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Double Youden rectangles — an update with examples of size 5×11

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Abstract

The literature of double Youden rectangles (DYRs) is reviewed, to indicate what is known about their existence and construction. The discovery is reported of some 5×11 DYRs and of some similarly generated 6×11 DYRs.

1. Introduction

As defined by Bailey [1, p. 40], a double Youden rectangle (DYR) of size $k\times v$ is an arrangement of kv ordered pairs x, y in k rows and v columns $(k<v)$ such that:

(i) each value x is drawn from a set S of v elements,

(ii) each value y is drawn from a set T of k elements,

(iii) each element from S occurs exactly once in each row and no more than once per column,

(iv) each element from T occurs exactly once in each column and either n or $n+1$ times in each row, where n is the integral part of v/k,

(v) each element from S is paired exactly once with each element from T,

(vi) each pair of elements from S occurs together in exactly λ columns, where $\lambda=k(k-1)/(v-1)$, i.e. the sets of elements of S in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD, also commonly known as a symmetric 2-design) with parameters $\{v, k, \lambda\}$,

(vii) if n occurrences of each element from T are removed from each row, leaving $m=v-nk$ elements from T in each row, then (a) the remaining sets of m elements of

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T in the rows are the blocks of a SBIBD with parameters $\{k, m, \mu\}$ where $\mu = m(m-1)/(k-1)$, or else (b) $m = 1$.

If the elements of T are omitted from a DYR, the remaining rectangular arrangement of elements from S is what is commonly but misleadingly called a 'Youden square' (see e.g. [15]). The elements within the blocks of any SBIBD can be reordered to produce a Youden square [20].

There are well-known tight restrictions on the pairs of values v, k for which an SBIBD can exist. The restrictions are perforce even tighter for the existence of DYRs. For example, there are 3 SBIBDs with $v = 16$, $k = 6$; however, with $n =$ (integral part of $v/k) = 2$ and $m = v - nk = 4$, the ratio $\mu = m(m-1)/(k-1)$ is not equal to an integer, so condition (vii) above cannot be met for $v = 16$, $k = 6$. Even for parameter-sets $\{v, k, \lambda, \mu\}$ for which SBIBDs with parameters $\{v, k, \lambda\}$ and $\{k, m, \mu\}$ are known to exist, no DYR may yet have been found. Indeed, notwithstanding the merit of DYRs as experimental designs with a particularly simple form of statistical analysis, and notwithstanding their merit for constructing other valuable experimental designs [2], their discovery and enumeration have proceeded slowly since Clarke [4] gave the first DYRs in 1963. The present paper reviews progress to date, and reports the finding of DYRs of size 5×11. First, however, the concepts of 'transformation set' and 'species' of DYRs must be introduced.

2. Transformation sets and species of DYRs

Consider the 4×5 DYR (2.1), where $S = \{A, B, C, D, E\}$ and $T = \{1, 2, 3, 4\}$ and where element i from T is duplicated in row i:

$$\begin{matrix} E1 & C3 & D4 & B2 & A1 \\ B3 & D1 & E2 & A4 & C2 \\ C4 & A2 & B1 & E3 & D3 \\ D2 & E4 & A3 & C1 & B4 \end{matrix} \tag{2.1}$$

This DYR can be rewritten as (2.2), as follows:

$$\begin{bmatrix} - & 32 & 43 & 24 & 11 \\ 23 & - & 31 & 12 & 44 \\ 34 & 13 & - & 41 & 22 \\ 42 & 21 & 14 & - & 33 \\ 11 & 44 & 22 & 33 & - \end{bmatrix}. \tag{2.2}$$

To obtain (2.2) from (2.1), replace each entry from column j ($= 1, 2, 3, 4, 5$) of (2.1) by an entry in column j of (2.2) as follows: replace each entry containing A by an entry in row 1 of (2.2), each entry containing B by an entry in row 2, and so on; when each replacement is made, replace the letter from the original entry by the number ($= 1, 2, 3$ or 4) of the row containing the original entry. The array (2.2) is simply the incidence matrix of the SBIBD from condition (vi) for a DYR with $k=4$ and $v=5$, except that each entry 1 from the incidence matrix has been replaced by an ordered pair z, y where each value z or y is drawn from a set of $k=4$ elements. An array similar to (2.2) can be obtained similarly for any DYR, and will be called the 'square incidence array' of the DYR.

The sets of rows and of columns of a DYR can be denoted by P and Q, respectively. Then P, Q, S and T can be described as the four 'constraints' (or 'factors') of the DYR. The v rows and v columns of square incidence arrays such as (2.2) pertain to constraints S and Q, respectively, whereas the entries z and y pertain to P and T, respectively. The four constraints of a DYR are analogous to the three constraints R (rows), C (columns) and L (letters) of a Latin square [10].

The properties of a SBIBD clearly imply that, if W is the square incidence array for a DYR X, then

(i) the transpose of W about its main diagonal is the square incidence array for a DYR $*X$, and

(ii) changing the order of all the ordered pairs z, y in W gives the square incidence array for a DYR $X*$.

Thus, given a DYR X, we have four 'adjugate' DYRs to consider, namely X, $*X$, $X*$ and $*X* = *(X*) = (*X)*$. Here, $*X$ is obtained from X by interchanging the roles of the constraints S and Q, whereas $X*$ is obtained by interchanging the roles of P and T. Thus, the relationship of adjugacy here is like that for Latin squares [10, p. 272], the difference being that, for a Latin square, the roles of any two of the constraints R, C and L can be interchanged to give six Latin squares that are adjugate to one another.

By analogy with a classic definition for Latin squares (see [6] and [10, p. 272]), two DYRs may be said to belong to the same 'transformation set' (or 'isotopy class') if one can be obtained from the other by a combination of some or all of the following operations:

(i) permuting the rows P,
(ii) permuting the columns Q,
(iii) permuting the elements of S.
(iv) permuting the elements of T.

If two DYRs X and Y come from the same transformation set, we shall describe them as 'equivalent' and we shall write $X \sim Y$. The transformation set containing a DYR X may or may not contain $*X$, $X*$, or $*X*$; indeed, prima facie, we may have any of the following situations:

(a) $X \sim *X \sim X* \sim *X*$;
(b) $X \sim *X$ and $X* \sim *X*$, but X not equivalent to $X*$;
(c) $X \sim X*$ and $*X \sim *X*$, but X not equivalent to $*X$;
(d) $X \sim *X*$ and $*X \sim X*$, but X not equivalent to $*X$;
(e) no two of X, $*X$, $X*$ and $*X*$ equivalent.

Thus, the four DYRs $X, *X, X*$ and $*X*$ may be comprised within 1, 2 or 4 transformation sets. Again by analogy with a definition for Latin squares [10, p. 272], these

1, 2 or 4 transformation sets may be said to constitute a 'species' (or 'main class') of DYRs.

If X is the DYR (2.1), its square incidence array (2.2) shows at once that $X = {}^*X^*$ and ${}^*X = X^*$. To show further that $X \sim X^*$, change the order of all the ordered pairs in (2.2) and then, in the resultant square incidence array,

(i) interchange rows 3 and 4,
(ii) interchange columns 3 and 4,
(iii) interchange the values 2 and 3 of z in the entries z, y,
(iv) interchange the values 2 and 3 of y in the entries z, y.

Thus, the DYR (2.1) belongs to a species of DYRs that comprises just a single transformation set.

3. Review of known DYRs

In 1963, Clarke [4, p. 98] gave a 5×6 DYR obtained by trial and error for use as the design of an orchard experiment. He also showed [4, p. 99] that there is no 2×3 or 3×4 DYR, and he gave a general method [4, Section 2] for obtaining a $k \times (k+1)$ DYR for any value of k such that there are at least three mutually orthogonal $k \times k$ Latin squares. Preece [11, Section 4.1] gave further 4×5 and 5×6 DYRs. Hedayat et al. [8] constructed $k \times (k+1)$ DYRs for any $k > 3$ for which there is a $k \times k$ Graeco-Latin square. The gap remaining at $k = 6$ was filled by Preece [12] with a 6×7 DYR that is implicit in work by Freeman [7]. Christofi [3] has enumerated $k \times (k+1)$ DYRs for $k = 4$ and 5 and has classified them into transformation sets and species.

In 1982, Preece [14] gave a 4×13 DYR. Its square incidence array can be written as in Fig. 1, where each of its sets P and T is $\{0, 1, 2, 3\}$ and where each element of P occurs 4 times with the corresponding element of T and 3 times with each other element from T. (The rows and columns in Fig. 1 are ordered more helpfully than in the original paper.) Within each 1×3 or 3×1 subarray bounded by horizontal and vertical lines in Fig. 1, each pair z, y is obtained from the previous pair by use of the cyclic permutation (1 2 3); within each 3×3 subarray, this permutation is used similarly on the diagonal and on broken diagonals parallel to it, the elements 0 from P and T being invariant. With the 4×13 DYR denoted by X, Fig. 1 shows at once that $X = {}^*X^*$ and ${}^*X = X^*$; however, X^* is not equivalent to X, so X comes from a species containing two transformation sets.

All other known DYRs are of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$ where p is an odd prime. In 1966, Preece [11, Section 4.2] showed that 3×7 DYRs do not exist. But in 1967, Clarke [5] produced some 4×7 DYRs that belong to more than one species. (Clarke himself used the term 'species' for Youden squares, not DYRs.)

In 1971, Preece [12] published some 7×15 DYRs and indicated that similar $p \times (2p+1)$ DYRs can be generated using cycles of degree p (just as the above 4×13 DYR is generated using cycles of degree 3) for all prime p of the form $4q-1$ where q is an integer >2. However, the formal general methods of construction

00	11	22	33	—	—	—	—	—	—	—	—	—
11	—	—	—	23	—	—	02	—	—	30	—	—
22	—	—	—	—	31	—	—	03	—	—	10	—
33	—	—	—	—	—	12	—	—	01	—	—	20
—	32	—	—	00	—	—	—	—	—	—	21	13
—	—	13	—	—	00	—	—	—	—	21	—	32
—	—	—	21	—	—	00	—	—	—	13	32	—
—	20	—	—	—	—	—	—	11	33	02	—	—
—	—	30	—	—	—	—	11	—	22	—	03	—
—	—	—	10	—	—	—	33	22	—	—	—	01
—	03	—	—	—	12	31	20	—	—	—	—	—
—	—	01	—	12	—	23	—	30	—	—	—	—
—	—	—	02	31	23	—	—	—	10	—	—	—

Fig. 1. Square incidence array for 4×13 double Youden rectangle.

have only now been supplied by Vowden [21]. For each value of p, there are Vowden–Preece designs based on each of three different (i.e. nonisomorphic) SBIBDs with parameters $\{2p+1, p, (p-1)/2\}$; one of these three is self-dual and the other two are the duals of one another. Clearly, if X is a $k \times v$ DYR based on a non-self-dual SBIBD with v blocks and k elements per block, X cannot be equivalent to $^*X$, and so X must come from a species containing 2 or 4 transformation sets of DYRs.

Very many species of 7×15 DYRs based on the self-dual SBIBD with parameters $\{15, 7, 3\}$ can be generated using cycles of degree 7 [17]. A species containing 4 transformation sets of 8×15 DYRs has also been found, by trial and error [18]; again the designs can be generated using cycles of degree 7, but no generalisation of the construction suggests itself for larger DYRs of size $(p+1) \times (2p+1)$.

DYRs of sizes $p \times (2p+1)$ and $(p+1) \times (2p+1)$, with p of the form $4q+1$ where q is a positive integer, are known only for $p=5$. Very many 6×11 DYRs can be generated using cycles of degree 5 [19]; an example was given by Preece [16] in 1991. However, Preece [13] reported in 1976 that no 5×11 DYRs can be generated using cycles of degree 5. This left no obvious method — other than trial and error — for generating a 5×11 DYR. But further consideration of the automorphisms of the SBIBD with parameters $\{11, 5, 2\}$ has now yielded some 5×11 DYRs, of which (3.1) is an example (the primes in which will be explained later):

$$\begin{array}{ccccccccccc} B1 & 'C1 & 'F2 & H5 & J2 & 'I1 & K3 & 'A4 & 'G4 & D5 & E3 \\ D2 & K2 & H4 & G1 & F3 & J5 & E1 & B3 & A2 & C4 & I5 \\ J3 & G3 & D1 & E2 & I4 & K4 & A5 & F1 & B5 & H3 & C2 \\ E5 & 'F5 & 'A3 & B4 & H1 & 'G2 & J4 & 'I2 & 'C3 & K1 & D4 \\ F4 & E4 & G5 & I3 & C5 & D3 & H2 & K5 & J1 & B2 & A1 \\ \\ a & b & c & d & e & f & g & h & i & j & k \end{array} \qquad (3.1)$$

	g	d	k	c	e	h	f	i	j	a	b
B	—	44	—	—	—	23	—	35	52	11	—
D	—	—	44	31	—	—	53	—	15	22	—
J	44	—	—	—	12	—	25	51	—	33	—
C	—	—	32	—	55	—	—	43	24	—	11
K	13	—	—	—	—	55	34	—	41	—	22
G	—	21	—	55	—	—	42	14	—	—	33
I	—	53	25	—	34	42	11	—	—	—	—
A	35	—	51	43	—	14	—	22	—	—	—
H	52	15	—	24	41	—	—	—	33	—	—
E	21	32	13	—	—	—	—	—	—	45	54
F	—	—	—	12	23	31	—	—	—	54	45

Fig. 2. Square incidence array for the 5 × 11 double Youden rectangle (3.1), but with the elements of Q and S reordered as indicated.

55	—	61	13	46	—	32	—	—	—	24
42	66	—	—	21	54	—	13	—	—	35
—	53	44	65	—	32	—	—	21	—	16
64	31	—	22	—	43	56	—	—	15	—
—	45	12	51	33	—	—	64	—	26	—
23	—	56	—	62	11	—	—	45	34	—
31	—	—	46	—	—	—	25	14	53	62
—	12	—	—	54	—	25	—	36	61	43
—	—	23	—	—	65	14	36	—	42	51
—	—	—	34	15	26	41	52	63	—	—
16	24	35	—	—	—	63	41	52	—	—

Fig. 3. Square incidence array for a 6 × 11 double Youden rectangle.

In (3.1), the columns $a, b, \ldots, k$ have been ordered so that the set of capital letters in any column can be obtained from the set in the previous column by using the cyclic permutation $(ABCDEFGHIJK)$ of degree 11. But (3.1) is based on cycles of degree 3, as its square incidence array shows in Fig. 2, where each of the sets P and T is taken as $\{1, 2, 3, 4, 5\}$, where each element of P occurs 3 times with the corresponding element of T and twice with each other member from T, and where the cyclic permutation $(1\,2\,3)$ can be used as in Fig. 1.

If X_1 denotes the DYR (3.1), then a DYR X_2 from a different 5 × 11 species can be obtained by swapping E and F in the bottom left-hand 2 × 2 corner of (3.1) to give (3.2) as follows:

$$\begin{array}{ccccccccccc} B1 & 'C1 & 'F2 & H5 & 'J2 & I1 & K3 & A4 & 'G4 & D5 & E3 \\ D2 & K2 & H4 & G1 & F3 & J5 & E1 & B3 & A2 & C4 & I5 \\ J3 & G3 & D1 & E2 & I4 & K4 & A5 & F1 & B5 & H3 & C2 \\ F5 & E5 & A3 & B4 & H1 & G2 & J4 & I2 & C3 & K1 & D4 \\ E4 & 'F4 & 'G5 & I3 & 'C5 & D3 & H2 & K5 & 'J1 & B2 & A1 \end{array} . \qquad (3.2)$$

X_i is not equivalent to $*X_i$ ($i=1, 2$), but $X_i \sim X_i^*$ and $*X_i \sim *X_i{}^*$, so each of the two species in question comprises two transformation sets. The salient distinction

Table 1
The extent of present knowledge of DYRs with $k<v-1$ and $k<13$ (Nd denotes the number of nonisomorphic SBIBDs with parameters $\{v, k, \lambda\}$).

$k \times v$	λ	Nd	Whether μ is integral	Whether DYRs exist; references
3×7	1	1	Yes	No; [11]
4×13	1	1	Yes	Yes; [14]
4×7	2	1	Yes	Yes; [5, 16]
5×21	1	1	Yes	?
5×11	2	1	Yes	Yes; Section 3 of present paper
6×31	1	1	Yes	?
6×16	2	3	No	No
6×11	3	1	Yes	Yes; [16, 19] and Section 3 above
7×43	1	0	Yes	No
7×22	2	0	Yes	No
7×15	3	5	Yes	Yes; [12, 17, 21]
8×57	1	1	Yes	?
8×29	2	0	No	No
8×15	4	5	Yes	Yes; [18]
9×73	1	1	Yes	?
9×37	2	4	Yes	?
9×25	3	78	No	No
9×19	4	6	Yes	?
9×13	6	1	No	No
10×91	1	4	Yes	?
10×46	2	0	No	No
10×31	3	$\geqslant 38$	Yes	?
10×19	5	6	Yes	?
10×16	6	3	No	No
11×111	1	0	Yes	No
11×56	2	$\geqslant 5$	Yes	?
11×23	5	1103	Yes	Yes; [12, 21]
12×133	1	$\geqslant 1$	Yes	?
12×67	2	0	No	No
12×45	3	$\geqslant 2649$	No	No
12×34	4	0	No	No
12×23	6	1103	Yes	?

between the two species is that a member of the first contains three 2×5 Latin rectangles in the Youden square, whereas a member of the second contains three 2×4 Latin rectangles in the Youden square; in each of the DYRs (3.1) and (3.2), the position of one such Latin rectangle is indicated by the primes.

Discovery of these 5×11 DYRs with cycles of degree 3 soon led to the discovery of 6×11 DYRs with cycles of degree 3. The square incidence array of one of these DYRs is in Fig. 3, where each of the sets P and T is taken as $\{1,2,3,4,5,6\}$, where each element of P occurs once with the corresponding element of T and twice with each other member from T, and where the cycles use the permutation (1 2 3)(4 5 6). The species containing this 6×11 DYR comprises four transformation sets.

Present knowledge of $k\times v$ DYRs with $k<v-1$ and $k<13$ is summarized in Table 1, which draws attention to 12 parameter-sets for which the existence of DYRs has been neither disproved nor established. Subject to the restriction $k<\min\{v-1,13\}$, Table 1 covers all parameter-sets $\{v,k,\lambda\}$ for which λ is an integer; the parameter sets thus include some for which there is known to be no SBIBD with parameters $\{v,k,\lambda\}$. For Table 1, the values of Nd, the number of non-isomorphic SBIBDs with parameters $\{v,k,\lambda\}$, were taken from Mathon and Rosa [9].

All known DYRs have $m=1$ or $m=k-1$. We know of no parameter-set $\{v,k,\lambda\}$ for which a $k\times v$ double Youden rectangle could exist that had $m\neq 1$ or $k-1$.

References

[1] R.A. Bailey, Designs: mappings between structured sets, in: J. Siemons, ed., Surveys in Combinatorics (Cambridge Univ. Press, Cambridge, 1989) 22–51.

[2] R.A. Bailey and H.D. Patterson, A note on the construction of row-and-column designs with two replicates, J. Roy. Statist. Soc. B 53 (1991) 645–648.

[3] C. Christofi, Enumerating non-isomorphic 4×5 and 5×6 double Youden rectangles, Discrete Math. 125 (this Vol.) (1994) 129–135.

[4] G.M. Clarke, A second set of treatments in a Youden square design, Biometrics 19 (1963) 98–104.

[5] G.M. Clarke, Four-way balanced designs based on Youden squares with 5, 6 or 7 treatments, Biometrics 23 (1967) 803–812.

[6] R.A. Fisher and F. Yates, The 6×6 Latin squares, Proc. Cambridge Philosoph. Soc. 30 (1934) 492–507.

[7] G.H. Freeman, Some non-orthogonal partitions of 4×4, 5×5 and 6×6 Latin squares, Ann. Math. Statist. 37 (1966) 666–681.

[8] A. Hedayat, E.T. Parker and W.T. Federer, The existence and construction of two families of designs for two successive treatments, Biometrika 57 (1970) 351–355.

[9] R. Mathon and A. Rosa, Tables of parameters of BIBDs with $r\leqslant 41$ including existence, enumeration and resolvability results: an update, Ars Combin. 30 (1990) 65–96.

[10] H.W. Norton. The 7×7 squares, Ann. Eugenics 9 (1939) 269–307.

[11] D.A. Preece, Some row and column designs for two sets of treatments, Biometrics 22 (1966) 1–25.

[12] D.A. Preece, Some new balanced row-and-column designs for two non-interacting sets of treatments, Biometrics 27 (1971) 426–430.

[13] D.A. Preece, Some designs based on 11×5 Youden 'squares', Utilitas Math. 9 (1976) 139–146.

[14] D.A. Preece, Some partly cyclic 13×4 Youden 'squares' and a balanced arrangment for a pack of cards, Utilitas Math. 22 (1982) 255–263.

[15] D.A. Preece, Fifty years of Youden squares: a review, Bull. IMA 26 (1990) 65–75.

[16] D.A. Preece, Double Youden rectangles of size 6 × 11, Math. Sci. 16 (1991) 41–45.
[17] D.A. Preece, Enumeration of some 7 × 15 Youden squares and construction of some 7 × 15 double Youden rectangles, Utilitas Math. 41 (1992) 51–62.
[18] D.A. Preece, A set of double Youden rectangles of size 8 × 15, Ars Combin., to appear.
[19] D.A. Preece, Some 6 × 11 Youden squares and some 6 × 11 double Youden rectangles, unpublished.
[20] C.A.B. Smith and H.O. Hartley, The construction of Youden squares, J. Roy. Statist. Soc. B 10 (1948) 262–263.
[21] B.J. Vowden, Infinite series of double Youden rectangles, Discrete Math. 125 (this Vol.) (1994) 385–391.