f((i,j)) = j.$$

So G consists of a clique $Q = [r] \times \{r+1\}$ and an independent set W of vertices of degree 1 such that every vertex of Q is adjacent to vertices colored $1,\ldots,r$ by f. Clearly, G is an r-colorable perfect graph and any extension of f to a coloring of V requires r new colors.

Next, we show that $\text{Pre}(r,d,0) \leqslant 2r$. Suppose that $G = (V,E)$ is an r-colorable graph, $W = W_1 \cup W_2 \cup \cdots \cup W_m \subset V$ where $\text{dist}(W_i, W_j) \geqslant 4$ whenever $i \neq j$, and f is a $[2r]$-coloring of W such that $f|W_i$ is an r-coloring for all i. For $v \in N(W)$, let $i(v)$ be the unique index i such that v is adjacent to a vertex in W_i. Let $h : V \xrightarrow{G} [r]$. Let $R_i = \text{range}(f|W_i)$ and $s_i : R_i \cap [r] \to [2r] \setminus (R_i \cup [r])$ be one to one. Define $g : V \to [2r]$ by

$$g(v) = \begin{cases} f(v) & \text{if } v \in W, \\ h(v) & \text{if } v \notin N[W] \text{ or } h(v) \notin R_{i(v)}, \\ s_{i(v)} \circ h(v) & \text{if } v \in N(W) \text{ and } h(v) \in R_{i(v)}. \end{cases}$$

Since $\text{dist}(W_i, W_j) \geqslant 4$ whenever $i \neq j$, g is a well defined coloring. □

3. Main results

We shall need the following lemma.

Lemma 4. *Let $G=(V,E)$ be a graph, $W\subset V$, and R and R' be two sets of r colors. For every partial coloring $f: N^t[W_i]\xrightarrow{G} R$ there exists a partial coloring $f': N^t[W_i]\xrightarrow{G} R\cup R'$ such that $f'|W=f|W$ and f' is an R'-coloring of $N^t[W]\setminus W$.*

Proof. Let s be a bijection between $R\setminus R'$ and $R'\setminus R$. Define f' by

$$f'(v)=\begin{cases} f(v) & \text{if } v\in W_i \text{ or } f(v)\in R',\\ s\circ f(v) & \text{if } v\notin W_i \text{ and } f(v)\notin R'. \end{cases}\qquad\square$$

Theorem 5. $\mathrm{Pre}(r,5,2)\leqslant 2r-1$.

Proof. Let $G=(V,E)$ be an r-colorable graph and let $W\subset V$ be such that $W=W_1\cup W_2\cup\cdots\cup W_m$ and $\mathrm{dist}(W_i,W_j)\geqslant 5$ whenever $i\neq j$. Suppose further that $f_i: N^2[W_i]\xrightarrow{G}[2r-1]$ is an r-coloring for all $i\in[m]$. We must show that there exists a $[2r-1]$-coloring g of G such that $g|W_i=f_i|W_i$ for all $i\in[m]$.

By Lemma 4 we may assume that f_i is an $[r]$-coloring of $N^2[W_i]\setminus W_i$ for all $i\in[m]$. Let $h: V\xrightarrow{G}\{r,r+1,\dots,2r-1\}$. Define g by

$$g(v)=\begin{cases} h(v) & \text{if } v\notin \bigcup_{i\in[m]} N^2[W_i],\\ f_i(v) & \text{if } v\in N[W_i],\\ h(v) & \text{if } v\in \bigcup_{i\in[m]} N^2(W_i) \text{ and } h(v)\neq r,\\ f_i(v) & \text{if } v\in N^2(W_i) \text{ and } h(v)=r. \end{cases}$$

If $i\neq j$, then $N^2[W_i]\cap N^2[W_j]=\emptyset$, since $\mathrm{dist}(W_i,W_j)>4$. It follows that g is well defined. It remains to show that g is a coloring. Consider two adjacent vertices x and y. Our only concern is that x may be colored by f_i and y may be colored by some other function. This could only happen if $x\in N(W_i)\cup N^2(W_i)$. If $x\in N(W_i)$ then $y\in N^2(W_i)$. So if y is not colored by f_i then $g(y)=h(y)>r\geqslant f_i(x)=g(x)$. Otherwise $x\in N^2(W_i)$. Since $g(x)=f_i(x)$, it must be the case that $h(x)=r$. Thus $h(y)\neq r$. Since $y\notin N[W_i]$ and for all $j\neq i$, $\mathrm{dist}(W_i,W_j)>4$, $y\notin\bigcup_{j\in[m]}N[W_j]$. Thus $g(y)=h(y)>r\geqslant f_i(x)=g(x)$. $\square$

A very similar argument shows that $\mathrm{Pre}(r,6,1)\leqslant 2r-1$. The main difference is that instead of modifying the range of f_i on $N^2[W_i]\setminus W_i$ we modify the range of h on $N^2[W_i]\setminus W_i$.

Theorem 6. $\mathrm{Pre}(r,d,t)\geqslant 2r-1$.

Proof. For all positive integers $r\geqslant 2$ and d, we will show that there exists an r-colorable graph $G=(V,E)$ with a partial coloring f of a subset $W=W_1\cup W_2\subset V$ such that $\mathrm{dist}(W_1,W_2)\geqslant d$ and f can be extended to a partial r-coloring of $N^2[W]$, but cannot be extended to an $(2r-1)$-coloring of G.

For $i \in [2d]$ let Q^i be an r-clique on the vertices $X^i = \{x_1^i, x_2^i, \ldots, x_r^i\}$. Let $P^i = Q^i \times Q^i$ be the graph defined on $X^i \times X^i$ by (x_m^i, x_n^i) is adjacent to (x_s^i, x_t^i) iff $m \neq s$ and $n \neq t$. Then the diagonal of P is an r-clique and $\chi(P) = r$, since assigning color i to every vertex in the ith row (column) of P produces a coloring. We call such a coloring a row (column) coloring. Call a coloring f of P row (column) colorful if the vertices of some row (column) have r different colors. We will need the following easily proved lemma. □

Lemma 7 (Schmerl [16]). *If f is a $(2r-2)$-coloring of P then f is either row colorful or column colorful, but not both.*

Proof. If f were neither row nor column colorful then f would have to repeat a color on each row and on each column of P. Since f is a proper coloring of P the same color cannot be repeated on two different rows or on two different columns or on a row and a column. Thus f would have to use at least $2r$ colors, a contradiction. On the other hand, the ranges of a row of P and a column of P have exactly one color, the color of their common vertex, in common. If f were both row and column colorful the union of the ranges of the colorful row and the colorful column would contain $2r-1$ colors, a contradiction. □

Let G be the graph obtained from $P^1, \ldots, P^{2d+2}$ by adding edges between vertices of the form (x_m^i, x_n^i) and (x_s^{i+1}, x_t^{i+1}) iff $m \neq t$ and $n \neq s$. The next lemma has a similar proof to that of the previous lemma.

Lemma 8 (Schmerl [16]). *If f is a $(2r-2)$-coloring of G then P^i is row colorful iff P^{i+1} is column colorful.*

Now, let $W_1 = P^1$ and $W_2 = P^{2d+2}$ and define a partial r-coloring $f : W \to [r]$ so that f is a row coloring of W_1 and a row coloring of W_2. Then f is column colorful on W_1 and on W_2, and so by Lemma 8, f cannot be extended to a coloring of G. However, it is easy to extend $f|W_i$ to a coloring of $N^t[W_i]$ by alternately row and column coloring the P^i.

If G is a perfect graph then fewer extra colors suffice. First, we prove a lemma and then use it to show that $\mathrm{PPre}(r, 4r, 2r-1) = r+1$.

Lemma 9. *Let $G = (V, E)$ be a perfect r-colorable graph and $W \subset V$. If $f : N^{2r-2}[W] \xrightarrow{G} \{2, 3, \ldots, r+1\}$, then there exists an $[r+1]$-coloring g of G such that $g|W = f|W$.*

Proof. We shall show by induction on k that for all $U \subset V$, if $f : U \cap N^{2k-2}[W] \xrightarrow{G} [k]$ and $\chi(G[U]) \leqslant k$, then there exists a $(k+1)$-coloring g of U such that

$$g|U \cap W = f|U \cap W$$

and

$$g^{-1}(k) \subset U \cap N^{2k-2}[W].$$

The base step $k=1$ is trivial, so suppose the result holds for $k-1$. Let

$$h : U \xrightarrow{G} \{0,1,\ldots,k-1\},$$

$$I = \{v \in U \cap N^{2k-2}[W]\colon\ f(v) = k\},$$

$$J = \{v \in U \setminus N^{2k-3}[W]\colon\ h(v) = k-1\}$$

and

$$U' = U \setminus (I \cup J).$$

Clearly $f' = f|U' \cap N^{2k-4}[W]$ is a $[k-1]$-coloring. We claim that $\chi(G[U']) \leqslant k-1$. Since G is perfect it suffices to show that $\omega(G[U']) \leqslant k-1$. Suppose that K is a k-clique in U. If $K \subset U \cap N^{2k-2}[W]$, then $K \cap I \neq \emptyset$, since f is a $[k]$-coloring of $U \cap N^{2k-2}[W]$. Otherwise $K \subset U \setminus N^{2k-3}[W]$ and so $K \cap J \neq \emptyset$, since h is a $\{0,1,\ldots,k-1\}$ -coloring of U. Regardless, $K \not\subset U'$, establishing the claim. By the induction hypothesis there exists a $\{0,1,\ldots,k-1\}$-coloring g' of $G[U']$ such that $g'|U' \cap W = f'|U' \cap W$ and $g'^{-1}(k-1) \subset U' \cap N^{2k-4}[W]$. Define $g : U \to \{0,1,\ldots,k\}$ by

$$g(v) = \begin{cases} k & \text{if } v \in I, \\ k-1 & \text{if } v \in J, \\ g'(v) & \text{if } v \in U'. \end{cases}$$

Note that if $g'(v) = k-1$ then v is not adjacent to any vertex in J. It follows easily that g is a coloring of U. Also $g|U \cap W = f|U \cap W$, since $U \cap W = (U' \cup I) \cap W$. □

Theorem 10. $\mathrm{PPre}(r, 4r, 2r-1) \leqslant r+1$.

Proof. Let $G = (V,E)$ be an r-colorable perfect graph and let $W \subset V$ be such that $W = W_1 \cup W_2 \cup \cdots \cup W_m$ and $d(W_i, W_j) \geqslant 4r$ whenever $i \neq j$. Suppose further that $f_i : N^{2r-1}[W_i] \xrightarrow{G} [r+1]$ is an r-coloring for all $i \in [m]$. We must show that there exists an $[r+1]$-coloring g of G such that $g|W_i = f_i|W_i$ for all $i \in [m]$.

By relaxing the assumption that f_i is an r-coloring and applying Lemma 4 we may assume that

$$\mathrm{range}(f_i|N^{2r-1}[W_i] \setminus W_i) \subset [r].$$

Now apply the Lemma 9 to $G' = G[V \setminus W]$, $W' := N(W)$, and

$$f' := \bigcup_{i \in [m]} f_i|N^{2r-1}[W_i] \setminus W_i.$$

Note that f' is a coloring, since $d(W_i, W_j) \geqslant 4r$ whenever $i \neq j$. This yields a coloring $g' : V \setminus W \xrightarrow{G'} [r+1]$ such that $g'|N(W) = f'|N(W)$. Clearly g' can be extended to $g : V \xrightarrow{G} [r+1]$ so that $g|W_i = f_i|W_i$ for all $i \in [m]$. □

Theorem 11. $r + 1 \leqslant \mathrm{PPre}(r, 4r, 2r - 1)$.

Proof. For all positive integers r we shall construct an r-colorable graph $G=(V,E)$ with a partial coloring f of a subset $W = W_1 \cup W_2 \subset V$ such that $\mathrm{dist}(W_1, W_2) \geqslant d$ and $f|W_i$ can be extended to a partial r-coloring of $N^{2r-1}[W_i]$, but f cannot be extended to a r-coloring of G. Let $P=(v_1,\ldots,v_{4r+1})$ be a path. Form G by replacing each vertex v_{2i-1}, $i \in [2r+1]$, by a $(k-1)$-clique Q_i. Then G is an r-colorable perfect graph. Let $W_1 = Q_1$ and $W_2 = Q_{2r+1}$. Let $f : W \to [r]$ such that $f|W_1$ is an $[r-1]$-coloring and $f|W_2$ is a $\{2,\ldots,r\}$-coloring. Clearly $f|W_i$ can be extended to an r-colorings of $N^{2r-1}[W_i]$ for $i \in [2]$, but f cannot be extended to an r-coloring of G. □

Finally, we mention a result of Schmerl [17]. Let **N** be the set of natural numbers.

Theorem 12. *There exists a (computable) function* $\phi : \mathbf{N} \to \mathbf{N}$ *such that whenever* $3 \leqslant k$, $G=(V,E)$ *is a graph such that* $\Delta(G) \leqslant k$ *and* G *does not contain a* $(k+1)$*-clique,* $W \subset V$ *is such that* $|W| \leqslant n$, *and* $f : N^{\phi(n)}[W] \xrightarrow{G} [k]$, *then* $f|W$ *can be extended to a* k*-coloring of* G.

References

[1] M.O. Albertson, You can't paint yourself into a corner, J. Combin. Theory Ser. B 73 (1998) 189–194.
[2] M.O. Albertson, E.H. Moore, Extending graph colorings, manuscript.
[3] D. Bean, Effective coloration, J. Symbolic Logic 41 (1976) 469–480.
[4] A. Gyárfás, J. Lehel, On-line and first-fit coloring of graphs, J. Graph Theory 12 (1988) 217–227.
[5] A. Gyárfás, J. Lehel, Effective on-line coloring of P5-free graphs, Combinatorica 11 (1991) 181–184.
[6] H.A. Kierstead, Recursive colorings of highly recursive graphs, Canad. J. Math. 33 (1981) 1279–1290.
[7] H.A. Kierstead, An effective version of Dilworth's theorem, Trans. Amer. Math. Soc. 268 (1981) 63–77.
[8] H.A. Kierstead, A polynomial time approximation algorithm for Dynamic Storage Allocation, Discrete Math. 88 (1991) 231–237.
[9] H.A. Kierstead, On-line coloring k-colorable graphs, Israel J. Math (1998) 93–104.
[10] H.A. Kierstead, K. Kolossa, On-line coloring of perfect graphs, Combinatorica 16 (1996) 479–491.
[11] H.A. Kierstead, W.T. Trotter, An extremal problem in recursive combinatorics, Congr. Numer. 33 (1981) 143–153.
[12] H.A. Kierstead, S.H. Penrice, W.T. Trotter, On-line graph coloring and recursive graph theory, SIAM J. Discrete Math. 7 (1994) 72–89.
[13] A. Kostochka, private communication.
[14] L. Lovász, M. Saks, W.T. Trotter, An online graph coloring algorithm with sublinear performance ratio, Discrete Math. (1989) 319–325.
[15] A. Manaster, J. Rosenstein, Effective matchmaking and k-chromatic graphs, Proc. Amer. Math. Soc. 39 (1973) 371–378.
[16] J.H. Schmerl, Recursive colorings of graphs, Canad. J. Math. 32 (1980) 821–831.
[17] J.H. Schmerl, The effective version of Brooks' Theorem, Canad. J. Math. 34 (1982) 1036–1046.
[18] C. Thomassen, Color-critical graphs on a fixed surface, manuscript.
[19] Zs. Tuza, Graph colorings with local constraints — a survey, Discuss. Math. 17 (1997) 161–228.