JOURNAL OF COMBINATORIAL THEORY, Series A **66**, 17–27 (1994)

# A Constant Term Identity Featuring the Ubiquitous (and Mysterious) Andrews–Mills–Robbins–Rumsey Numbers 1, 2, 7, 42, 429, ...

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*Communicated by George Andrews*

Received February 28, 1991

Andrews's recent proof of the Mills–Robbins–Rumsey conjectured formula for the number of totally symmetric self-complementary plane partitions is used to derive a new multi-variate constant term identity, reminiscent of, but not implied by, Macdonald's $BC_n$-Dyson identity. The method of proof consists in translating to the language of constant terms an expression of Doran for the desired number in terms of sums of minors of a certain matrix. The question of a direct proof of the identity, which would furnish an alternative proof of the Mills–Robbins–Rumsey conjecture, is raised, and a prize is offered for its solution. 

## PROLOGUE

Sometimes, it may occur to mathematician X, in his attempt at proving a conjectured equality $A=B$, to introduce another quantity $C$, and to attempt to prove the two lemmas $A=C$ and $C=B$. The original conjecture $A=B$ would then follow by the transitivity of the $=$ relation. Alas, it might happen that, after the successful completion by X of the first part of his program, but before finishing the second part, the conjecture $A=B$ is proved by his rival Y by a completely different method. Should X let thousands of hours, 50 yellow pads, and 10 ball-point pens go unrecorded in the archival literature? Certainly not! All that X has to do is promote the equality $C=B$ from the status of lemma to that of theorem and observe that its proof follows immediately from his own lemma $A=C$, Y's theorem $A=B$, and the symmetry and transitivity of the equality relation. To be on the safe side, X should argue also that in addition to the intrinsic interest of $C=B$, the *method of proof* of the lemma $A=C$ is interesting, and might lead to the proof of other conjectures.

* Supported in part by NSF Grant DMS8800663.
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0097-3165/94 $6.00

The above scenario happened with the following specialization.

X = Your faithful servant.

Y = George Andrews.

$A$ = The number of totally symmetric self complementary plane partitions (TSSCPP) whose 3D Ferres diagrams fit inside $[0, 2n]^3$.

$$B = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!},$$

the Andrews–Mills–Robbins–Rumsey sequence $\{1, 2, 7, 42, 429, ...\}$ [A3, MRR1, MRR2, Ro1].

$$C = CT \prod_{1 \leqslant i < j \leqslant n} (1 - x_i/x_j) \prod_{i=1}^{n} (1 + x_i^{-1})^{n-i} \times \left\{ \prod_{i=1}^{n} (1 - x_i)^{-1} \prod_{1 \leqslant i < j \leqslant n} (1 - x_i x_j)^{-1} \right\},$$

where "$CT$" stands for the phrase "the constant term of," i.e., the coefficient of $x_1^0 \cdots x_n^0$.

## 1. Introduction

Andrews [A1] has recently proved the following conjecture of Mills, Robbins, and Rumsey [MRR2] (see also [Ro1, S]).

THEOREM $A = B$ (Andrews [A1]). $A = B$.

In this paper we prove

THEOREM $C = B$. $C = B$.

This identity closely resembles Macdonald's [Macd2] celebrated $BC_n$-Dyson identity, but does not seem to be implied by it. Its mere existence seems to indicate that Macdonald's identities and conjectures are far from being the only explicitly evaluable constant term expressions, and that there are still many more waiting to be discovered. Even more exciting is the possibility that there exists a common generalization of Macdonald's constant term identities and the present one. Section 7 presents yet another such identity. We refer the readers to [Gu] and [Ga-Go] for an update on the status of Macdonald's constant term conjectures.

In the constant term $C$, we view the rational function inside the braces as a power series in positive power of $x_1, ..., x_n$. When this is multiplied by the Laurent polynomial in front of the braces, one obtains a well-defined Laurent series, wih a well-defined constant term.

Theorem $C=B$ would follow from Andrews's theorem $A=B$ and the following lemma:

LEMMA $A=C$. $A=C$.

I am offering 25 U.S. dollars[1] for a direct proof of theorem $C=B$ that does not use Andrews's theorem $A=B$. Such a proof, combined with my lemma $A=C$ would, of course, give a new proof of theorem $A=B$.

The readers are welcome to look up the definition of TSSCPP in [MRR2] or [S], if they wish, but the present paper can be understood without it, provided one is willing to believe Doran's [Do] result, to be recalled shortly, that the number of these creatures, whatever they are, equals the sum of minors of a certain matrix. In fact the larger message of this paper is the introduction of a new and potentially useful *method* for expressing sums of minors of matrices whose entries are binomial coefficients in terms of constant terms of rational functions, the Mills–Robbins–Rumsey conjecture being the instructive example by which this method is illustrated. The present method can be viewed as a determinantal extension of Egorychev's [E] method of "integral representation" (which is tantamount to "constant term") for binomial coefficients sums. MacMahon's celebrated master theorem [MacM] that expresses a certain determinant as a certain constant term also comes to mind, but at first sight appears to be only a distant cousin.

## 2. DORAN'S SUM OF MINORS EXPRESSION FOR THE NUMBER OF TSSCPP

To prove lemma $A=C$, we need a result of Doran [Do] (no relation). Let $D$ be the sum of all the $n\times n$ minors of the following $n\times(2n-1)$ matrix:

$$X_{i,j}:=\binom{i-1}{j-i}, \qquad 1\leqslant i\leqslant n, \quad 1\leqslant j\leqslant 2n-1.$$

THEOREM $A=D$ (Doran [Do]. $A=D$.

It remains to prove that $D=C$. I prove a more general result. Let $D'$ be the sum of all the $n\times n$ minors of the $n\times(2n+m-1)$ matrix $X$ given by

$$X_{i,j}:=\binom{m+i-1}{j-i}, \qquad 1\leqslant i\leqslant n, \quad 1\leqslant j\leqslant 2n+m-1,$$

[1] Because the prize is being offered by the author and not by the publisher, readers should contact the author directly.

and let $C'$ be given by

$$C' := CT \prod_{1 \leqslant i<j \leqslant n} (1-x_i/x_j) \prod_{i=1}^{n} (1+x_i^{-1})^{m+n-i} \times \left\{ \prod_{i=1}^{n} (1-x_i)^{-1} \prod_{1 \leqslant i<j \leqslant n} (1-x_i x_j)^{-1} \right\}.$$

I prove

LEMMA $D'=C'$. $D'=C'$.

The proof of this lemma is given in the next section. Assuming it for the moment, we have

COROLLARY $D=C$. $D=C$.

*Proof.* Take $m=0$ in Lemma $D'=C'$.

*Completion of the proof of Lemma $A=C$.* Combine Doran's [Do] Lemma $A=D$ and Corollary $D=C$.

## 3. PROOF OF LEMMA $D'=C'$

We have a fairly long string of equalities. Whenever the equality requires explanation, we label the equal sign with an integer and give the explanation in Section 4, under the heading of that integer. Some readers may prefer to consult Section 4 simultaneously.

$$D' := \sum_{1 \leqslant j_1<j_2< \cdots <j_n \leqslant 2n+m-1} \det \binom{m+i-1}{j_r - i}_{1 \leqslant i \leqslant n,\, 1 \leqslant r \leqslant n}$$

$$= \sum_{0 \leqslant j_1<j_2< \cdots <j_n \leqslant 2n+m-2} \det \binom{m+i-1}{j_r - i+1}_{1 \leqslant i \leqslant n,\, 1 \leqslant r \leqslant n}$$

$$=^{(1)} \sum_{0 \leqslant j_1<j_2< \cdots <j_n \leqslant 2n+m-2} \sum_{\pi \in S_n} sgn(\pi) \prod_{r=1}^{n} \binom{m+\pi(r)-1}{j_r - \pi(r)+1}$$

$$=^{(2)} \sum_{0 \leqslant j_1<j_2< \cdots <j_n \leqslant 2n+m-2} \sum_{\pi \in S_n} sgn(\pi)\, CT \left[ \prod_{r=1}^{n} \frac{(1+x_r)^{m+\pi(r)-1}}{x_r^{j_r-\pi(r)+1}} \right]$$

$$= \sum_{0 \leqslant j_1<j_2< \cdots <j_n \leqslant 2n+m-2} CT \left[ \prod_{r=1}^{n} \frac{(1+x_r)^m}{x_r^{j_r}} \times \sum_{\pi \in S_n} sgn(\pi) \prod_{r=1}^{n} [(1+x_r)x_r]^{\pi(r)-1} \right]$$

$$\stackrel{(3)}{=} CT\left[\prod_{r=1}^{n} (1+x_r)^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i)x_i - (1+x_j)x_j] \times \sum_{0 \leqslant j_1 < j_2 < \cdots < j_n \leqslant 2n+m-2} x_1^{-j_1} \cdots x_n^{-j_n}\right]$$

$$\stackrel{(4)}{=} CT\left[\prod_{r=1}^{n} (1+x_r)^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i)x_i - (1+x_j)x_j] \times \sum_{0 \leqslant j_1 < j_2 < \cdots < j_n < \infty} x_1^{-j_1} \cdots x_n^{-j_n}\right]$$

$$\stackrel{(5)}{=} CT\left[\prod_{r=1}^{n} (1+x_r^{-1})^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i^{-1})x_i^{-1} - (1+x_j^{-1})x_j^{-1}] \times \sum_{0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_n < \infty} x_1^{j_1} \cdots x_n^{j_n}\right]$$

$$\stackrel{(6)}{=} CT\left[\prod_{r=1}^{n} (1+x_r^{-1})^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i^{-1})x_i^{-1} - (1+x_j^{-1})x_j^{-1}] \times (1-x_n)^{-1}(1-x_n x_{n-1})^{-1} \cdots (1-x_n x_{n-1} \cdots x_1)^{-1}\right]$$

$$\stackrel{(7)}{=} \frac{1}{n!} CT\left[\prod_{r=1}^{n} (1+x_r^{-1})^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i^{-1})x_i^{-1} - (1+x_j^{-1})x_j^{-1}] \times \sum_{\pi \in S_n} sgn(\pi)\pi[(1-x_n)^{-1}(1-x_n x_{n-1})^{-1} \cdots \cdot (1-x_n x_{n-1} \cdots x_1)^{-1}]\right]$$

$$\stackrel{(8)}{=} \frac{1}{n!} CT\left[\prod_{r=1}^{n} (1+x_r^{-1})^m \prod_{1 \leqslant i<j \leqslant n} [(1+x_i^{-1})x_i^{-1} - (1+x_j^{-1})x_j^{-1}] \times \frac{\prod_{1 \leqslant i<j \leqslant n}(x_j - x_i)}{\prod_{i=1}^{n}(1-x_i)\prod_{1 \leqslant i<j \leqslant n}(1-x_i x_j)}\right]$$

$$\stackrel{(3)}{=} \frac{1}{n!} CT\left[\prod_{r=1}^{n} (1+x_r^{-1})^m \prod_{1 \leqslant i<j \leqslant n} (x_j - x_i) \times \prod_{i=1}^{n} (1-x_i)^{-1} \prod_{1 \leqslant i<j \leqslant n} (1-x_i x_j)^{-1} \times \sum_{\pi \in S_n} sgn(\pi) \prod_{r=1}^{n} [(1+x_r^{-1})x_r^{-1}]^{\pi(r)-1}\right]$$

$$=^{(9)} CT\left[\prod_{r=1}^{n}(1+x_r^{-1})^m \prod_{1\leqslant i<j\leqslant n}(x_j-x_i)\right.$$

$$\times\prod_{i=1}^{n}(1-x_i)^{-1}\prod_{1\leqslant i<j\leqslant n}(1-x_ix_j)^{-1}$$

$$\left.\times\prod_{r=1}^{n}[(1+x_r^{-1})x_r^{-1}]^{r-1}\right]$$

$$= CT\left[\prod_{r=1}^{n}(1+x_r^{-1})^{m+r-1}\prod_{1\leqslant i<j\leqslant n}(1-x_i/x_j)\right.$$

$$\left.\times\prod_{i=1}^{n}(1-x_i)^{-1}\prod_{1\leqslant i<j\leqslant n}(1-x_ix_j)^{-1}\right].$$

## 4. Explanations of the above Equalities

1. The definition of the determinant.

2. The binomial theorem and the fact that $CT[f(x)\,g(y)] = CT[f(x)]\,CT[g(y)]$.

3. The Vandermonde determinant identity:

$$\sum_{\pi\in S_n} sgn(\pi)\prod_{r=1}^{n} y_r^{\pi(r)-1} = \prod_{1\leqslant i<j\leqslant n}(y_i-y_j). \tag{V}$$

See [Ge] and [A2, Sect. 4.4] for Gessel's beautiful combinatorial proof.

4. The added terms are of higher degree than the polynomial in front of the sigma, and so add 0 to the constant term.

5. We make the transformation $x_i \to x_i^{-1}$, $i=1, ..., n$, which obviously does not change the constant term. We also changed the summation range from $\{0<j_1<\cdots<j_n<\infty\}$ to $\{0\leqslant j_1\leqslant\cdots\leqslant j_n<\infty\}$, which does not change anything, since the extra monomials all have at least two of their exponents equal, and hence their contribution to the constant term is zero, thanks to the anti-symmetry of the kernel.

6. Here we use

$$\sum_{0\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_n<\infty} x_1^{j_1}\cdots x_n^{j_n}$$

$$= (1-x_n)^{-1}(1-x_nx_{n-1})^{-1}\cdots(1-x_nx_{n-1}\cdots x_1)^{-1}.$$

For $n=1$ this is just the sum of an infinite geometric series. Assuming this is true for $n-1$ variables, $x_2, ..., x_n$, we have

$$\sum_{0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_n < \infty} x_1^{j_1} \cdots x_n^{j_n}$$
$$= \left[ \sum_{j_1=0}^{\infty} (x_1 \cdots x_n)^{j_1} \right] \cdot \left[ \sum_{0 \leqslant j_2 - j_1 \leqslant \cdots \leqslant j_n - j_1 < \infty} x_2^{j_2 - j_1} \cdots x_n^{j_n - j_1} \right]$$
$$= (1 - x_1 \cdots x_n)^{-1} (1 - x_2 \cdots x_n)^{-1} \cdots (1 - x_n)^{-1}.$$

7. Here we "average" over all the images under the symmetric group, noting that the constant term is not affected. More explicitly, for any Laurent series $f(x_1, ..., x_n)$ let the symmetric group act naturally by

$$\pi(f)(x_1, ..., x_n) = f(x_{\pi(1)}, ..., x_{\pi(n)}),$$

and let

$$f^{\#} = \frac{1}{n!} \sum_{\pi \in S_n} \pi(f);$$

then $CT(f) = CT(f^{\#})$. We also used the obvious fact that if $f^{\&}$ denotes the "anti-symmetrizer,"

$$f^{\&} = \frac{1}{n!} \sum_{\pi \in S_n} sgn(\pi)\, \pi(f);$$

then if $g$ is anti-symmetric,

$$(gf)^{\#} = g(f^{\&}).$$

8. Here we used the identity

$$\sum_{\pi \in S_n} sgn(\pi)\, \pi[(1 - x_n)^{-1} (1 - x_n x_{n-1})^{-1} \cdots (1 - x_n x_{n-1} \cdots x_1)^{-1}]$$
$$= \frac{\prod_{1 \leqslant i < j \leqslant n} (x_j - x_i)}{\prod_{i=1}^{n} (1 - x_i) \prod_{1 \leqslant i < j \leqslant n} (1 - x_i x_j)}.$$

This is easily seen to be equivalent to Schur's identity that sums all the Schur functions (e.g., [Macd1, Ex. I.5.4, p. 45]). Here I give another proof which, incidentally, also proves Schur's identity. Let us call the left side $f(x_1, ..., x_n)$, and the right side $g(x_1, ..., x_n)$. Separating, the sum over $S_n$ in the definition of $f$ into the $n$ subsets of $S_n$, according to the values of $\pi(n)$, we have (chopping $\pi(n)=i$ amounts to losing $n-i$ inversions)

$$f(x_1, ..., x_n)=(1-x_1x_2\cdots x_n)^{-1}\sum_{i=1}^{n}(-1)^{(n-i)}f(x_1, x_2, ..., \hat{x}_i, ..., x_n).$$

Since $f(x_1)=g(x_1)$ is obviously true, $f=g$ is true for $n=1$. The identity would follow by induction if we can prove that

$$g(x_1, ..., x_n)=(1-x_1x_2\cdots x_n)^{-1}\sum_{i=1}^{n}(-1)^{(n-i)}g(x_1, x_2, ..., \hat{x}_i, ..., x_n),$$

which is equivalent to

$$(1-x_1\cdots x_n)=\sum_{i=1}^{n}(-1)^{n-i}g(x_1, ..., \hat{x}_i, ..., x_n)/g(x_1, ..., x_n).$$

But it is readily seen that

$$(-1)^{n-i}\frac{g(x_1, ..., \hat{x}_i, ..., x_n)}{g(x_1, ..., x_n)}=(1-x_i)\prod_{1\leqslant j\leqslant n,\, j\neq i}(1-x_ix_j)/(x_i-x_j).$$

It thus remains to prove that

$$(1-x_1\cdots x_n)=\sum_{i=1}^{n}(1-x_i)\prod_{1\leqslant j\leqslant n,\, j\neq i}(x_i-x_j)/(1-x_ix_j).$$

This is a rational function identity resembling those of Good [Go], Gustafson and Milne [Gu–Mi], Gross and Richards [Gr–Ri], and Milne [Mi]. It is easily proved by Lagange-interpolating the degree-$n$ polynomial (in $z$) $f(z):=(1-zx_1)\cdots(1-zx_n)$ at the $n+1$ points $\{1, x_1, ..., x_n\}$, substituting $z=0$, and finally making trivial adjustments.

9. This is (7) in reverse.

## 5. Robbins's Conjectured Expression for $D'$ and a More General (and Hence Easier) Constant Term Conjecture

Robbins [Ro2], in a private communication, made the following conjecture regarding the sum of minors $D'$. Let $B'(m, n)$ be defined by $B'(0, n)=B(n)$, and

$$\frac{B'(m+1, n)}{B'(m, n)}=2\prod_{j=1}^{n-1}\frac{(2m+j+2)(3m+2n+2+j)}{(m+1+j)(3m+2+2j)};$$

then

CONJECTURE $D'=B'$ (Robbins [Ro2]). $D'=B'$.

Conjecture $D'=B'$ would, of course, follow from the following constant term conjecture:

*Conjecture* $C'=B'$. $C'=B'$.

$C'=B'$, being more general than $C=B$, should be easier to prove. Its proof will, in particular, solve the 25-dollar problem stated above.

## 6. Sums of Minors of Generalized Binomial Coefficients Matrices

Our approach to expressing the sums of minors of the matrix $X$ took advantage of the fact that its entries were representable as constant terms as

$$X_{i,j}=CT\frac{(1+x)^{m+i-1}}{x^{j-i}}=CT\left[(1+x)^m\frac{((1+x)x)^{i-1}}{x^{j-1}}\right].$$

Scanning the proof of $D'=C'$ given in Section 3, we see that it is still valied when $(1+x)^m$ is replaced by a general polynomial $f(x)$ and $(1+x)x$ is replaced by a general polynomial $g(x)$. We thus have

GENERAL THEOREM. *Let $f(x)$ and $g(x)$ be polynomials and consider the $n\times(\deg(f)+(n-1)\deg(g)+1)$ matrix whose entries are given by*

$$X_{i,j}:=CT\left[\frac{f(x)\,g(x)^{i-1}}{x^{j-1}}\right].$$

*The sum of all its $n\times n$ minors equals*

$$\begin{aligned}&CT\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)\prod_{i=1}^{n}f(x_i^{-1})\,g(x_i^{-1})^{n-i}\\&\qquad\times\left\{\prod_{i=1}^{n}(1-x_i)^{-1}\prod_{1\leqslant i<j\leqslant n}(1-x_ix_j)^{-1}\right\}\\&\quad=\frac{1}{n!}CT\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)\prod_{i=1}^{n}f(x_i^{-1})\prod_{1\leqslant i<j\leqslant n}(g(x_j^{-1})-g(x_i^{-1}))\\&\qquad\times\left\{\prod_{i=1}^{n}(1-x_i)^{-1}\prod_{1\leqslant i<j\leqslant n}(1-x_ix_j)^{-1}\right\}.\end{aligned}$$

*Proof.* Do a *global* "*replace*" in Section 3, replacing $(1+x\_)^m$ by $f(x)$ and $x\_(1+x\_)$ by $g(x)$. (Here we used the MATHEMATICA convention of transformation rules, where "$x\_$" means "anything, to be called $x$.") The second equality follows from explantion 4 of Section 4.

## Another Constant Term Identity

Taking $f(x)=(1+x)^m$, $g(x)=(1+x)$, the matrix $X$ becomes

$$X_{i,j} := \binom{m+i-1}{j-i}, \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n+m. \tag{$*$}$$

The sum of its $n \times n$ minors can be computed explicitly, since each minor is nothing but a Schur function evaluated at $x_1 = x_2 = \cdots = x_n = 1$, by the Jacobi–Trudi formula (e.g. [Macd1, p. 25, (3.5)]), and these can be summed ($q=1$ in [Macd1, p. 52, (4)]; see also [De1, De2] for a superb exposition about summing Schur functions) to yield

$$\prod_{j=0}^{n-1} \prod_{i=1}^{m} \frac{2i+j}{i+j},$$

as observed empirically by Robbins [Ro2]. Combining the general theorem with the above, we get the following elegant multi-variate constant term identity.

## Identity

$$\frac{1}{n!} CT \prod_{1 \leqslant i \neq j \leqslant n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=1}^{n} (1+x_i^{-1})^m$$
$$\times \left\{ \prod_{i=1}^{n} (1-x_i)^{-1} \prod_{1 \leqslant i < j \leqslant n} (1-x_i x_j)^{-1} \right\} = \prod_{j=0}^{n-1} \prod_{i=1}^{m} \frac{2i+j}{i+j}.$$

I am offering 5 U.S. dollars for a direct proof of this identity.

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