Trees, tight-spans and point configurations

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ABSTRACT

Tight-spans of metrics were first introduced by Isbell in 1964 and rediscovered and studied by others, most notably by Dress in 1984, who gave them this name. Subsequently, it has been found that tight-spans can be defined for more general maps, such as directed metrics and distances, and more recently for diversities. In this paper, we show that all of these tight-spans, as well as some related constructions, can be defined in terms of point configurations. This provides a useful way in which to study these objects in a unified and systematic way. We also show that by using point configurations we can recover results concerning one-dimensional tight-spans for all of the maps we consider, as well as extending these and other results to more general maps such as symmetric and asymmetric maps.

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1. Introduction

Let V be a real vector space with some fixed basis B, and $\langle \cdot, \cdot \rangle$ denote the standard scalar product with respect to B (i.e., $\langle v, w \rangle = \sum_{b \in B} \lambda_b \mu_b$ if $v = \sum_{b \in B} \lambda_b b$, $w = \sum_{b \in B} \mu_b b$). A *point configuration* $\mathcal{A}$ in V is a finite subset of V; for technical reasons we shall always assume that the affine hull of any such configuration has codimension 1. Given a function $w : \mathcal{A} \to \mathbb{R}$, we define the *envelope* of $\mathcal{A}$ with respect to w to be the polyhedron

$$\mathcal{E}_w(\mathcal{A}) = \{x \in V : \langle a, x \rangle \geq -w \text{ for all } a \in \mathcal{A}\},$$

and the *tight-span* $\mathcal{T}_w(\mathcal{A})$ of $\mathcal{A}$ to be the union of the bounded faces of $\mathcal{E}_w(\mathcal{A})$. Tight-spans of point configurations were introduced in [15] for vertex sets of polytopes, as a tool for studying subdivisions of polytopes. Even so, they first appeared several years ago in a somewhat different guise.

More specifically, let X be a finite set, $V = \mathbb{R}^X$ be the vector space of functions $X \to \mathbb{R}$ and, for $x \in X$, let e_x denote the elementary function assigning 1 to x and 0 to all other $y \in X$. In addition, let D be a metric on X, that is, a symmetric map on $X \times X$ that vanishes on the diagonal and satisfies the triangle inequality. Then, as first remarked by Sturmfels and Yu [24], by setting $w(e_x + e_y) = -D(x, y)$, the tight-span $\mathcal{T}_w(\bar{\mathcal{A}}(X))$ of $\bar{\mathcal{A}}(X) = \{e_x + e_y : x, y \in X, x \neq y\}$ is nothing other than the *injective hull* of D that was first introduced by Isbell [20]. This object was subsequently rediscovered by Dress [8], who called it the *tight-span of D*, as well as by Chrobak and Larmore [4,5].

Since its discovery, the tight-span of a metric on a finite set has been intensively studied (see, e.g., [10,12] for overviews) and various related constructions have been introduced. These include tight-spans of *directed metrics* and *directed distances* [19], tight-spans of *polytopes* [15] and more recently the tight-span of a so-called *diversity* [3]. Note that, in contrast to the tight-span of a metric, it is not known whether or not all of these constructions are necessarily injective hulls (i.e., injective objects in some appropriate category), but for simplicity we shall still refer to them as tight-spans. Here we shall show that, as with metrics on finite sets, tight-spans of directed distances, diversities and some related maps can also all be described in terms of point configurations. This provides a useful way to systematically study these objects.

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More specifically, after presenting some preliminary results concerning point configurations in Sections 2 and 3, in Section 4 we shall show that the tight-span of a distance on X can be defined in terms of the configuration $\mathcal{A}(X) = \bar{\mathcal{A}}(X) \cup \{2e_x: x \in X\} = \{e_x + e_y: x, y \in X\}$ (Proposition 4.1). For Y a finite set with $X \cap Y = \emptyset$, let $\bar{\mathcal{B}}(X, Y) \subseteq \mathbb{R}^{X \cup Y}$ be the configuration of all points $e_x + e_y$ with $x \in X, y \in Y$ and $\mathcal{B}(X, Y) = \bar{\mathcal{B}}(X, Y) \cup \{2e_x: x \in X \cup Y\}$. We also show that the tight-span of a directed metric (distance) can be defined in terms of $\bar{\mathcal{B}}(X) = \bar{\mathcal{B}}(X, Y)$ or $\mathcal{B}(X) = \mathcal{B}(X, Y)$ if Y is considered to be a disjoint copy of X (Proposition 5.1). Using these point configurations, we will also extend this analysis to include arbitrary symmetric and even asymmetric maps (Section 5).

In Sections 6 and 7 we shall consider tight-spans of diversities, which were recently introduced in [3]. Using a relationship that we shall derive between metrics and diversities, in Section 7 we show that the tight-span of a diversity on X can be expressed in terms of the point configuration $\mathcal{C}(X) = \{\sum_{i \in A} e_i: A \in \mathcal{P}(X)\}$ (the vertices of a cube). Intriguingly, we also show that a very closely related object can also be associated with a diversity on X by considering the point configuration $\mathcal{A}(\mathcal{P}(X) \setminus \{\emptyset\})$ and that, for a special class of diversities (split system diversities) this object and the tight-span are in fact the same (Theorem 7.4).

In addition to providing some new insights on tight-spans using point configurations, we shall also pay special attention to one-dimensional tight-spans. These are important since, for example, they provide ways to generate phylogenetic trees and, more generally, phylogenetic networks (see, e.g., [9,11]). Indeed, a one-dimensional tight-span associated with a point configuration $\mathcal{A}$ and weight function w can also be regarded as a graph, with vertex set equal to that of $\mathscr{E}_w(\mathcal{A})$ and edge set consisting of precisely those pairs of vertices that both lie in a one-dimensional face of $\mathscr{E}_w(\mathcal{A})$. Since the union of bounded faces of an unbounded polyhedron is contractible (see, e.g., [18, Lemma 4.5]) it follows that a one-dimensional tight-span considered as a graph is, in fact, a tree.

The archetypal characterisation for one-dimensional tight-spans was first observed by Dress for metrics [8]:

Theorem 1.1 (*Tree Metric Theorem*). *The tight-span of a metric D on a finite set X is a tree if and only if D satisfies*

$$D(x, y) + D(u, v) \leq \max\{D(x, u) + D(y, v), D(x, v) + D(y, u)\}$$

for any $x, y, u, v \in X$.

In this paper we will use point configurations to give various conditions for when tight-spans are trees in more general settings (Theorems 4.5, 5.5 and 7.3). This allows us to recover and extend various theorems connecting tight-spans and trees that arise in the literature. We conclude in Section 8 with a discussion on some possible future directions.

2. Tight-spans and splits of point configurations

In this section we shall recall some definitions and results concerning tight-spans and splits of general point configurations, as well as giving some elementary properties of these objects that we shall use later. For details, we refer the reader to [15] and [14, Section 2]. First we present a characterisation of the tight-span as the set of minimal elements of the envelope of a configuration for configurations that satisfy certain conditions. These conditions are fulfilled by all of the configurations that we shall consider. When tight-spans (of metric spaces, but also of diversities) are considered and thought of in a non-polyhedral way, this characterisation is normally used as a definition instead.

Now, as in the introduction, let V be a finite-dimensional vector space with a fixed basis B. An element of $v \in V$ is called *positive* (with respect to B) if in its representation $v = \sum_{b \in B} \lambda_b^v b$ with respect to B one has $\lambda_b^v \geq 0$ for all $b \in B$. In particular, we have a partial order $\preceq$ on V defined by $v \preceq v'$ if and only if $\lambda_b^v \leq \lambda_b^{v'}$ for all $b \in B$ (or, equivalently, $v' - v$ is positive). For a subset $A \subseteq V$ an element $a \in A$ is called *minimal* if $a \preceq a'$ implies $a = a'$ for all $a' \in A$. The set A is called *bounded from below* if there exists some $M \in \mathbb{R}$ such that $\lambda_b^v \geq M$ for all $b \in B$ and $v \in A$.

Let now $e \in \mathbb{N}$ and $\varphi : V \to \mathbb{R}^e$ be a linear map and $b \in \mathbb{R}^e$. In general, for a polyhedron $P = \{x \in V: \varphi(x) \geq b\}$, an element $x \in P$ is contained in a bounded face of P if and only if there does not exist some (non-trivial) $r \in \{x \in V: \varphi(x) \geq 0\}$ (a *ray* of P) and some $\lambda \in \mathbb{R}_{>0}$ with $x - \lambda r \in P$. Note that P is bounded from below if and only if all rays of P are positive. We now give a characterisation of tight-spans using these concepts.

Lemma 2.1. *Let* $\mathcal{A} \subseteq V$ *be a configuration of positive points. Then* $\mathscr{T}_w(\mathcal{A})$ *is a subset of the set of minimal elements of* $\mathscr{E}_w(\mathcal{A})$. *If, additionally,* $\mathscr{E}_w(\mathcal{A})$ *is bounded from below, then* $\mathscr{T}_w(\mathcal{A})$ *equals the set of minimal elements of* $\mathscr{E}_w(\mathcal{A})$.

Proof. Let $x \in \mathscr{T}_w(\mathcal{A})$ be non-minimal, that is, there exist $b \in B$ and $\lambda \in \mathbb{R}_{>0}$ such that $x - \lambda b \in P$. By positivity, we have $\langle a, b \rangle \geq 0$ for all $a \in \mathcal{A}$ and hence b is a ray of $\mathscr{E}_w(\mathcal{A})$ contradicting the assumption $x \in \mathscr{T}_w(\mathcal{A})$.

Conversely, let $x \in \mathscr{E}_w(\mathcal{A}) \setminus \mathscr{T}_w(\mathcal{A})$, r be a ray of $\mathscr{E}_w(\mathcal{A})$ and $\lambda \in \mathbb{R}_{>0}$ be such that $x - \lambda r \in \mathscr{E}_w(\mathcal{A})$. Since $\mathscr{E}_w(\mathcal{A})$ is bounded from below, r is positive and hence $x - \lambda r \preceq x$, so x is not minimal. However, then we have $x - \lambda_b^r b \in P$ which implies that x is not minimal. $\square$

Another simple but useful observation is the following:

Lemma 2.2. *Let* $\mathcal{A} \subseteq V$ *be a point configuration,* $w : \mathcal{A} \to \mathbb{R}$ *a weight function,* $v \in V$, *and* $w' = w + \langle \cdot, v \rangle$. *Then* $\mathscr{T}_w(\mathcal{A}) = \mathscr{T}_{w'}(\mathcal{A}) + v$.
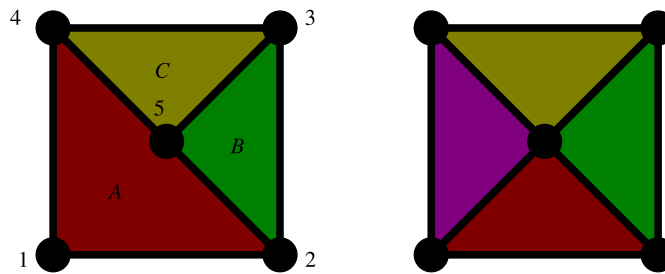
Fig. 2.1. Two collections of subconfigurations of the five points {1, 2, 3, 4, 5}, as indicated by the triangles. The collection of subconfigurations on the left is not a subdivision, as it violates (SD3): The intersection of the interior of convex hull of the edge {2, 4} (which is a face of the triangle $A = \{1, 2, 4\}$) and the convex hull of the edge {4, 5} is non-empty. In contrast, the collection of subconfigurations on the right is a subdivision of {1, 2, 3, 4, 5}. It contains four maximal faces (triangles) of cardinality 3 and eight edges of cardinality 2.

Proof. For all $x \in V$, we have

$$\langle a, x \rangle \geq -w'(a) = -(w(a) + \langle a, v \rangle) \Leftrightarrow \langle a, x + v \rangle \geq -w \quad \text{for all } a \in \mathcal{A}.$$

Hence

$$\mathscr{E}_{w'}(\mathcal{A}) = \{x \in V : \langle a, x + v \rangle \geq -w \text{ for all } a \in \mathcal{A}\}$$
$$= \{y - v \in V : \langle a, y \rangle \geq -w \text{ for all } a \in \mathcal{A}\} = \mathscr{E}_w(\mathcal{A}) - v.$$

Obviously, this equation carries over to the unions of the bounded faces, that is, the tight-spans. □

Tight-spans of a point configuration $\mathcal{A}$ equipped with certain weight functions are closely related so-called regular subdivisions of the configuration with respect to these weight functions, as we shall now recall. The convex hull of $\mathcal{A}$ is denoted by conv$\mathcal{A}$ and the relative interior of a set $A \subseteq V$ is denoted by relint$A$. For a point configuration $\mathcal{A}$ we call $F \subseteq \mathcal{A}$ a *face* of $\mathcal{A}$ if there exists a supporting hyperplane H of conv$\mathcal{A}$ such that $F = \mathcal{A} \cap H$. An *edge* of $\mathcal{A}$ is a face of size 2. A *subdivision* Σ of a point configuration $\mathcal{A}$ is a collection of subconfigurations of $\mathcal{A}$ satisfying the following three conditions (see [6, Section 2.3]):

- (SD1) If $F \in \Sigma$ and $\bar{F}$ is a face of F, then $\bar{F} \in \Sigma$.
- (SD2) conv$\mathcal{A} = \bigcup_{F \in \Sigma}$ conv$F$.
- (SD3) If $F, \bar{F} \in \Sigma, F \neq F'$, then relint(conv$F$) ∩ relint(conv$\bar{F}$) = ∅.

In addition, a subdivision is called a *triangulation* if all faces are simplices, that is, configurations formed by the vertices of a simplex. If $\mathcal{A}$ is a simplex the only possible subdivision of $\mathcal{A}$ is the trivial subdivision $\mathcal{P}(\mathcal{A})$ with sole maximal cell being $\mathcal{A}$ itself. See Fig. 2.1 for some examples illustrating these concepts.

Now, given a weight function $w : \mathcal{A} \rightarrow \mathbb{R}$, consider the *lifted polyhedron*:

$$\mathscr{L}_w(\mathcal{A}) = \text{conv}\{(w(a), a) : a \in \mathcal{A}\} + \mathbb{R}_{\geq 0}(1, 0) \subseteq \mathbb{R} \times V.$$

Then the *regular subdivision* $\Sigma_w(\mathcal{A})$ of $\mathcal{A}$ with respect to w is obtained by taking the configurations $\{b \in \mathcal{A} : (w(b), b) \in F\}$ for all lower faces F of $\mathscr{L}_w(\mathcal{A})$ (with respect to the first coordinate; by definition, these are exactly the bounded faces). So the elements of $\Sigma_w(\mathcal{A})$ are the projections of the bounded faces of $\mathscr{L}_w(\mathcal{A})$ to the last d coordinates. For more details see [6, Chapter 5].

We can now state the relationship between tight-spans and regular subdivisions of point configurations:

Proposition 2.3 (*Proposition 2.1 in [14]*). *The polyhedron $\mathscr{E}_w(\mathcal{A})$ is affinely equivalent to the polar dual of the polyhedron $\mathscr{L}_w(\mathcal{A})$. Moreover, the face poset of $\mathscr{T}_w(\mathcal{A})$ is anti-isomorphic to the face poset of the interior lower faces (with respect to the first coordinate) of $\mathscr{L}_w(\mathcal{A})$.*

We shall not define all the notions of this proposition, but note that, as a consequence, the (inclusion) maximal faces of the tight-span $\mathscr{T}_w(\mathcal{A})$ correspond to the (inclusion) minimal interior faces of $\Sigma_w(\mathcal{A})$. Here, a face of $\Sigma_w(\mathcal{A})$ is an *interior face* if it is not entirely contained in the boundary of conv$\mathcal{A}$. In particular, the structure of $\mathscr{T}_w(\mathcal{A})$ determines the structure of $\Sigma_w(\mathcal{A})$ and vice versa.

We now consider a useful concept, a so-called split of a point configuration, that is useful for understanding the structure of tight-spans of point configurations (see [15] for details on splits of polytopes and [14] for generalisations). A *split* T of a point configuration $\mathcal{A}$ is a subdivision of $\mathcal{A}$ which has exactly two maximal faces denoted by T_+ and T_- (see e.g. Fig. 2.2). Note that the affine hull of $T_+ \cap T_-$ is a hyperplane H_T (in the affine hull of $\mathcal{A}$), called the *split hyperplane* of T with respect to $\mathcal{A}$. Conversely, it follows from (SD2) and (SD3) that a hyperplane defines a split of $\mathcal{A}$ if and only if its intersection with the (relative) interior of $\mathcal{A}$ is nontrivial and it does not separate the endpoints of any edge of $\mathcal{A}$. It is straightforward to check that a split T of a point configuration $\mathcal{A}$ is a regular subdivision of $\mathcal{A}$. In particular, it follows that there is a lifting function w_T such that $\Sigma_{w_T}(\mathcal{A}) = T$; see [15, Lemma 3.5].
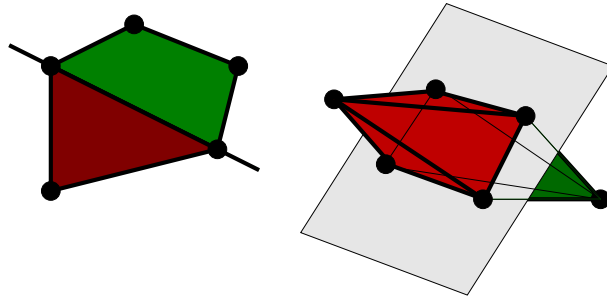
Fig. 2.2. To the left a split T of a configuration of five points forming a pentagon together with its split hyperplane (line) H_T. To the right a split of the point configuration $\bar{\mathcal{A}}(X)$ for $|X| = 4$ whose convex hull is an octahedron.

Now, a set $\mathcal{T}$ of splits of a point configuration $\mathcal{A}$ is called *compatible* if for all $T_1, T_2 \in \mathcal{T}$ the intersection of $H_{T_1} \cap H_{T_2}$ with the relative interior of conv$\mathcal{A}$ is empty. The following observation characterises when the tight-span of a point configuration is a tree in terms of compatible splits, and will be the key to proving some of our results. It is a slight generalisation of [15, Proposition 4.6].

Proposition 2.4. *Let $\mathcal{A}$ be a point configuration and $w : \mathcal{A} \to \mathbb{R}$ a weight function. Then the tight-span $\mathscr{T}_w(\mathcal{A})$ is a tree if and only if the subdivision $\Sigma_w(\mathcal{A})$ is a common refinement of compatible splits of $\mathcal{A}$.*

Another important result concerning splits that we shall also use later is the so-called Split Decomposition Theorem; see [15, Theorem 3.10] and [17, Theorem 2.2]. It states that each weight function w inducing a subdivision $\Sigma_w(\mathcal{A})$ of a point configuration $\mathcal{A}$ can be uniquely decomposed in a certain special way into a sum of a "split prime" weight function plus a sum of split weight functions. Here, we only need the following direct corollary of this fact:

Corollary 2.5. *Let $\mathcal{A}$ be a point configuration and $w : \mathcal{A} \to \mathbb{R}$ a weight function such that $\Sigma_w(\mathcal{A})$ is a common refinement of a set $\mathcal{T}$ of compatible splits of $\mathcal{A}$. Then there exists a function $\alpha : \mathcal{T} \to \mathbb{R}_{>0}$ such that*

$$w = \sum_{T \in \mathcal{T}} \alpha(T) w_T.$$

3. Splits of sets and point configurations

A key observation used in [17] to relate the tight-span of a metric on X with the point configuration $\bar{\mathcal{A}}(X)$ defined in the introduction is that a bipartition of the set X corresponds to a split of the point configuration $\bar{\mathcal{A}}(X)$ and vice versa. We now explain how this fact leads to some further relationships between bipartitions of X and splits of $\bar{\mathcal{A}}(X)$.

Let X be a finite set. A set $\{A, B\}$, with $A, B \subseteq X$ two non-empty, disjoint subsets of X, is a *partial split* of X, and a pair (A, B) is called a *directed partial split* of X. If in addition $A \cup B = X$, we call the bipartition $\{A, B\}$ of X a *split* of X and (A, B) a *directed split* of X. For a subset $C \subseteq X$ a (partial) split $\{A, B\}$ of X is said to *split* C if neither of the intersections $A \cap C$ or $B \cap C$ is empty.

Now, two partial splits $\{A, B\}$, $\{C, D\}$ of X are called *compatible* if one of the following four conditions is satisfied:

$$
\begin{aligned}
A &\subseteq C \quad \text{and} \quad B \supseteq D, \\
A &\subseteq D \quad \text{and} \quad B \supseteq C, \\
A &\supseteq C \quad \text{and} \quad B \subseteq D, \\
A &\supseteq D \quad \text{and} \quad B \subseteq C.
\end{aligned}
\tag{3.1}
$$

Note that this implies that there exist $E \in \{A, B\}$ and $F \in \{C, D\}$ such that $E \cap F = \emptyset$, a fact which is often used to define compatibility for splits (see, e.g., [23]). Two directed partial splits (A, B), (C, D) of X are called *compatible* if one of the following four conditions is satisfied:

$$
\begin{aligned}
A &\subseteq C \quad \text{and} \quad B \supseteq D, \\
A &\subseteq X \setminus C \quad \text{and} \quad B \supseteq X \setminus D, \\
A &\supseteq C \quad \text{and} \quad B \subseteq D, \\
X \setminus A &\supseteq C \quad \text{and} \quad X \setminus B \subseteq D,
\end{aligned}
\tag{3.2}
$$

Note that Condition (3.2) implies Condition (3.1) and that Conditions (3.1) and (3.2) are equivalent for (directed) splits. A set $\mathcal{S}$ of partial (directed) splits of X is called *compatible* if each two elements of $\mathcal{S}$ are compatible. Furthermore, a set $\mathcal{S}$ of directed splits of X is called *strongly* compatible if there exists an ordering $(A_1, B_1), \ldots, (A_l, B_l)$ of the elements of $\mathcal{S}$ such that $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ for all $1 \leq i < l$.

We now investigate the relation of these different kinds of compatibility with compatibility of splits of the point configurations defined in the introduction.

First, we consider the point configuration $\mathcal{A}(X)$. Splits of this point configuration were first studied by Hirai [18,17].

Proposition 3.1 (*Proposition* 4.4 *in* [17]). *Let X be a finite set. For a partial split* $\{A, B\}$ *of X the hyperplane given by the equation*

$$\sum_{i \in A} f(i) = \sum_{i \in B} f(i)$$

defines a split of the point configuration $\mathcal{A}(X)$. *Moreover, all splits of* $\mathcal{A}(X)$ *arise in this way.*

Compatibility of splits of sets and of point configurations and can be related as follows:

Proposition 3.2 (*Theorem* 2.3 *in* [18]). *A set* $\mathcal{S}$ *of partial splits of X is compatible if and only if* $\{T_S : S \in \mathcal{S}\}$ *is a compatible set of splits of* $\mathcal{A}(X)$.

We now consider the point configurations $\bar{\mathcal{B}}(X, Y)$, first describing their splits.

Proposition 3.3. *Let X and Y be two disjoint finite sets with* $|X|, |Y| \geq 2$ *and* $A \subsetneq X, B \subsetneq Y$ *be non-empty subsets of X and Y, respectively. Then the hyperplane given by*

$$\sum_{i \in A} f(i) = \sum_{j \in B} f(j) \tag{3.3}$$

defines a split of the point configuration $\bar{\mathcal{B}}(X, Y)$. *Moreover, all splits of* $\bar{\mathcal{B}}(X, Y)$ *arise in this way.*

Note that the partial split of X obtained by taking the complements of A and B yields the same split of $\bar{\mathcal{B}}(X, Y)$ (but no other split yields this split of $\bar{\mathcal{B}}(X, Y)$).

Proof. First we remark that for all non-empty $A \subsetneq X, B \subsetneq Y$ the function $f \in \mathbb{R}^{X \cup Y}$ defined by

$$f(i) = \begin{cases} \dfrac{1}{2|A|}, & \text{if } i \in A, \\[2mm] \dfrac{1}{2|X \setminus A|}, & \text{if } i \in X \setminus A, \\[2mm] \dfrac{1}{2|B|}, & \text{if } i \in B, \\[2mm] \dfrac{1}{2|Y \setminus B|}, & \text{if } i \in Y \setminus B, \end{cases}$$

is in the interior of conv$\bar{\mathcal{B}}(X, Y)$ since $0 < f(i) < 1$ for all $i \in X \cup Y$ and $\sum_{i \in X \cup Y} f(i) = 2$. It is also in the hyperplane defined by Eq. (3.3), since $\sum_{i \in A} f(i) = 1, \sum_{i \in B} f(i) = 1$. Hence all those hyperplanes meet the interior of conv$\bar{\mathcal{B}}(X, Y)$.

By the definition of $\bar{\mathcal{B}}(X, Y)$, two vertices $u = (f_1, f_2), v = (g_1, g_2) \in \mathbb{R}^X \times \mathbb{R}^Y$ of $\bar{\mathcal{B}}(X, Y)$ are connected by an edge if and only if $f_1 = g_1$ or $f_2 = g_2$. So, by going from u to v along an edge, the value on at most one side of Eq. (3.3) changes by at most 1. Since for all elements of $\bar{\mathcal{B}}(X, Y)$ all values occurring in Eq. (3.3) are integers, the corresponding hyperplane does not cut an edge of $\bar{\mathcal{B}}(X, Y)$ and hence defines a split of $\bar{\mathcal{B}}(X, Y)$.

Now, let $H = \{f \in \mathbb{R}^{X \cup Y} : \sum_{i \in X \cup Y} \alpha_i f(i) = 0\}$ define a split of $\bar{\mathcal{B}}(X, Y)$ for some $\alpha_i \in \mathbb{R}$. We can assume that the first non-zero α_i is equal to 1. However, since the matrix of vertices of a product of simplices is totally unimodular (i.e., all the determinants of all square submatrices are in $\{0, 1, -1\}$ [2]), all other non-zero α_j have to be equal to ± 1. Also, note that the product of simplices conv$\bar{\mathcal{B}}(X, Y)$ has $|Y|$ facets that are isomorphic to $|X|$-dimensional simplices. The hyperplane H has to meet at least one of these facets non-trivially. Since simplices have no splits, H has to define a face of this facet F. So we can conclude that we cannot have $\alpha_i = -\alpha_j$ for $i, j \in X$ since H would then meet the interior of F. The only remaining possibility for H is therefore Eq. (3.3) for arbitrary A and B. Since for $A = \emptyset, B = \emptyset, A = X$, or $B = Y$ this hyperplane would not intersect the relative interior of conv$\bar{\mathcal{B}}(X, Y)$, the proof is complete. □

In view of the last result, a split of the point configuration $\bar{\mathcal{B}}(X, Y)$ is defined by two sets $\emptyset \neq A \subsetneq X$ and $\emptyset \neq B \subsetneq Y$, and this representation is unique up to simultaneously taking the complements of A and B. Compatibility can now be characterised as follows.

Proposition 3.4. *Let T be a split of* $\bar{\mathcal{B}}(X, Y)$ *defined by* $A \subsetneq X$ *and* $B \subsetneq Y$, *and U a split of* $\bar{\mathcal{B}}(X, Y)$ *defined by* $C \subsetneq X$ *and* $D \subsetneq Y$. *Then T and U are compatible if and only if one of the following conditions is satisfied:*

$$\begin{aligned} A \subseteq C \quad &\text{and} \quad B \supseteq D, \\ A \subseteq X \setminus C \quad &\text{and} \quad B \supseteq Y \setminus D, \\ A \supseteq C \quad &\text{and} \quad B \subseteq D, \text{ or} \\ X \setminus A \supseteq C \quad &\text{and} \quad Y \setminus B \subseteq D. \end{aligned} \tag{3.4}$$

Proof. We define the sets

$$A_1 = A \setminus C, \qquad A_2 = C \setminus A, \qquad A_3 = A \cap C, \qquad A_4 = X \setminus (A \cup C),$$
$$B_1 = B \setminus D, \qquad B_2 = D \setminus B, \qquad B_3 = B \cap D, \qquad B_4 = Y \setminus (B \cup D);$$

and set

$$X_i = \sum_{i \in A_i} f(i), \qquad Y_i = \sum_{i \in B_i} f(i).$$

Then the hyperplanes for T and U are defined by

$$X_1 + X_3 = Y_1 + Y_3 \quad \text{and} \quad X_2 + X_3 = Y_2 + Y_3, \tag{3.5}$$

respectively. If we subtract these two equations, we get

$$X_1 - X_2 = Y_1 - Y_2. \tag{3.6}$$

We first prove that Condition (3.4) is sufficient for compatibility of T and U. So, suppose T and U are not compatible, $A \subseteq C$, and $D \subseteq B$. This implies that there exists some $x \in H_T \cap H_U$ in the interior of $\text{conv}\bar{\mathcal{B}}(X, Y)$ with $X_1 = Y_2 = 0$. From this, Eq. (3.6) and the fact that $\bar{\mathcal{B}}(X, Y) \subseteq \mathbb{R}^{X \cup Y}_{\geq 0}$, we can further conclude that $X_2 = Y_1 = 0$. So $x_i = 0$ for all $i \in A_2 \cup B_1$. However, if $x_i = 0$ for some $i \in X \cup Y$, the point x would be contained in the boundary facet of $\text{conv}\bar{\mathcal{B}}(X, Y)$ defined by $x_i = 0$. So A_2 and B_1 are empty, and hence $T = U$. The second case follows in a similar manner.

For the necessity, assume that (3.4) does not hold. This is equivalent to

$$A_1 \neq \emptyset \quad \text{or} \quad B_2 \neq \emptyset, \qquad A_3 \neq \emptyset \quad \text{or} \quad B_4 \neq \emptyset, \qquad A_2 \neq \emptyset \quad \text{or} \quad B_1 \neq \emptyset, \quad \text{and} \quad A_4 \neq \emptyset \quad \text{or} \quad B_3 \neq \emptyset. \tag{3.7}$$

We now consider five possible cases, depending on the number of sets A_j, B_j which are empty. In each case we will give a point $x \in \text{relint}(\text{conv}\bar{\mathcal{B}}(X, Y)) \cap H_S \cap H_T$. This will be done by assigning values in the interval $(0, 1)$ to all X_j, Y_j for which A_j, B_j, respectively, are non-empty such that Eq. (3.6) holds and $\sum X_j = \sum Y_j = 1$. The explicit values of $f \in \mathbb{R}^{X \cup Y}$ are then obtained by setting $f(i) = \frac{X_j}{|A_j|}, f(i) = \frac{Y_j}{|B_j|}$, for $i \in A_j, i \in B_j$, respectively.

Case 1: None of the sets A_j, B_j is empty. Then we simply set $X_j, Y_j = \frac{1}{4}$ for all $j \in \{1, 2, 3, 4\}$.

Case 2: One of the sets A_j, B_j is empty. We assume without loss of generality that $A_1 = \emptyset$. Then we set $X_3 = \frac{1}{2}, X_4 = Y_2 = \frac{3}{8}, Y_1 = Y_3 = \frac{1}{4}$, and $X_2 = Y_4 = \frac{1}{8}$.

Case 3: Two of the sets A_j, B_j are empty. As in Case 2, we assume that one of these sets is A_1. Using (3.7), and taking into account that neither A, B, C, D nor their complements (in X and Y, respectively) can be empty, we get the following possibilities:

- $A_1 = A_2 = \emptyset$: Set $X_3 = X_4 = \frac{1}{2}$ and $Y_i = \frac{1}{4}$ for all $i \in \{1, 2, 3, 4\}$.
- $A_1 = B_1 = \emptyset$: Set $X_3 = Y_3 = \frac{1}{2}$ and $X_2 = X_4 = Y_2 = Y_4 = \frac{1}{4}$.
- $A_1 = B_3 = \emptyset$: Set $X_4 = Y_2 = \frac{1}{2}$ and $X_2 = X_3 = Y_1 = Y_4 = \frac{1}{4}$.
- $A_1 = B_4 = \emptyset$: Set $X_3 = Y_2 = \frac{1}{2}$ and $X_2 = X_4 = Y_1 = Y_3 = \frac{1}{4}$.

Case 4: Three of the sets A_j, B_j are empty. We again assume that A_1 is one of the sets. There remain three possibilities:

- $A_1 = A_2 = B_3 = \emptyset$: Set $X_4 = \frac{2}{3}$ and $X_3 = Y_1 = Y_2 = Y_4 = \frac{1}{3}$.
- $A_1 = A_2 = B_4 = \emptyset$: Set $X_3 = \frac{2}{3}$ and $X_4 = Y_1 = Y_2 = Y_3 = \frac{1}{3}$.
- $A_1 = B_3 = B_4 = \emptyset$: Set $Y_2 = \frac{2}{3}$ and $X_2 = X_3 = X_4 = Y_1 = \frac{1}{3}$.

Case 5: Four of the sets A_j, B_j are empty. By assuming that A_1 is one of them, this yields $A_1 = A_2 = B_3 = B_4 = \emptyset$. Set $X_3 = X_4 = Y_1 = Y_2 = \frac{1}{2}$. $\quad\square$

Note that in the special case where Y is a disjoint copy of X, a directed partial split $S = (A, B)$ of X gives rise to a split T_S of $\bar{\mathcal{B}}(X)$, namely the one defined by A and B. In general, not all splits of $\bar{\mathcal{B}}(X)$ arise in this way, since we assume that A and B are disjoint. Proposition 3.4 then gives us the following.

Corollary 3.5. *A collection $\mathcal{S}$ of directed partial splits of X is compatible if and only if $\{T_S : S \in \mathcal{S}\}$ is a compatible set of splits for $\bar{\mathcal{B}}(X)$.*

Note that a characterisation of the splits of the point configuration $\mathcal{C}(X)$ (the vertices of the cube) and their compatibility is given in [13, Propositions 3.15 and 3.16]; we refrain from stating or proving it here since we will not use it later.

4. Tight-spans of symmetric maps, distances and metrics

We begin by recalling some key definitions. A symmetric map $D : X \times X \to \mathbb{R}$ with $D(x, x) = 0$ for all $x \in X$ and $D(x, y) \geq 0$ for all $x, y \in X$ is called a *distance* (or dissimilarity map) on X. It is called a *metric* on X if it additionally satisfies $D(x, y) + D(y, z) \geq D(x, y)$ for all $x, y, z \in X$ (the *triangle inequality*).

Now, consider the vector space $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$ of all functions $X \to \mathbb{R}$ with the natural basis $\{e_x : x \in X\}$ where $e_x \in \mathbb{R}^X$ denotes the function sending x to 1 and all other elements of X to 0. Then, the tight-span T_D of a symmetric map $D : X \times X \to \mathbb{R}$ is defined to be the set of all minimal elements of the polyhedron

$$P_D = \left\{ f \in \mathbb{R}^X : f(x) + f(y) \geq D(x, y) \text{ for all } x, y \in X \right\}$$

(see [18, Section 2.3]). Note that if D is a metric, then T_D corresponds to the tight-span of the metric D as defined by Isbell [20] and Dress [8].

Now, given a symmetric map $D : X \times X \to \mathbb{R}$ we define a weight function $w^D : \mathcal{A}(X) \to \mathbb{R}$ on the point configuration $\mathcal{A}(X)$ via $w^D(e_x + e_y) = -D(x, y)$. The following proposition is the key to deriving the relation between tight-spans of symmetric functions and tight-spans of the point configuration $\mathcal{A}(X)$. In the special case where D is a metric, this was the observation of Sturmfels and Yu [24] mentioned in the introduction.

Proposition 4.1. *Let $D : X \times X \to \mathbb{R}$ be a symmetric function. Then we have:*

(a) $P_D = \mathscr{E}_{w^D}(\mathcal{A}(X))$, *and*
(b) $T_D = \mathscr{T}_{w^D}(\mathcal{A}(X))$.

Proof. (a) We have

$$\begin{aligned}
P_D &= \{f : X \to \mathbb{R} : f(x) + f(y) \geq D(x, y) \text{ for all } x, y \in X\} \\
&= \left\{f : X \to \mathbb{R} : \langle e_x + e_y, f \rangle \geq D(x, y) \text{ for all } x, y \in X \right\} \\
&= \left\{f : X \to \mathbb{R} : \langle a, f \rangle \geq -w^D(a) \text{ for all } a \in \mathcal{A}(X) \right\} \\
&= \mathscr{E}_{w^D}(\mathcal{A}(X)).
\end{aligned}$$

(b) Obviously, $\mathcal{A}(X)$ is positive and $\mathscr{E}_{w^D}(\mathcal{A}(X))$ is bounded from below since we have the inequalities $2e_x \geq -D(x, x)$ for all $x \in X$. Thus, the claim follows from (a) and Lemma 2.1. □

We will now see that tight-spans of general symmetric maps and tight-spans of distances are essentially the same in the sense that they only differ by a simple shift.

Proposition 4.2. *Let $D : X \times X \to \mathbb{R}$ be a symmetric function, D' defined via*

$$D'(x, y) = D(x, y) - \frac{1}{2} \left(D(x, x) + D(y, y) \right),$$

and $v : X \to \mathbb{R}$ defined by $v(x) = \frac{1}{2}D(x, x)$. Then $T_D = T_{D'} + v$.

Proof. Our definitions imply that for $a = e_x + e_y \in \mathcal{A}(X)$, we have

$$w^{D'}(a) = -D(x, y) + \frac{1}{2} \left(D(x, x) + D(y, y) \right) = w^D(a) + \langle a, v \rangle.$$

Hence the claim follows from Lemma 2.2. □

Obviously, for an arbitrary symmetric map $D : X \times X \to \mathbb{R}$ we have $D'(x, x) = 0$ for all $x \in X$. However, D' is not necessarily a distance, since D' need not to be positive, even if D is. Even so, the following lemma shows that the negative values of D can be essentially ignored when looking at the tight-span:

Lemma 4.3. *Let $D : X \times Y \to \mathbb{R}$ be a symmetric function with $D(x, x) = 0$ for all $x \in X$, and D_+ be defined by $D_+(x, y) = \max(0, D(x, y))$ for all $x, y \in X$. Then $T_D = T_{D_+}$.*

Proof. For two symmetric maps $E, F : X \to \mathbb{R}$ with $E \geq F$ (pointwise) one directly sees that $P_E \subseteq P_F$, so $P_{D_+} \subseteq P_D$. On the other hand, for all $f \in P_D$ and $x \in X$, we have $2f(x) \geq D(x, x) = 0$, hence $f(x) + f(y) \geq 0 \geq \max(0, D(x, y)) = D_+(x, y)$ for all $x, y \in X$. This implies $P_D \subseteq P_{D_+}$. In consequence, we have $P_D = P_{D_+}$ and hence $T_D = T_{D_+}$. □

So, by Proposition 4.2 and Lemma 4.3, we get:

Corollary 4.4. *For any symmetric map $D : X \times X \to \mathbb{R}$ there exists a distance map (namely $(D')_+ : X \times X \to \mathbb{R}$) and some $v \in \mathbb{R}^X$ such that $T_D = T_{(D')_+} + v$.*

We now turn to one-dimensional tight-spans: Given a tree $T = (V, E)$, an edge-length function $\alpha : E \to \mathbb{R}$ and a family $\mathcal{F} = \{F_x : x \in X\}$ of subtrees of T, we can define a distance D on X by setting $D(x, y) = \min\{D_T(x, y) : x \in F_x, y \in F_y\}$. A distance arising in that way is called a *distance between subtrees of a tree*. Those distances can also be characterised in terms of partial splits as follows. Given a partial split $P = \{A, B\}$ of X, a corresponding distance on X is then defined by

$$d_P(i, j) = \begin{cases} 1, & \text{if } i \in A, \ j \in B \text{ or } i \in B, \ j \in A, \\ 0, & \text{else}, \end{cases} \tag{4.1}$$

for all $i, j \in X$. Furthermore, for a set $\mathcal{P}$ of partial splits of X and a function $\alpha \to \mathbb{R}_{>0}$, we define $d_{(\mathcal{P}, \alpha)} = \sum_{P \in \mathcal{P}} \alpha(P) d_P$.

Using the relation between splits of the point configuration $\mathcal{A}(X)$ and partial splits of X (Propositions 3.1 and 3.2), we obtain the following result characterising when the tight-span of a distance is a tree. This can also be inferred from [18, Theorem 2.3] together with Proposition 4.1.

Theorem 4.5. *Let D be a distance. Then the following are equivalent:*

(a) *The tight-span $\mathcal{T}_{w^D}(\mathcal{A}(X))$ is a tree.*
(b) *D is a distance of weights between subtrees of a tree.*
(c) *There exists a compatible set $\mathcal{P}$ of partial splits of X and $\alpha : \mathcal{P} \to \mathbb{R}_{>0}$ such that $D = d_{(\mathcal{P}, \alpha)}$.*

5. Tight-spans of non-symmetric maps

Hirai and Koichi [19] introduced the concept of tight-spans of directed distances in order to study certain multicommodity flow problems. In this section, we will show that these tight-spans can also be considered as tight-spans of point configurations. We will also generalise these results to general non-symmetric maps.

Let X and Y be finite sets with $X \cap Y = \emptyset$ and $D : X \times Y \to \mathbb{R}$ be an (arbitrary) map. We define the polyhedra

$$\Pi_D = \{f : X \cup Y \to \mathbb{R} : f(x) + f(y) \geq D(x, y) \text{ for all } x \in X, \ y \in Y\}, \quad \text{and}$$
$$P_D = \Pi_D \cap \mathbb{R}_{\geq 0}^{X \cup Y}.$$

We let Θ_D and T_D denote the sets of minimal elements of Π_D and P_D, respectively, and we call the set T_D the *tight-span* of D.

Now, recall that $\bar{\mathcal{B}}(X, Y) \subseteq \mathbb{R}^{X \cup Y}$ is defined to be the configuration of all points $e_x + e_y$ with $x \in X, y \in Y$ and also that $\mathcal{B}(X, Y) = \bar{\mathcal{B}}(X, Y) \cup \{2e_x : x \in X \cup Y\}$. Note that $\mathcal{B}(X, Y), \bar{\mathcal{B}}(X, Y) \subseteq \mathcal{B}(X \cup Y)$ and that $\text{conv} \mathcal{B}(X, Y)$ is the simplex $\text{conv}\{2e_x : x \in X \cup Y\}$, whereas all elements of $\bar{\mathcal{B}}(X, Y)$ are vertices of $\text{conv} \bar{\mathcal{B}}(X, Y)$, which is the product of a $(|X| - 1)$- and a $(|Y| - 1)$-dimensional simplex.

We now state an analogue of Proposition 4.1 for arbitrary maps $D : X \times Y \to \mathbb{R}$ (whose proof is similar and therefore omitted). With such a map D associate the weight functions $\bar{w}^D : \bar{\mathcal{B}}(X, Y) \to \mathbb{R}$ given by $\bar{w}^D(e_x + e_y) = -D(x, y), x \in X, y \in Y$ and $w^D : \mathcal{B}(X, Y) \to \mathbb{R}$ given by

$$w^D(a) = \begin{cases} \bar{w}^D(a), & \text{if } a = e_x + e_y, \ x \in X, \ y \in Y, \\ 0, & \text{else}. \end{cases}$$

Proposition 5.1. *Let $D : X \times Y \to \mathbb{R}$. Then*

(a) $\Pi_D = \mathcal{E}_{\bar{w}^D}(\bar{\mathcal{B}}(X, Y))$,
(b) $\Theta_D = \mathcal{T}_{\bar{w}^D}(\bar{\mathcal{B}}(X, Y))$,
(c) $P_D = \mathcal{E}_{w^D}(\mathcal{B}(X, Y))$, *and*
(d) $T_D = \mathcal{T}_{w^D}(\mathcal{B}(X, Y))$.

It was shown in [7, Lemma 22] that Θ_D is piecewise-linear isomorphic to the tropical polytope (see, e.g., Develin and Sturmfels [7]) with vertex set $\{(D(x, y))_{y \in Y} : x \in X\}$. Similarly, using a proof like that of Lemma 4.3, we have:

Lemma 5.2. *Let $D : X \times Y \to \mathbb{R}$ and D_+ defined by $D_+(x, y) = \max(0, D(x, y))$ for all $x \in X, y \in Y$. Then $T_D = T_{D_+}$.*

Of particular interest is the case where Y is a disjoint copy of X, that is, D is a (not necessarily symmetric) function from $X \times X$ to $\mathbb{R}$. We denote the two distinct copies of X by X_l and X_r and set $X_d = X_l \cup X_r$. If in this case $D \geq 0$ and $D(x, x) = 0$ for all $x \in X$, then D is called a *directed distance*; if D also satisfies the triangle inequality, it is called a *directed metric*. These were considered by Hirai and Koichi [19]. In fact, our definitions of Π_D, Θ_D, P_D and T_D generalise their definitions of these objects to general maps.

As in the case of symmetric maps, it can be deduced from Lemma 2.2 that a map $D : X \times X \to \mathbb{R}$ can be transformed into a map with $D(x, x) = 0$ for all $x \in X$ by shifting it by a vector $v \in \mathbb{R}^{X_d}$ defined by $v(x) = 1/2D(x, x)$ for $x \in X_d$ (here x is identified with its copy in X). Together with Lemma 5.2 this leads to the following corollary.

Corollary 5.3. *For each map $D : X \times X \to \mathbb{R}$ there exists a directed distance D' on X and some $v \in \mathbb{R}^{X_d}$ such that $T_D = T_{D'} + v$.*

With a directed distance D on X we associate a (symmetric) distance $D^u : X_d \times X_d \to \mathbb{R}_{\geq 0}$ given by setting

$$D^u(x, y) = \begin{cases} D(x, y), & \text{if } x \in X_l \text{ and } y \in X_r \text{ or } x \in X_r \text{ and } y \in X_l, \\ 0, & \text{else}. \end{cases}$$

It follows that $P_{D^u} = P_D$ and $T_{D^u} = T_D$. In particular, it follows that tight-spans of undirected distances are just tight-spans of certain special directed distances.

We no turn again to one-dimensional tight-spans. In [19], Hirai and Koichi give explicit conditions on D for when T_D and Θ_D are trees. We now give new and (we feel) somewhat conceptually simpler proofs of these results using point configurations. We begin by recalling some basic definitions from [19, Section 3].

An *oriented tree* $\Gamma = (V(\Gamma), E(\Gamma))$ is a directed graph whose underlying undirected graph is a tree. For an oriented tree Γ and an edge-length function $\alpha : E(\Gamma) \to \mathbb{R}_{\geq 0}$, we define a directed distance $D_{\Gamma,\alpha}$ on $V(\Gamma)$ by setting $D_{\Gamma,\alpha}(x, y) = \sum_{e \in \vec{P}(x,y)} \alpha(e)$ for all $x, y \in V(\Gamma)$, where $\vec{P}(x, y)$ is the set of all edges on the unique (undirected) path from x to y that are directed from x to y. For $A, B \subseteq V(\Gamma)$ we set $D_{\Gamma,\alpha}(A, B) = \min\{D_{\Gamma,\alpha}(a, b): a \in A, b \in B\}$. For an undirected distance $D : X \times X \to \mathbb{R}$ and a family $\mathcal{F} = \{F_x: x \in X\}$ of subtrees of Γ, we say that $(\Gamma, \alpha, \mathcal{F})$ is an *oriented tree realisation* of D if

$$D(x, y) = D_{\Gamma,\alpha}(F_x, F_y) \quad \text{for all } x, y \in X.$$

Now, let $S = (A, B)$ be a directed partial split of X. We define a directed distance D_S on X by setting

$$D_S(x, y) = \begin{cases} 1, & \text{if } x \in A \text{ and } y \in B, \\ 0, & \text{else.} \end{cases}$$

Note that D_S is a directed metric if and only if S is a directed split. Also note that the oriented tree Γ consisting of a single edge (v, w) with weight λ and

$$F_x = \begin{cases} \{v\} & \text{if } x \in A, \\ \{w\} & \text{if } x \in B, \\ \Gamma & \text{else,} \end{cases}$$

for all $x \in X$, is an oriented tree realisation of D_S.

Now, let D be an arbitrary directed distance with oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$. For each $e = (a, b) \in E(\Gamma)$ we define a directed partial split $S_e = (A_e, B_e)$ of X, where A_e is the set of all $x \in X$ whose subtree $F_x \in \mathcal{F}$ is entirely contained in the same connected component the forest obtained from Γ by deleting the edge e as a, and B_e is defined in the same way for b. It easily follows that $D = \sum_{e \in E(\Gamma)} \alpha(e)D_{S_e}$.

Proposition 5.4. *Let $\mathcal{S}$ be a set of directed partial splits, $\alpha : \mathcal{S} \to \mathbb{R}_{>0}$ a function, and D the directed distance defined by*

$$D = \sum_{S \in \mathcal{S}} \alpha(S)D_S.$$

Then:

(a) *D has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ where each $F \in \mathcal{F}$ is a directed path if and only if $\mathcal{S}$ is compatible.*
(b) *D has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that Γ is a directed path if and only if $\mathcal{S}$ is strongly compatible.*

Proof. (a) Let e and f be two edges of Γ and S_e and S_f the associated directed partial splits. We have to show that S_e and S_f are compatible. Suppose first that in any undirected path in Γ containing e and f these two edges are directed in the same direction. (Since Γ is a tree, this is the case if there exists some undirected path with this property.) Then (possibly after exchanging e and f) we have $A_e \subseteq A_f$ and $B_f \subseteq B_e$, which implies that S_e and S_f are compatible by the first condition of (3.2).

Now, suppose that in any undirected path in Γ containing e and f these two edges are directed in the different directions. Because each $F \in \mathcal{F}$ is a directed path, this implies that there does not exist an $F \in \mathcal{F}$ that contains e as well as f. Hence, we get (again after a possible exchange of e and f) $X \setminus A_e \subseteq A_f$ and $B_e \subseteq X \setminus B_f$, implying that S_e and S_f are compatible by the third condition of (3.2). On the other hand, given a compatible set of directed partial splits of X, we can construct a corresponding tree Γ with a family of subtrees $\mathcal{F}$ that are directed paths such that $(\Gamma, \alpha, \mathcal{F})$ is an oriented tree realisation of D.

(b) Follows directly from the definition. □

We now characterise directed distances whose tight-spans are trees.

Theorem 5.5 (*Theorems* 3.1 *and* 3.2 *in* [19])**.** *Let D be a directed distance on X. Then:*

(a) *Θ_D is a tree if and only if D has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that each $F \in \mathcal{F}$ is a directed path.*
(b) *T_D is a tree if and only if D has an oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ such that Γ is a directed path.*

Proof. (a) Let D be a directed distance on X such that D has on oriented tree realisation $(\Gamma, \alpha, \mathcal{F})$ in which each $F \in \mathcal{F}$ is a directed path. By Proposition 5.4(a), this implies that there exists a set $\mathcal{S}$ of directed partial splits of X such that

$$D = \sum_{S \in \mathcal{S}} \alpha(S)D_S.$$

For each $S \in \mathscr{S}$ and $a \in \bar{\mathcal{B}}(X)$ we have

$$\bar{w}^{D_S}(a) = \begin{cases} -1, & \text{if } a = e_x + e_y \text{ and } x \in A \text{ and } y \in B, \\ 0, & \text{else}, \end{cases}$$

and so it immediately follows that $\bar{w}^{D_S}$ defines the split T_S of $\bar{\mathcal{B}}(X)$. By Corollary 3.5, the set $\{T_S: S \in \mathscr{S}\}$ is a compatible set of splits for $\bar{\mathcal{B}}(X)$, and so the subdivision $\Sigma_{\bar{w}^D}(\bar{\mathcal{B}}(X))$ is a common refinement of compatible splits. Proposition 2.4 now implies that $\Theta_D = \mathscr{T}_{\bar{w}^D}(\bar{\mathcal{B}}(X))$ is a tree.

Conversely, let D be a directed distance on X such that Θ_D is a tree. By Proposition 2.4 there exists a compatible set $\mathcal{T}$ of splits of $\bar{\mathcal{B}}(X)$ such that $\Sigma_{\bar{w}^D}(\bar{\mathcal{B}}(X))$ is the common refinement of all splits in $\mathcal{T}$. Since D is a directed distance, we have $w^D(e_x + e_x) = 0$ for all $x \in X$, which implies that each $T \in \mathcal{T}$ is defined by two disjoint subsets of X. Hence there exists a directed partial split S with $T = T_S$. Let $\mathscr{S}$ be the set of all such splits, which is compatible by Corollary 3.5. By Corollary 2.5, there exists $\alpha_S \in \mathbb{R}_{\geq 0}$ such that $\bar{w}^D = \sum_{T \in \mathcal{T}} \alpha_S w^{D_S}$, so $D = \sum_{S \in S} \alpha_S D_S$ and the claim now follows from Proposition 5.4 (a).

(b) The splits of the point configuration $\mathcal{B}(X)$ are given by partial splits of the set X_d. However, given a directed distance D, by definition, the value $w^D(e_x + e_y)$ can only be non-zero if $x \in X_l$ and $y \in X_r$. This implies that the only possible partial splits $\{A, B\}$ of X_d that may occur are those with $A \subseteq X_l$ and $B \subseteq X_r$ (or vice versa) and $A \cap B = \emptyset$ (where A and B are considered as subsets of X). The splits of this type are in bijection with directed partial splits (A, B) of X.

Now, given two such directed partial splits (A, B), (C, D) of X, the corresponding splits of $\mathcal{B}(X)$ are compatible if and only if

$$A \subseteq C \quad \text{and} \quad B \supseteq D, \quad \text{or} \quad A \supseteq C \quad \text{and} \quad B \subseteq D.$$

Proposition 5.4 (b) now implies the claim in a similar way as Proposition 5.4 (a) implies Part (a). □

In the case where D is a directed metric, that is, satisfies the triangle inequality, it is obvious that in each directed tree realisation $(\Gamma, \alpha, \mathcal{F})$ of D all of the subtrees F_x are single vertices. In this special case, given a directed tree Γ, an edge length function α and a map $\varphi : X \to V(\Gamma)$, we call $(\Gamma, \alpha, \varphi)$ an *oriented tree realisation* of D if and only if $(\Gamma, \alpha, \{\varphi(x): x \in X\})$ is an oriented tree realisation of D.

Corollary 5.6. *Let D be a directed metric on X. Then we have:*

(a) *Θ_D is a tree if and only if D has an oriented tree realisation.*
(b) *T_D is a tree if and only if D has an oriented tree realisation $(\Gamma, \alpha, \varphi)$ such that Γ is a directed path.*

Note that this result concerning oriented tree realisations is somewhat related to those presented in [21,22]. However it differs since we take only one edge for each undirected edge of a tree rather than two.

6. Diversities as distances

For a finite set Y, we denote the powerset of Y by $\mathcal{P}(Y)$ and let $\mathcal{P}^\star(Y) = \mathcal{P}(Y) \setminus \{\emptyset\}$. A *diversity* on Y is a function $\delta : \mathcal{P}(Y) \to \mathbb{R}$ satisfying:

(D1) $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$ for all $A, C \in \mathcal{P}(Y)$ and $B \in \mathcal{P}^\star(Y)$, and
(D2) $\delta(A) = 0$ for all $A \in \mathcal{P}(Y)$ with $|A| \leq 1$.

Diversities were introduced by Bryant and Tupper in [3].[1] In this section, we will show that a diversity can also be thought of as a distance on $\mathcal{P}(Y)$. This will allow us to apply the results in Section 4 to tight-spans of diversities which we shall consider in the next section.

We first note two trivial properties of diversities; for a proof see [3, Proposition 2.1].

Lemma 6.1. *Let δ be a diversity on Y and $A, B \subseteq Y$.*

(a) *If $A \cap B \neq \emptyset$, then $\delta(A) + \delta(B) \geq \delta(A \cup B)$.*
(b) *If $A \subseteq B$, then $\delta(A) \leq \delta(B)$.*

Now, for an arbitrary symmetric map $D : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$, we define the following properties:

(A1) $D(A, B) + D(B, C) \geq D(A, C)$ for all $A, B, C \in \mathcal{P}^\star(Y)$, (triangle inequality)
(A2) $D(\{x\}, \{x\}) = 0$ for all $x \in Y$, and
(A3) $D(A, B) = \frac{1}{2}D(A \cup B, A \cup B)$ for all $A, B \in \mathcal{P}^\star(Y)$ with $A \neq B$.

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[1] Note that Bryant and Tupper required $\delta(A) = 0 \iff |A| \leq 1$ in (D2). In particular, our maps could be regarded as "pseudo-diversities", but for simplicity we shall just call our maps diversities, too.

Given a map $\delta : \mathcal{P}(Y) \to \mathbb{R}$, let $D_\delta : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$ be given by

$$D_\delta(A, B) = \begin{cases} 2\delta(A), & \text{if } A = B, \\ \delta(A \cup B), & \text{else.} \end{cases}$$

Obviously, D_δ is symmetric. Conversely, given a symmetric map $D : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$, we define the function $\delta(D) : \mathcal{P}(Y) \to \mathbb{R}$ by setting $\delta(D)(A) = \frac{1}{2}D(A, A)$ for $A \in \mathcal{P}^\star(Y)$ and $\delta(D)(\emptyset) = 0$.

Proposition 6.2. (a) *Given an arbitrary function $\delta : \mathcal{P}(Y) \to \mathbb{R}$, the map D_δ satisfies* (A3). *Moreover, if δ is a diversity, then D_δ also satisfies* (A1) *and* (A2).
(b) *If $D : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$ is a symmetric map fulfilling* (A1)–(A3), *then $\delta(D)$ is a diversity.*
(c) *Let $\delta : \mathcal{P}(Y) \to \mathbb{R}$. Then $\delta(D_\delta) = \delta$.*
(d) *Let $D : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$ be a symmetric map. Then $D_{\delta(D)} = D$ if and only if* (A3) *holds.*

Proof. (a) For all $A, B \in \mathcal{P}^\star(Y)$ with $A \neq B$ we have $D_\delta(A, B) = \delta(A \cup B) = \frac{1}{2}D_\delta(A \cup B, A \cup B)$, that is, (A3) holds. If in addition δ is a diversity, then for all $A, B, C \in \mathcal{P}^\star(Y)$ with $A \neq C$ we have

$$D_\delta(A, B) + D_\delta(B, C) \geq \delta(A \cup B) + \delta(B \cup C) \overset{(D1)}{\geq} \delta(A \cup C) = D_\delta(A, C).$$

So, if $A = C$, using Lemma 6.1, we get

$$D_\delta(A, B) + D_\delta(B, C) = 2D_\delta(A, B) = 2\delta(A \cup B) \geq 2\delta(A) = D_\delta(A, C).$$

Hence (A1) holds. Furthermore, $D_\delta(\{x\}, \{x\}) = 2\delta(\{x\}) = 0$ for all $x \in Y$ by (D2), so (A1) and (A2) also hold.
(b) We have

$$\begin{aligned} \delta(D)(A \cup B) + \delta(D)(B \cup C) &= \frac{1}{2}D(A \cup B, A \cup B) + \frac{1}{2}D(B \cup C, B \cup C) \\ &\overset{(A3)}{=} D(A, B) + D(B, C) \overset{(A1)}{\geq} D(A, C) \\ &= \frac{1}{2}D(A \cup C, A \cup C) = \delta(D)(A \cup C), \end{aligned}$$

which is (D1). From (A2), we conclude that $\delta(D)(\{x\}) = D(\{x\}, \{x\}) = 0$ for all $x \in Y$, hence (D2) holds.
(c) By definition, $\delta(D_\delta)(A) = \frac{1}{2}D_\delta(A, A) = \frac{2}{2}\delta(A) = \delta(A)$ for all $A \in \mathcal{P}(Y)$.
(d) For all $A, B \in \mathcal{P}^\star(Y)$ with $A \neq B$ we have

$$D_{\delta(D)}(A, B) = \delta(D)(A \cup B) = \frac{1}{2}D(A \cup B, A \cup B)$$

and $D_{\delta(D)}(A, A) = 2\delta(D)(A) = D(A, A)$. Hence $D_{\delta(D)} = D$ if and only if condition (A3) is satisfied. $\square$

Corollary 6.3. *Diversities on Y are in bijective correspondence with symmetric maps $\mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$ satisfying* (A1)–(A3).

To obtain not only a symmetric map, but even a distance on $\mathcal{P}^\star(Y)$, one can use the process described in Section 4 to arrive at the distance $d_\delta : \mathcal{P}^\star(Y) \times \mathcal{P}^\star(Y) \to \mathbb{R}$ defined by $d_\delta = ((D_\delta)')_+$, or, equivalently, define

$$d_\delta(A, B) = \begin{cases} \max(0, \delta(A \cup B) - (\delta(A) + \delta(B))), & \text{if } A \neq B, \\ 0, & \text{if } A = B. \end{cases} \tag{6.1}$$

Lemma 6.4. *The mapping from the set of diversities on Y to the set of all distances on $\mathcal{P}^\star(Y)$ defined by $\delta \mapsto d_\delta$ is injective.*

Proof. Let δ, δ' be diversities on Y with $d_\delta(A, B) = d_{\delta'}(A, B)$ for all $A, B \in \mathcal{P}^\star(Y)$. We will show that $\delta(A) = \delta'(A)$ implies $\delta(A \cup \{i\}) = \delta'(A \cup \{i\})$ for all $i \in Y$, which will prove the claim by induction. Since we have

$$\max(0, \delta(A \cup \{i\}) - \delta(A) - \delta(\{i\})) = d_\delta(A, \{i\}) = d_{\delta'}(A, \{i\}) = \max(0, \delta'(A \cup \{i\}) - \delta'(A) - \delta'(\{i\})),$$

by Lemma 6.1 we also have $\delta(A \cup \{i\}) - \delta(A) = \delta'(A \cup \{i\}) - \delta'(A)$, which completes the proof. $\square$

We conclude with one final observation which is of independent interest.

Lemma 6.5. *Let Y be a finite set, δ a diversity on Y and d_δ the associated distance on $\mathcal{P}^\star(Y)$. Then $d_\delta(A, B) = 0$ for all $A, B \in \mathcal{P}^\star(Y)$ with $A \cap B \neq \emptyset$.*

Proof. By Lemma 6.1, we have $\delta(A \cup B) - \delta(A) - \delta(B) \leq 0$ which implies $d_\delta(A, B) = 0$ by the definition of d_δ. $\square$

7. Tight-spans of diversities

In [3], the tight-span of a diversity is introduced, which is defined as follows. Given a diversity δ on a finite set Y, the tight-span of δ is the set $\hat{T}_\delta$ of all minimal elements of the polyhedron

$$\hat{P}_\delta = \left\{ f \in \mathbb{R}^{\mathcal{P}(Y)} : f(\emptyset) = 0 \text{ and } \sum_{A \in \mathcal{A}} f(A) \geq \delta\left(\bigcup \mathcal{A}\right) \text{ for all } \mathcal{A} \subseteq \mathcal{P}(Y) \right\}.$$

We shall also consider the related set T_δ consisting of all minimal elements of the polyhedron

$$P_\delta = \left\{ f \in \mathbb{R}^{\mathcal{P}(Y)} : f(\emptyset) = 0 \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in \mathcal{P}(Y) \right\}$$

which was mentioned in a preliminary version of [3]. Note that Bryant and Tupper showed that the set $\hat{T}_\delta$ is an injective hull in an appropriately defined category. Although a similar result has not been shown to hold for T_δ, this set will also be useful in our investigations of diversities.

We first prove that T_δ arises from the point configuration $\mathcal{A}(\mathcal{P}^\star(Y))$, the configuration of integer points in a simplex with edge-length 2, and that $\hat{T}_\delta$ arises from $\mathcal{C}(\mathcal{P}^\star(Y))$, the configuration of vertices of a cube. More specifically, let $w^\delta : \mathcal{C}(\mathcal{P}^\star(Y)) \to \mathbb{R}$ be given by $w^\delta(\sum_{i \in \mathcal{A}} e_i) = \delta(\cup \mathcal{A})$ for all $\mathcal{A} \subseteq \mathcal{P}^\star(Y)$. Then:

Proposition 7.1. *Let $\delta : \mathcal{P}(Y) \to \mathbb{R}$. Then:*

(a) $P_\delta = \{0\} \times \mathcal{E}_{w^{D_\delta}}(\mathcal{A}(\mathcal{P}^\star(Y)))$, *and* $T_\delta = \{0\} \times \mathcal{T}_{w^{D_\delta}}(\mathcal{A}(\mathcal{P}^\star(Y)))$.
(b) $\hat{P}_\delta = \{0\} \times \mathcal{E}_{w^\delta}(\mathcal{C}(\mathcal{P}^\star(Y)))$, *and* $\hat{T}_\delta = \{0\} \times \mathcal{T}_{w^\delta}(\mathcal{C}(\mathcal{P}^\star(Y)))$.

Proof. (a) We have

$$\begin{aligned}
P_\delta &= \left\{ f \in \mathbb{R}^{\mathcal{P}(Y)} : f(\emptyset) = 0 \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in \mathcal{P}(Y) \right\} \\
&= \left\{ f \in \mathbb{R}^{\mathcal{P}(Y)} \mid f(\emptyset) = 0, f(A) + f(\emptyset) = f(A) \geq \delta(A) \text{ for all } A \in \mathcal{P}^\star(Y) \text{ and } f(A) \right. \\
&\quad \left. + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in \mathcal{P}^\star(Y) \right\} \\
&= \{0\} \times \left\{ f \in \mathbb{R}^{\mathcal{P}^\star(Y)} \mid f(A) + f(A) \geq 2\delta(A) \text{ for all } A \in \mathcal{P}^\star(Y), f(A) + f(A) \right. \\
&\quad \left. \geq \delta(A \cup A) \text{ for all } A \in \mathcal{P}^\star(Y), \text{ and } f(A) + f(B) \geq \delta(A \cup B) \text{ for all } A, B \in \mathcal{P}^\star \text{ with } A \neq B \right\} \\
&= \{0\} \times \mathcal{E}_{w^{D_\delta}}(\mathcal{A}(\mathcal{P}^\star(Y))).
\end{aligned}$$

The statement now follows from Proposition 4.1.
(b) We have

$$\begin{aligned}
\hat{P}_\delta &= \left\{ f \in \mathbb{R}^{\mathcal{P}(Y)} : f(\emptyset) = 0 \text{ and } \sum_{A \in \mathcal{A}} f(A) \geq \delta\left(\bigcup \mathcal{A}\right) \text{ for all } \mathcal{A} \subseteq \mathcal{P}(Y) \right\} \\
&= \{0\} \times \left\{ f \in \mathbb{R}^{\mathcal{P}^\star(Y)} : \left\langle \sum_{A \in \mathcal{A}} e_A, f \right\rangle \geq \delta\left(\bigcup \mathcal{A}\right) \text{ for all } \mathcal{A} \subseteq \mathcal{P}(Y) \right\} \\
&= \{0\} \times \left\{ f \in \mathbb{R}^{\mathcal{P}^\star(Y)} : \langle a, f \rangle \geq w^\delta(a) \text{ for all } a \in \mathcal{C}(\mathcal{P}^\star(Y)) \right\} \\
&= \{0\} \times \mathcal{E}_{w^\delta}(\mathcal{C}(\mathcal{P}^\star(Y))).
\end{aligned}$$

The statement now follows from Lemma 2.1 since $\mathcal{C}(\mathcal{P}^\star(Y))$ is positive and $\mathcal{E}_{w^\delta}(\mathcal{C}(\mathcal{P}^\star(Y)))$ is bounded from below by the inequalities $e_A \geq \delta(A) = 0$ for all $A \in \mathcal{P}^\star(Y)$. $\square$

We turn now to one-dimensional tight-spans. First note that, by considering the distance map d_δ defined in (6.1), we get

$$T_\delta = \{0\} \times T_{d_\delta} + (\delta(A))_{A \in \mathcal{P}(X)},$$

which, in particular, implies the following:

Corollary 7.2. *Let δ be a diversity. Then T_δ is a tree if and only if the tight-span T_{d_δ} of the distance d_δ is a tree.*

We now characterise diversities whose tight-spans are trees. A diversity δ on a set Y is called a *phylogenetic diversity* if there is a weighted tree T with leaf set Y such that $\delta = \delta_T$, where δ_T is the diversity associated with T given by mapping any subset $A \subseteq Y$ to the length $\delta_T(A)$ of the smallest subtree of T connecting taxa in A.

Theorem 7.3. *Let δ be diversity on Y. Then the following are equivalent:*

(a) δ *is a phylogenetic diversity.*
(b) *The tight-span $\hat{T}_\delta$ is a tree.*
(c) *The tight-span T_δ is a tree.*

The fact that (a) implies (b) follows from [3, Theorem 5.8], since the tight-span of a phylogenetic diversity is isomorphic to the tight-span of the metric space associated with the same tree. To show that (b) is equivalent (c), we will prove that $\hat{T}_\delta$ and T_δ are equal for a much larger class of diversities, the so-called *split-system diversities* (Theorem 7.4). The proof that (c) implies (a) will then take up the remainder of this section.

Let S be a split of Y. We define the *split diversity* $\delta_S : \mathcal{P}(Y) \to \mathbb{R}$ of S as

$$\delta_S(A) = \begin{cases} 1, & \text{if } S \text{ splits } A, \\ 0, & \text{else.} \end{cases}$$

Given a set $\mathcal{S}$ of splits of Y and a function $\alpha : \mathcal{S} \to \mathbb{R}_{>0}$ assigning weights to the splits in $\mathcal{S}$, the *split system diversity* $\delta_{(\mathcal{S},\alpha)}$ of $(\mathcal{S}, \alpha)$ is defined as

$$\delta_{(\mathcal{S},\alpha)}(A) = \sum_{S \in \mathcal{S}} \alpha(S)\delta_S(A) = \sum_{S \in \mathcal{S}, S \text{ splits } A} \alpha(S).$$

A phylogenetic diversity is a special case of a split system diversity where the set $\mathcal{S}$ is compatible. We now show that in case δ is a split diversity, the tight-spans T_δ and $\hat{T}_\delta$ are equal:

Theorem 7.4. *Let $\mathcal{S}$ be a split system, $\alpha : \mathcal{S} \to \mathbb{R}_{>0}$ and $\delta_{(\mathcal{S},\alpha)}$ the associated split system diversity. Then*

$$\hat{P}_{\delta_{(\mathcal{S},\alpha)}} = P_{\delta_{(\mathcal{S},\alpha)}} \quad \text{and} \quad \hat{T}_{\delta_{(\mathcal{S},\alpha)}} = T_{\delta_{(\mathcal{S},\alpha)}}.$$

Proof. We will show that $\hat{P}_{\delta_{(\mathcal{S},\alpha)}} = P_{\delta_{(\mathcal{S},\alpha)}}$ which obviously implies $\hat{T}_{\delta_{(\mathcal{S},\alpha)}} = T_{\delta_{(\mathcal{S},\alpha)}}$. First note that, by definition, for any diversity δ, one has $\hat{P}_\delta \subseteq P_\delta$ and furthermore, for all $f \in P_\delta$ and $A \in \mathcal{P}(Y)$, one has $f(A) + f(\emptyset) = f(A) \geq \delta(A)$. It now suffices to show that for any $\mathcal{A} \subseteq \mathcal{P}(Y)$ with $|\mathcal{A}| \geq 2$ and $f \in P_{\delta_{(\mathcal{S},\alpha)}}$ the system of inequalities

$$f(A) + f(B) \geq \delta_{(\mathcal{S},\alpha)}(A \cup B) \quad \text{for all distinct } A, B \in \mathcal{A}, \tag{7.1}$$

implies the inequality

$$\sum_{A \in \mathcal{A}} f(A) \geq \delta_{(\mathcal{S},\alpha)}\left(\bigcup \mathcal{A}\right). \tag{7.2}$$

Summing up all the Inequalities (7.1) we get

$$\sum_{A,B \in \mathcal{A}, A \neq B} (f(A) + f(B)) \geq \sum_{A,B \in \mathcal{A}, A \neq B} \delta_{(\mathcal{S},\alpha)}(A \cup B)$$

$$\iff (|\mathcal{A}| - 1)\sum_{A \in \mathcal{A}} f(A) \geq \sum_{A,B \in \mathcal{A}, A \neq B} \sum_{S \in \mathcal{S}, S \text{ splits } A \cup B} \alpha(S) = \sum_{S \in \mathcal{S}} \alpha(S)\text{SP}(S, \mathcal{A}, 2),$$

where $\text{SP}(S, \mathcal{A}, 2)$ denotes the number of unordered pairs of distinct $A, B \in \mathcal{A}$ such that S splits $A \cup B$. We now show that $\text{SP}(S, \mathcal{A}, 2) \geq |\mathcal{A}| - 1$ for all $S \in \mathcal{S}$ that split $\cup \mathcal{A}$. Dividing the above inequality by $|\mathcal{A}| - 1$ then gives Inequality (7.2) as desired.

Let $S \in \mathcal{S}$ be a split that splits $\cup \mathcal{A}$. First suppose that there exists some $A \in \mathcal{A}$ that is split by S. Then $\text{SP}(S, \mathcal{A}, 2) \geq |\mathcal{A}| - 1$ since obviously for all $B \in \mathcal{A}$ distinct from A the set $A \cup B$ is also split by S. So we can assume that, for $S = (C, D)$ and all $A \in \mathcal{A}$, we have either $A \subseteq C$ or $A \subseteq D$. Let s be the number of $A \in \mathcal{A}$ with $A \subseteq C$. Since S splits $\cup \mathcal{A}$, both s and $|\mathcal{A}| - s$ have to be at least 1, so we get $\text{SP}(S, \mathcal{A}, 2) = s(|\mathcal{A}| - s) \geq \mathcal{A} - 1$, which finishes the proof. $\quad \square$

In preparation for completing the proof of Theorem 7.3, we now examine the distance d_δ in case δ is a split diversity. Let $S = \{C, D\}$ be a split of Y. For each $A, B \subseteq Y$ we have

$$\delta_S(A \cup B) - \delta_S(A) - \delta_S(B) = \begin{cases} 1, & \text{if } C \subseteq A \text{ and } D \subseteq B \text{ or } C \subseteq B \text{ and } D \subseteq A \\ -1, & \text{if } A \cap C \neq \emptyset, \ A \cap D \neq \emptyset, \ B \cap C \neq \emptyset, \text{ and } B \cap D \neq \emptyset, \\ 0, & \text{else,} \end{cases}$$

so that

$$d_{\delta_S}(A, B) = \begin{cases} 1, & \text{if } C \subseteq A \text{ and } D \subseteq B \text{ or } C \subseteq B \text{ and } D \subseteq A, \\ 0, & \text{else.} \end{cases}$$

Hence $d_{\delta_S} = d_P$ (as defined in Eq. (4.1)), where P is the partial split $\{\mathcal{P}^\star(A), \mathcal{P}^\star(B)\}$ of $\mathcal{P}^\star(Y)$. By Proposition 3.1, the corresponding weight function defines a split of $\mathcal{A}(\mathcal{P}^\star(Y))$. We will show that a diversity whose tight-span is a tree comes from a compatible set of splits by considering these steps in reverse order. The following lemmas will be the key to our proof.

Lemma 7.5. *Let Y be a finite set, $\{\mathcal{A}, \mathcal{B}\}, \{\mathcal{C}, \mathcal{D}\}$ two compatible partial splits of $\mathcal{P}(Y)$ and $A \in \mathcal{A}, A' \in \mathcal{C}, B \in \mathcal{B} \cap \mathcal{D}$. Then either $A' \in \mathcal{A}$ or $A \in \mathcal{C}$.*

Proof. Since $\{\mathcal{A}, \mathcal{B}\}, \{\mathcal{C}, \mathcal{D}\}$ are compatible and $B \in \mathcal{B} \cap \mathcal{D}$, by definition, we must have $\mathcal{A} \subseteq \mathcal{C}$ or $\mathcal{C} \subseteq \mathcal{A}$, which implies the claim. $\quad \square$

Lemma 7.6. *Let $\mathcal{P}$ be a compatible set of partial splits of $\mathcal{P}^\star(Y)$, $\{\mathcal{A}, \mathcal{B}\} \in \mathcal{P}$, $\alpha : \mathcal{P} \to \mathbb{R}_{>0}$ and δ be a diversity on Y such that $d_\delta = d_{(\mathcal{P},\alpha)}$. Then the following statements hold:*

(a) *If $A \in \mathcal{A}$, $B \in \mathcal{B}$, $A' \subseteq A$, and $d_\delta(A', B) \geq d_\delta(A, B)$, then $A' \in \mathcal{A}$.*
(b) *If $A \in \mathcal{A}$, then $\{i\} \in \mathcal{A}$ for all $i \in A$.*
(c) *If $A, A' \in \mathcal{A}$, then $d_\delta(A, A') = 0$.*
(d) *If $i \notin A \in \mathcal{P}(Y)$ with $\{i\}, A \in \mathcal{A}$, then $\{i\} \cup A \in \mathcal{A}$.*

Proof. (a) For $A, B \subseteq Y$, set $\mathcal{P}(A, B) = \{\{\mathcal{A}, \mathcal{B}\} \in \mathcal{P}\colon A \in \mathcal{A}, B \in \mathcal{B}\}$. Then we have

$$d_\delta(A', B) = d_{(\mathcal{P},\alpha)}(A', B) = \sum_{P \in \mathcal{P}(A',B)} \alpha(P)$$

$$\geq \sum_{P \in \mathcal{P}(A,B)} = d_\delta(A, B).$$

So, either $\mathcal{P}(A, B) = \mathcal{P}(A', B)$, which trivially implies the claim, or there exists some partial split $\{\mathcal{C}, \mathcal{D}\} \in \mathcal{P}(A', B) \setminus \mathcal{P}(A, B)$. Lemma 7.5 now gives us $A' \in \mathcal{A}$ (since $A \in \mathcal{C}$ would imply $\{\mathcal{C}, \mathcal{D}\} \in \mathcal{P}(A, B)$, as desired).
(b) Let $B \in \mathcal{B}$. By (D1), we get

$$\delta(A \cup \{i\}) + \delta(\{i\} \cup B) \geq \delta(A \cup B).$$

This implies that

$$\delta(\{i\} \cup B) - \delta(B) \geq \delta(A \cup B) - \delta(A) - \delta(B).$$

By definition of d_δ (and (D2)), this is equivalent to $d_\delta(\{i\}, B) \geq d_\delta(A, B)$. The claim now follows from Part (a).
(c) If $d_\delta(A, A') \neq 0$ this would imply that there exists a partial split $\{\mathcal{C}, \mathcal{D}\} \in \mathcal{P}$ with $A \in \mathcal{C}$ and $A' \in \mathcal{D}$ (or vice versa). However, this split could not be compatible with $\{\mathcal{A}, \mathcal{B}\}$.
(d) By Part (c), we have $d_\delta(\{i\}, A) = 0$ which implies $\delta(A \cup \{i\}) = \delta(A)$ by the definition of d_δ. Let $B \in \mathcal{B}$. Since $d_\delta(A, B) > 0$ by assumption, we get

$$\begin{aligned}
d_\delta(A, B) &= \delta(A \cup B) - \delta(A) - \delta(B) \\
&= \delta(A \cup B) - \delta(A \cup \{i\}) - \delta(B) \\
&\leq \delta(A \cup \{i\} \cup B) - \delta(A \cup \{i\}) - \delta(B) \\
&= d_\delta(A \cup \{i\}, B).
\end{aligned}$$

The claim now follows from Part (a). $\quad\square$

Corollary 7.7. *Let $\mathcal{P}$ be a compatible set of partial splits of $\mathcal{P}^\star(Y)$, $\alpha : \mathcal{P} \to \mathbb{R}_{\geq 0}$ and δ a diversity on Y such that $d_\delta = d_{(\mathcal{P},\alpha)}$. Then for all $P = \{\mathcal{A}, \mathcal{B}\} \in \mathcal{P}$ there exists a partial split $p(P) = \{A, B\}$ of Y such that $\mathcal{A} = \mathcal{P}^\star(A)$ and $\mathcal{B} = \mathcal{P}^\star(B)$. Furthermore the set $\{p(P)\colon P \in \mathcal{P}\}$ of partial splits of Y is compatible.*

Proof. The existence of $p(P)$ follows by iteratively applying Lemma 7.6 (b) and (d). The compatibility follows from the compatibility of $\mathcal{P}$ by Proposition 3.2. $\quad\square$

We now complete the proof that was promised above.

Proof of Theorem 7.3. We only have to show that (c) implies (a). So, let δ be a diversity on Y such that the tight-span T_δ is a tree. Corollary 7.2 implies that T_{d_δ} is a tree and Theorem 4.5 gives us a compatible set $\mathcal{P}$ of partial splits of $\mathcal{P}^\star(Y)$ and a function $\alpha : \mathcal{P} \to \mathbb{R}_{>0}$ such that $d_\delta = d_{(\mathcal{P},\alpha)}$. By Corollary 7.7, there exists a compatible set $\mathcal{S} = \{p(P)\colon P \in \mathcal{P}\}$ of partial splits of Y such that $\mathcal{A} = \mathcal{P}^\star(A)$ and $\mathcal{B} = \mathcal{P}^\star(B)$. It remains to show that all $S \in \mathcal{S}$ are splits.

Suppose one of these partial splits, say $\{A, B\}$, is not a split, and let $i \in A$, $j \in B$ and $k \in Y \setminus (A \cup B)$. By (D1), we get

$$\delta(\{i, k\}) + \delta(\{k, j\}) \geq \delta(\{i, j\}),$$

which is equivalent to

$$d_\delta(\{i\}, \{k\}) + d_\delta(\{k\}, \{j\}) \geq d_\delta(\{i\}, \{j\}) \tag{7.3}$$

by the definition of d_δ and (D2). Now any partial split $\{\mathcal{P}^\star(C), \mathcal{P}^\star(D)\}$ that separates k from either i or j must also separate i and j since it is compatible to $\{\mathcal{P}^\star(A), \mathcal{P}^\star(B)\}$. So each split making a contribution to the left of Eq. (7.3) makes the same contribution to the right of the equation and $\{\mathcal{P}^\star(A), \mathcal{P}^\star(B)\}$ only contributes to the right, a contradiction.

So $d_\delta = d_{\delta(\mathcal{S},\alpha)}$ which implies $\delta = \delta_{(\mathcal{S},\alpha)}$. Hence δ is a phylogenetic diversity. $\quad\square$

8. Discussion

8.1. k-dissimilarity maps

We have seen how to define the tight-span of various maps that generalise metrics. Another kind of map that we could consider taking the tight-span of is a *k-dissimilarity map* on a set X, that is, a function $D : \binom{X}{k} \to \mathbb{R}$. In this case, to obtain a tight-span one could take the set of vertices of the so-called hypersimplex $\Delta(k, X) \subseteq \mathbb{R}^X$ as the corresponding point configuration, that is, the set of all functions $\sum_{x \in A} e_x$ for all $A \in \binom{X}{k}$. More specifically, motivated by Proposition 4.1 for the case $k = 2$, given a k-dissimilarity map D on X, define the function $w^D : \Delta(k, X) \subseteq \mathbb{R}^X \to \mathbb{R}$ that sends $\sum_{x \in A} e_x$ to $-D(A)$, set $P_D = \mathcal{E}_{w^D}(\Delta(k, X))$, and $T_D = \mathcal{T}_{w^D}(\Delta(k, X))$. It follows from Lemma 2.1 that T_D is the set of minimal elements of P_D.

Even though one might expect that T_D has similar properties to the tight-spans we have so far considered, this is not the case. Indeed, given a weighted tree T with leaf set X, one can define a k-dissimilarity map D_T^k by assigning to each k-subset $A \subseteq X$ the total length of the induced subtree. However, the tight-span $T_{D_T^k}$ does not in general also have to be a tree, and so there is no obvious generalisation of the Tree Metric theorem. Even so, the tree T can be reconstructed from $T_{D_T^k}$ [16, Section 8.1], and so it could still be of interest to further study these tight-spans.

8.2. Coherent decompositions

Coherent decompositions of metrics were introduced by Bandelt and Dress [1] and are intimately related to tight-spans. Thus the question arises whether a similar decomposition theory could be developed for the different generalisations of metrics that we have considered.

In [15], the concept of coherent decompositions of metrics was generalised to weight functions of polytopes (as discussed in Section 2). For directed distances, and also symmetric and non-symmetric functions, this directly leads to a theory of coherent decompositions. Moreover, a decomposition theorem for k-dissimilarities in terms of "split k-dissimilarities" was recently derived in [16], which might be extended if an appropriate theory was worked out for tight-spans of k-dissimilarities, as suggested above. For diversities, as we have seen, a diversity on a set Y can be considered as a distance on $\mathcal{P}^\star(Y)$, but such diversities form only a subset of all such distances. So, to develop a theory of coherent decomposition for diversities, one could maybe first try to answer the following question:

Question 8.1. Given a diversity δ how can one compute coherent decompositions of the distance d_δ? Moreover, which coherent components of such a decomposition are of the form $d_{\delta^\star}$ for some diversity $\delta^\star$?

8.3. Infinite sets and injective hulls

In this paper, we have only considered tight-spans arising from finite sets. However, many of the results concerning tight-spans (and not point configurations) can be translated to infinite sets. For example, much of the theory for the tight-span of a metric space was originally developed for arbitrary metric spaces [8,20], which is important since the tight-span of a metric (diversity) – which is of course an infinite set – comes equipped with a canonical metric (diversity), such that the tight-span of this metric is nothing other than itself (see [20,3], respectively). Stated differently, this means that tight-spans are injective objects in the appropriate category [20], a property that would be interesting to understand in the setting of point configurations. However, if the theory for point configurations is to be extended to infinite sets, a first crucial step would be to understand how to generalise splits of polytopes, which does not appear to have an obvious appropriate generalisation in the infinite setting.

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