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Journal of Pure and Applied Algebra 186 (2004) 77–89
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

The limiting behavior on the restriction of divisor classes to hypersurfaces

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Received 4 October 2002; received in revised form 3 April 2003
Communicated by A.V. Geramita

Abstract

Let A be an excellent local normal domain and $\{f_n\}_{n=1}^{\infty}$ a sequence of elements lying in successively higher powers of the maximal ideal, such that each hypersurface A/f_nA satisfies R_1. We investigate the injectivity of the maps $\mathrm{Cl}(A) \to \mathrm{Cl}((A/f_nA)')$, where $(A/f_nA)'$ represents the integral closure. The first result shows that no non-trivial divisor class can lie in every kernel. Secondly, when A is, in addition, an isolated singularity containing a field of characteristic zero, $\dim A \geqslant 4$, and A has a small Cohen–Macaulay module, then we show that there is an integer $N > 0$ such that if $f_n \in \mathfrak{m}^N$, then $\mathrm{Cl}(A) \to \mathrm{Cl}((A/f_nA)')$ is injective. We substantiate these results with a general construction that provides a large collection of examples.

MSC: Primary: 13F40; 13B22; secondary: 13C14; 14C20

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1. Introduction

The development of the divisor class group of a Noetherian normal domain A is due, in large part, to Samuel's work [23,24] with unique factorization domains (UFDs) in the 1960s. Roughly speaking, the divisor class group of A, denoted by $\mathrm{Cl}(A)$, is a measure of the extent to which A fails to be a UFD. In particular, $\mathrm{Cl}(A)$ is trivial if and only if A is a UFD. Samuel [24, p. 171] conjectured the following: If B is a complete local UFD, then $B[[T]]$ is a UFD. However, without additional restrictions, this conjecture is false. Perhaps surprisingly, counterexamples to this conjecture, as well as subsequent research in the subject of divisor class groups, relied heavily upon methods from algebraic geometry. For instance, using projective schemes, Danilov

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doi:10.1016/S0022-4049(03)00123-3

[8, Proposition 1.1] established a map j^*: $\mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$. Then in a series of articles [6–8], he characterized the injectivity of j^*. These results in some ways parallel those of Grothendieck [14, Lemma 3.16], who found conditions under which the homomorphism from the Picard group of the punctured spectrum of A to that of a hypersurface is injective.

Let f be a prime element such that A/fA is normal. Lipman [17, pp. 205–206] generalized Danilov's map by showing that there is a homomorphism of divisor class groups j^*: $\mathrm{Cl}(A) \to \mathrm{Cl}(A/fA)$. Many examples exist where j^* is not injective. When A is, in addition, an excellent $\mathbb{Q}$-algebra, Griffith and Weston [13, Corollary 1.3] gave conditions for the kernel of j^* to be torsion free. Then, in 1996, Miller [19, Sections 4 & 5], generalized the notion of divisor class group to rings satisfying the Serre condition S_2 and proved that $\bigcap_{n=1}^{\infty} \mathrm{Ker}(\mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A[[T]]/(T^n)))$ is trivial. (Actually, Miller's generalization of the class group is subsumed by that of Call [3, appendix].)

This motivates the investigation into whether a similar result will hold more generally for a sequence of distinct elements. To be specific, let $(A, \mathfrak{m})$ be a Noetherian local normal domain and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of elements such that each $A_n = A/f_nA$ satisfies R_1 and $\lim_{n\to\infty} f_n = 0$ in the $\mathfrak{m}$-adic topology (i.e. $f_n \in \mathfrak{m}^{e_n}$ where $e_n \to \infty$ as $n \to \infty$). Then there is a map $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$, where A'_n represents the integral closure of A_n. We consider the following two questions:

1. Must it be the case that $\bigcap_{n=1}^{\infty} \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n))$ is trivial?
2. Are there situations where an integer $N > 0$ exists such that if $f_n \in \mathfrak{m}^N$, then $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$ is monic? In other words, if the answer to (1) is yes, must it be true that all but finitely many of the kernels are null? And if so, are there effective methods to determine N?

We take questions (1) and (2) as "principles" which govern the behavior of the group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$.

In Section 2, we begin by stating definitions and giving a review of several concepts that will be used in proving our results. This section provides the background information and references that the reader may find useful.

In Section 3, we answer (1) affirmatively when the ambient ring is excellent. Although this first result shows that no divisor class can be in all of the kernels of $\mathrm{Cl}(A) \to \mathrm{Cl}(A'_n)$, it does not give much of a connection between a given divisor class of the ambient ring and its image in the divisor class group of any specific hypersurface. However, it does suggest that the pathology of the map $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ lies near the "top" of the maximal ideal, where $f \in \mathfrak{m}$ is any element such that A/fA satisfies R_1.

In Section 4, our second theorem seeks to make a connection between divisor classes on the ambient ring and a hypersurface—at least concerning injectivity. We show the existence of an integer $N > 0$, such that if f is a element in $\mathfrak{m}^N$ with A/fA satisfying R_1, then the group homomorphism $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ is injective. In this case, we add the assumptions that A is an isolated singularity of dimension greater than three which contains the rationals. We also assume that A has a small Cohen–Macaulay module M. This result supplies evidence for an affirmative answer to the following question: For an excellent, normal, local isolated singularity A containing the rationals and $f \in \mathfrak{m}$ such that $(A/fA)'$ satisfies R_1, is $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ injective? We end with several examples that elucidate our results.

2. Preliminaries

There are several basic references for the material appearing in this section. The standard results of commutative and homological algebra appear in the ubiquitous Matsumura [18] and Rotman [22], respectively. For material on the divisor class group, one should refer to Bourbaki [2, Chapter 7] and Fossum [10].

Let A be a Noetherian normal domain. The *dual* of an A-module $\mathfrak{a}$ is $\mathrm{Hom}_A(\mathfrak{a}, A)$, denoted $\mathfrak{a}^\star$. Note that $\mathfrak{a}^{\star\star} := (\mathfrak{a}^\star)^\star$. There is a map σ: $\mathfrak{a} \to \mathfrak{a}^{\star\star}$, where $\sigma(x)$ is defined by $\sigma(x)(g) = g(x)$, for $x \in \mathfrak{a}$ and $g \in \mathfrak{a}^\star$. We say that $\mathfrak{a}$ is *reflexive* if σ is an isomorphism.

One formulation of the *divisor class group* of A is the group of isomorphism classes of reflexive ideals of A, or equivalently, reflexive A-modules of rank one. An element $[\mathfrak{a}] \in \mathrm{Cl}(A)$ is called a *divisor class*. Multiplication is defined by $[\mathfrak{a}] \cdot [\mathfrak{b}] = [(\mathfrak{a} \otimes \mathfrak{b})^{\star\star}]$, the identity element is $[A]$, and the inverse of $[\mathfrak{a}]$ is $[\mathfrak{a}^\star]$. This definition is equivalent to the classical additive definition of the divisor class group appearing in [2,10, p. 489, p. 29]. In particular, a reflexive height one ideal $\mathfrak{a}$ can be written uniquely as the primary decomposition $\bigcap_{j=1}^{s} \mathfrak{p}_j^{(e_j)}$, where the $\mathfrak{p}_j$ are height one prime ideals containing $\mathfrak{a}$. The notation $\mathfrak{a}^{(d)}$ means $\bigcap_{j=1}^{s} \mathfrak{p}_j^{(e_j d)}$.

There is also a notion of divisor for modules which are not necessarily of rank one. In particular, for a finitely generated A-module M, there exists a free submodule L of M such that M/L is a torsion module. Set $\chi(M/L) = \Sigma_{\mathfrak{p}}\ l_{\mathfrak{p}}(M/L) \cdot \mathfrak{p}$, where the sum is taken over all height one primes, and where $l_{\mathfrak{p}}$ denotes the length of $(M/L)_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module. This is a finite sum. The class of $\chi(M/L)$ in $\mathrm{Cl}(A)$ is called the *divisor class attached to M* and is denoted by $[M]$. In [2, Section 4.7, Proposition 16], it is demonstrated that $[M]$ is independent of the choice of L. For an ideal $\mathfrak{a}$ of A, the two definitions of divisor coincide, so there is no confusion in notation.

The following facts concerning attached divisors, taken from [19, Lemma 6.3], can be a useful tool for comparing divisor classes:

(2.1) *If I is an ideal of a normal domain A and M is a finitely generated torsion-free A-module of rank r, then* $[\mathrm{Hom}_A(I, M)] = -r[I] + [M]$.

Another important subject for our purposes is the S_2-ification of a ring. A ring S is an *S_2-ification* of A if, (i) it is module-finite over A, (ii) it satisfies the Serre condition S_2 over A, and (iii) $\mathrm{Coker}(A \to S)$ has no support in codimension one in A. If A has a canonical module, for example, if A is the homomorphic image of a Gorenstein ring, then A has an S_2-ification. Furthermore, when A satisfies R_1, the S_2-ification is the integral closure. This fact is instrumental in obtaining the maps $\mathrm{Cl}(A) \to \mathrm{Cl}((A_n)')$. (See Hochster–Huneke [16] for more details on S_2-ifications.) We collect a few facts concerning S_2-ifications, the third of which has a proof similar to the one given for [1, Proposition 4.1].

(2.2) *Let* $(A,\mathfrak{m})$ *be an excellent local domain and* f *an element of* $\mathfrak{m}$ *such that* A/fA *satisifes* R_1. *Then the integral closure of* A/fA *is local*; *in particular*, f *is a prime element*. (See [15, Section XIII], or [16, Proposition 3.9].)

(2.3) *Let* A *be a local ring satisfying* R_1 *such that* A *has an* S_2*-ification* A'. *Let* M *be a finitely generated torsion-free* A*-module. If* M *satisfies the condition* S_2, *then* M *is an* A'*-module.*

(2.4) *Let* A *be a normal ring. If* L *and* N *are finitely generated* A*-modules such that* N *satisfies* S_2, *then the module* $\mathrm{Hom}_A(L,N)$ *satisfies* S_2, *and there is an isomorphism* $\mathrm{Hom}_A(L^{**},N)\xrightarrow{\cong}\mathrm{Hom}_A(L,N)$.

We end this section with some additional definitions and two lemmas which will be useful in the proof of our first theorem. The proofs of the lemmas rely on the concepts introduced here.

Recall that a submodule N of M is *pure* if the sequence $0\to N\otimes L\to M\otimes L$ is exact for every A-module L. A module N is called *pure injective* if, whenever the injection $N\to M$ is pure, then it splits. Warfield [27] and Griffith [12, Section 3] are good references for the preceding definitions. Next, an A-module M is said to be $\mathfrak{m}$-*divisible* if $\mathfrak{m}\cdot M=M$. Note that if an A-submodule N of M is pure, then the unique maximal $\mathfrak{m}$-divisible submodule of M/N is $\overline{N}/N$, where $\overline{N}$ represents the $\mathfrak{m}$-adic closure of N in M. As a result, $M/\overline{N}$ has no $\mathfrak{m}$-divisible submodule.

Lemma 2.5. *Let* $(A,\mathfrak{m})$ *be a local ring,* $M=\prod A$ *the countable direct product of copies of* A, *and* $N=\coprod A$ *the direct sum. Then* $M/\overline{N}$ *is faithfully flat.* (*Note that* $\overline{N}=\{\langle a_n\rangle\in M\,|\,a_n\to 0$ *in the* $\mathfrak{m}$*-adic topology*}.)

Proof. By Griffith [11, Lemma 1.7], $\overline{N}$ is a pure submodule of the flat module M. Therefore, it is flat, and $M/\overline{N}$ is flat. Since $\overline{N}/N$ is the maximal $\mathfrak{m}$-divisible submodule in M/N, $\mathfrak{m}\cdot M/\overline{N}\neq M/\overline{N}$. □

Lemma 2.6. *Let* $f_1,f_2,f_3,\ldots$ *be a sequence of prime elements in a complete local ring* $(A,\mathfrak{m})$ *such that* $\lim_{n\to\infty} f_n=0$ *in the* $\mathfrak{m}$*-adic topology. Set* $P=\prod A_n$. *Then the map* $A\hookrightarrow P$ *splits.*

Proof. Let M be a finitely generated A-module of finite length; say $\mathfrak{m}^r M=0$ for $r\gg 0$. Choose $n\gg 0$ such that $e_n\geqslant r$. Consider the map $M\to\prod M/f_nM$, where for $x\in M, x\mapsto(x+f_1M,x+f_2M,\ldots)$. The nth component $x+f_nM$ equals x, which shows that the map is an injection. Consequently, by Griffith [12, Corollary 3.2], A is a pure submodule of P. Let $E=E_A(k)$. By Griffith [12, Proposition 3.6], $\mathrm{Hom}_A(E,E)=A$ is pure injective, which gives the result. □

3. First theorem

We begin with a statement of our first theorem, motivated by the first principle in the introduction.

Theorem 3.1. *Let* $(A,\mathfrak{m},k)$ *be an excellent, normal, local domain and let* $f_1, f_2, f_3, \ldots$ *be a sequence of elements in* A *such that*

(a) $\lim_{n\to\infty} f_n = 0$ *in the* $\mathfrak{m}$*-adic topology, and*
(b) $A_n = A/f_nA$ *satisfies* R_1, *for each* n.

Then $\bigcap_{n=1}^{\infty} \operatorname{Ker}(\operatorname{Cl}(A) \to \operatorname{Cl}(A'_n))$ *is trivial, where* A'_n *represents the integral closure of* A.

Before beginning the proof, we provide some necessary discussion. Because of the assumption of excellence on A, we can assume that A is complete. The only detail in passing to the completion that is a possible cause for concern is that $\widehat{A_n}$ remains a domain. But this follows from (2.2).

There exists a regular local ring $R \subset A$ such that A is a finite R-module. Set $A^* = \operatorname{Hom}_R(A,R)$. Since A satisfies S_2 as an R-module, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ of codimension less than or equal to two, $A_{\mathfrak{p}}$ is a maximal Cohen–Macaulay module over the regular local ring $R_{\mathfrak{p}}$. As a result, the height of $\operatorname{ann}_A \operatorname{Ext}^1_R(A^*,R)$ is greater than or equal to three.

For each n, there is a short exact sequence $0 \to A \xrightarrow{\cdot f_n} A \to A_n \to 0$, and thus an exact sequence $0 \to \operatorname{Hom}_R(A,R) \xrightarrow{\cdot f_n} \operatorname{Hom}_R(A,R) \to \operatorname{Ext}^1_R(A_n,R)$. Likewise, the short exact sequence $0 \to A^* \xrightarrow{\cdot f_n} A^* \to \overline{A^*} \to 0$, where $\overline{A^*} = A^*/f_nA^*$, yields the long exact sequence

$0 \to \operatorname{Hom}_R(\overline{A^*},R) \to \operatorname{Hom}_R(A^*,R) \xrightarrow{\cdot f_n} \operatorname{Hom}_R(A^*,R) \to \operatorname{Ext}^1_R(\overline{A^*},R) \to \cdots$. Now $\operatorname{Hom}_R(A^*,R) = A^{**} \cong A$ since R is normal and A satisfies S_2 as an R-module. These results are summarized in the commutative diagram below, where the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & A^{**} \arrow[r, "\cdot f_n"] \arrow[dd, "\cong"'] & A^{**} \arrow[rr] \arrow[dd, "\cong"'] \arrow[dr, two heads] & & \operatorname{Ext}^1_R(\overline{A^*},R) \arrow[r] & \operatorname{Ext}^1_R(A^*,R) \\
& & & \operatorname{Coker}(\cdot f_n) \arrow[ur, hook] & & \\
0 \arrow[r] & A \arrow[r, "\cdot f_n"] & A \arrow[r] & A_n \arrow[r] & 0 &
\end{tikzcd}$$

The sequence $0 \to A_n \to \operatorname{Ext}^1_R(\overline{A^*},R) \to \operatorname{Ext}^1_R(A^*,R)$ is exact. Set $\bar{R} = R/(f_nA \cap R)$. We claim that $\operatorname{Ext}^1_R(\overline{A^*},R)$, or equivalently $\operatorname{Hom}_R(\overline{A^*},\bar{R})$, is an S_2-ification of A_n. Since it is straightforward to show that $\operatorname{Hom}_R(\overline{A^*},\bar{R})$ satisfies S_2 and is finitely generated over A_n, we need only establish that $\operatorname{Coker}(A_n \to \operatorname{Hom}_R(\overline{A^*},\bar{R}))$ has no support in codimension one in A_n. But this follows from the fact that $\operatorname{ht}_A \operatorname{ann}_A \operatorname{Ext}^1_R(A^*,R) \geqslant 3$.

Lemma 3.2. *There is a finitely generated* A*-module* W, *independent of* n, *with* $ht_A \operatorname{ann} W \geqslant 3$, *such that for every* n, A'_n/A_n *is isomorphic to a submodule of* W.

Proof. Since $\operatorname{Hom}_R(\overline{A^*},\bar{R})) = A'_n$, take W to be $\operatorname{Ext}^1_A(A^*,R)$. $\square$

From (2.6), recall that $P := \prod A_n$ and that $A \to P$ splits. Set $P' = \prod A'_n, S = \coprod A_n$, and $S' = \coprod A'_n$.

Lemma 3.3. *There is a commutative diagram*:

$$\begin{tikzcd}
P \arrow[rr, hook] \arrow[dr] & & P' \arrow[dl] \\
& A &
\end{tikzcd}$$

Proof. $\operatorname{Ann} W$ contains an A-sequence of length two, which by (3.2), is in $\operatorname{Ann}(P'/P)$ as well. Thus, $\operatorname{Ext}^i_A(P'/P, A) = 0$ for $i = 0, 1$. The claim follows by applying $\operatorname{Hom}_A(-, A)$ to the exact sequence $0 \to P \to P' \to P'/P \to 0$. □

Remark 3.4. A is also a direct summand of P/S since the image of $A \to P$ has a trivial intersection with S and the splitting map sends S to 0 in A. Consequently, the argument of (3.3) can be applied to $P/S \hookrightarrow P'/S'$ in order to conclude that A is also a direct summand of P'/S'.

Proof of Theorem 3.1. Let $[\mathfrak{a}] \in \bigcap_{n=1}^{\infty} \operatorname{Ker}(\operatorname{Cl}(A) \to \operatorname{Cl}(A'_n))$. The maps $\operatorname{Cl}(A) \to \operatorname{Cl}(A'_n)$ are defined by $[\mathfrak{a}] \mapsto [(\mathfrak{a} \otimes_A A'_n)^{**}]$, where the duals are taken with respect to A'_n. It suffices to show that $[\mathfrak{a}^\star]$ is trivial, where $\mathfrak{a}^\star = \operatorname{Hom}_A(\mathfrak{a}, A)$. Note that for each n, $\operatorname{Hom}_A(\mathfrak{a}, A'_n) \cong A'_n$. Thus, $\operatorname{Hom}_A(\mathfrak{a}, P') \cong P'$, and since $\mathfrak{a}$ is finitely generated, $\operatorname{Hom}_A(\mathfrak{a}, S') \cong S'$. As a result, $\operatorname{Hom}_A(\mathfrak{a}, P'/S') \cong P'/S'$, since the sequence $0 \to S' \to P' \to P'/S' \to 0$ is pure exact.

Because the f_n's go to zero in the $\mathfrak{m}$-adic topology, the $\mathfrak{m}$-adic closure of S in P, denoted by $\overline{S}$, is $\{\langle \overline{a_n} \rangle \in P \mid a_n \to 0$ in the $\mathfrak{m}$-adic topology on $A\}$.

Thus, the map $\prod A / \coprod A \to P/\overline{S}$, defined by $\langle a_n \rangle + \coprod A \mapsto \langle \overline{a_n} \rangle + \bar{S}$, is an isomorphism. By (2.5), $P/\overline{S}$ is faithfully flat over A. Consequently, one can see that sequence $0 \to P/\overline{S} \to P'/\overline{S} \to P'/P \to 0$ is split by applying $\operatorname{Hom}_A(-, P/\overline{S})$ and using the methods of (3.3). □

Lemma 3.5. *Any finitely generated torsion-free direct summand N of P'/S' is a direct summand of $P/\overline{S}$.*

Proof. Let $P'/S' = N \oplus K$. Making use of the fact that $S = S' \cap \overline{S}$, the short exact sequence $0 \to (S' + \overline{S})/(S') \to P'/S' \to P'/(S' + \overline{S}) \to 0$ can be rewritten as $0 \to (S' + \overline{S})/S' \to N \oplus K \to P/\overline{S} \oplus T \to 0$, where T is a torsion A-module. Note that $(S' + \overline{S})/S'$ is $\mathfrak{m}$-divisible since it is isomorphic to $\overline{S}/S$. Consequently, it must map into K. Thus, N is a direct summand of $P/\overline{S}$. □

Conclusion: As per (3.4), because A is a direct summand of P'/S', $\mathfrak{a}^\star$ is a direct summand of $\operatorname{Hom}_A(\mathfrak{a}, P'/S') \cong P'/S'$. By the previous claim, $\mathfrak{a}^\star$ is a direct summand of $P/\overline{S}$, which is faithfully flat. Consequently, $\mathfrak{a}^\star$ is flat, or equivalently, A-free. In other words, $[\mathfrak{a}^\star]$ is trivial.

4. Second theorem

As we stated in the introduction, our second theorem will provide a connection between a divisor class on the ambient ring and its image in the divisor class group of a specific hypersurface—a connection that Theorem 3.1 does not address. However, 3.1 does suggest that the pathology of the map $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ lies near the "top" of the maximal ideal. In fact, we put forth the following question: Let A be an excellent, normal, local $\mathbb{Q}$-algebra such that A is an isolated singularity of dimension at least four. For any $f \in \mathfrak{m}$ such that $(A/fA)'$ satisfies R_1, is the map $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ injective? We supply evidence for an affirmative answer to this query in the case where A has a *small Cohen–Macaulay module*. Such a module is finitely generated and has depth equal to the dimension of A. For such a ring A, we can identify an integer $N > 0$ having the distinction that, when $f \in \mathfrak{m}^N$ is such that A/fA satisfies R_1, then the map $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ is injective. This is our next result.

Theorem 4.1. *Let $(A, \mathfrak{m}, k)$ be an excellent, normal, local $\mathbb{Q}$-algebra such that A is an isolated singularity of dimension at least four. In addition, suppose that A has a small Cohen–Macaulay module M. Then there is an $N > 0$, depending only on the ring A, such that the following holds: If $f \in \mathfrak{m}^N$ is such that A/fA satisfies R_1, then $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ is injective.*

As in Section 3, we give some discussion before proceeding with the proof. Again, we can assume that A is complete. Set $\dim A = d$. For every system of parameters of A, there is a regular local ring R which is a subring of A and over which A is module-finite. Let Λ be the enveloping algebra. In this case, $\Lambda = A \otimes_R A$. Let $\mu : \Lambda \to A$ be the surjection defined by $\mu(a \otimes b) = ab$. Set $\mathfrak{J} = \mathrm{Ker}(\mu)$ and $\eta = \mathrm{ann}_\Lambda \mathfrak{J}$. Then the *Noetherian different* of the R-algebra A, as defined by the eponymous Noether [20], is the ideal $\mu(\eta)$, denoted by $\mathfrak{N}_{A/R}$. Let $\mathfrak{R}$ be the set of all regular local rings R obtained as above. Set $\mathfrak{N}_A = \Sigma_R \mathfrak{N}_{A/R}$, where the sum is taken over all $R \in \mathfrak{R}$. This ideal will play a central role in the proof of Theorem 4.1, as evinced by the following fact:

(4.2) *The ideal $\mathfrak{N}_A$ defines the singular locus of* $\mathrm{Spec}(A)$; *i.e., for $\mathfrak{P} \in \mathrm{Spec}(A), A_{\mathfrak{P}}$ is regular if and only if $\mathfrak{P}$ does not contain $\mathfrak{N}_A$.* (The proof of this uses the fact that A is an isolated singularity and is similar to the one that appears in [28, Lemma 6.12].)

As a result of (4.2), $\mathfrak{N}_A$ is an $\mathfrak{m}$-primary ideal; say $\mathfrak{m}^N \subset \mathfrak{N}_A$, for some $N > 0$. This is the integer in Theorem 4.1 that we wanted to identify. Let $0 \neq f \in \mathfrak{m}^N$ be an element such that A/fA satisfies R_1, and set $\overline{M} = M/fM$, where M is the small Cohen–Macaulay module in the statement of the theorem.

It happens that f, by virtue of belonging to $\mathfrak{N}_A$, annihilates $\mathrm{Ext}^1_A(L, -)$, for any lift L of $\overline{M}$. A finitely generated A-module L is a *lift* of $\overline{M}$ if there is a short exact sequence $0 \to L \xrightarrow{\cdot f} L \to \overline{M} \to 0$. The fact that f annihilates $\mathrm{Ext}^1_A(L, -)$ is important because it will allow us to establish the existence of only finitely many lifts of $\overline{M}$, which is a key part of our proof.

Before demonstrating all of this, we need a few facts about Hochschild cohomology, since it plays a crucial role in the annihilation of $\mathrm{Ext}^1_A(L, -)$. For any A-bimodule

W, the nth *Hochschild cohomology* module, $\mathrm{HH}^n_R(A,W)$, is obtained by taking the homology of the complex:

$$W \xrightarrow{d^0} \mathrm{Hom}_R(A,W) \xrightarrow{d^1} \mathrm{Hom}_R(A\otimes_R A,W) \xrightarrow{d^2} \cdots.$$

In particular, $\mathrm{HH}^0_R(A,W)=\ker(d^0)=W^{(A)}=\{w\in W \mid aw=wa,\ \forall a\in A\}$. For details, refer to [21, Chapter 11]. We are now equipped to prove the following preliminary lemma.

Claim 4.3. *For each $R\in\mathfrak{R}$, the Noetherian different $\mathfrak{N}_{A/R}$ annihilates $\mathrm{HH}^1_R(A,-)$.*

Proof. For any A-module W, by applying $\mathrm{Hom}_\Lambda(-,W)$ to the short exact sequence $0\to\mathfrak{J}\to\Lambda\xrightarrow{\mu}A\to 0$, one obtains a long exact sequence:

$$0\to\mathrm{Hom}_\Lambda(A,W)\to W\to\mathrm{Hom}_\Lambda(\mathfrak{J},W)\xrightarrow{\delta}\mathrm{Ext}^1_\Lambda(A,W)\to 0.$$

By the surjectivity of δ and the definition of η, it is easy to see that $\eta\cdot\mathrm{Ext}^1_\Lambda(A,W)=0$. Thus, $\mathfrak{N}_{A/R}$ annihilates $\mathrm{Ext}^1_\Lambda(A,W)$, which is isomorphic to $\mathrm{HH}^1_R(A,W)$. (See [4, p. 169] for details.) □

Claim 4.4. *For any lift L of $\overline{M}$, $\mathfrak{N}_A$ annihilates $\mathrm{Ext}^1_A(L,-)$.*

Proof. Let $R\in\mathfrak{R}$. Then any lift L is R-free. Let $0\to K\to F\to L\to 0$ be a short exact sequence of A-modules, where F is A-free. For any A-module W, $0\to\mathrm{Hom}_R(L,W)\to\mathrm{Hom}_R(F,W)\to\mathrm{Hom}_R(K,W)\to 0$ is a short exact sequence. Using the notation $\mathrm{Hom}_R(L,W)=[L,W]_R$, there is a long exact sequence of Hochschild cohomology: $0\to\mathrm{HH}^0_R(A,[L,W]_R)\to\mathrm{HH}^0_R(A,[F,W]_R)\to\mathrm{HH}^0_R(A,[K,W]_R)\to\mathrm{HH}^1_R(A,[L,W]_R)\to\cdots$

By definition, $\mathrm{HH}^0_R(A,[L,W]_R)=([L,W]_R)^{(A)}=[L,W]_A$. Therefore, the claim follows from (4.3) and the commutative diagram below, where the rows are exact: □

$$\begin{tikzcd}
([F,W]_R)^{(A)} \arrow[r] \arrow[dd, equal] & ([K,W]_R)^{(A)} \arrow[rr] \arrow[dd, equal] \arrow[dr, two heads] & & \mathrm{HH}^1_R(A,[L,W]_R) \\
 & & \mathrm{Coker} \arrow[ur, hook] & \\
{[F,W]_A} \arrow[r] & {[K,W]_A} \arrow[r] & \mathrm{Ext}^1_A(L,W) \arrow[r] & 0
\end{tikzcd}$$

Claim 4.5. *There are only finitely many lifts of $\overline{M}$.*

Proof. Let L be a lift of $\overline{M}$ and let F be a free A-module with rank equal to the number of minimal generators of L. There is a pullback diagram for the homomorphisms $\cdot f$

and π as seen below:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\epsilon f\colon & 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & K \oplus L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\cdot f"] & 0 \\
\epsilon\colon & 0 \arrow[r] & K \arrow[r] & F \arrow[r, "\pi"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & & \overline{M} \arrow[r, equal] \arrow[d] & \overline{M} \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

Note that the top row is split exact, since it is obtained by multiplying the extension ϵ by f. Therefore, $K \oplus L \cong Z_1(\overline{M})$, the first syzygy of $\overline{M}$. $Z_1(\overline{M})$ is unique up to isomorphism of complexes since F maps onto $\overline{M}$ minimally. Likewise, $K \cong Z_1(L)$. As a result, $Z_1(L) \oplus L \cong Z_1(\overline{M})$ for any lift L of $\overline{M}$. Since A satisfies the Krull–Schmidt Theorem, as per Swan [26, p. 566], $Z_1(\overline{M}) = N_1 \oplus \cdots \oplus N_t$, where each N_i is indecomposable and unique up to isomorphism. Consequently, up to isomorphism, there can be only finitely many L. □

Proof of Theorem 4.1. The idea of the proof is to contradict the finite number of lifts of $\overline{M}$ just established. For simplicity, set $B = (A/fA)'$. Let $[\mathfrak{a}]$ be a non-trivial element in $\mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(B))$. From the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot f} M \longrightarrow \overline{M} \longrightarrow 0,$$

there is a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\mathfrak{a}, M) \xrightarrow{\cdot f} \mathrm{Hom}_A(\mathfrak{a}, M) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}, \overline{M}) \xrightarrow{\delta} \mathrm{Ext}^1_A(\mathfrak{a}, M).$$

Claim 4.6. $\mathrm{Hom}_A(\mathfrak{a}, \overline{M}) \cong \overline{M}$.

Proof. By (2.3), $\overline{M}$ is a B-module. Therefore

$$\mathrm{Hom}_A(\mathfrak{a}, \overline{M}) \cong \mathrm{Hom}_A(\mathfrak{a}, \mathrm{Hom}_B(B, \overline{M})) \cong \mathrm{Hom}_B(\mathfrak{a} \otimes_A B, \overline{M}).$$

Since $\overline{M}$ satisfies S_2 over B, by (2.4)

$$\mathrm{Hom}_B(\mathfrak{a} \otimes_A B, \overline{M}) \cong \mathrm{Hom}_B((\mathfrak{a} \otimes_A B)^{**}, \overline{M}),$$

where the dual is taken with respect to B. Since $[\mathfrak{a}] \in \mathrm{Ker}(\mathrm{Cl}(A) \to \mathrm{Cl}(B)), (\mathfrak{a} \otimes_A B)^{**} \cong B$. Thus, $\mathrm{Hom}_B((\mathfrak{a} \otimes_A B)^{**}, \overline{M}) \cong \mathrm{Hom}_B(B, \overline{M}) \cong \overline{M}$. □

Claim 4.7. $\mathrm{Ext}^1_A(\mathfrak{a}, M) = 0$.

Proof. Assume $\mathrm{Ext}^1_A(\mathfrak{a}, M) \neq 0$. Then it has finite length as an A-module, since $\mathfrak{a}_\mathfrak{p} \cong A_\mathfrak{p}$ for every prime $\mathfrak{p} \neq \mathfrak{m}$. In the long exact sequence preceding (4.6), set

$C = \operatorname{Coker}(\operatorname{Hom}_A(\mathfrak{a}, M) \to \overline{M})$. Then we have the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r, dash] & \operatorname{Hom}_A(\mathfrak{a}, M) \arrow[r, "\cdot f"] & \operatorname{Hom}_A(\mathfrak{a}, M) \arrow[rr] \arrow[dr, two heads] & & \overline{M} \arrow[r] & C \arrow[r] & 0 \\
& & & K \arrow[ur, hook] & & &
\end{tikzcd}$$

Depth $\operatorname{Hom}_A(\mathfrak{a}, M) \geqslant 2$ since M, and hence $\operatorname{Hom}_A(\mathfrak{a}, M)$, satisfies S_2 as an A-module. Since $\operatorname{depth}_A(\overline{M}) \geqslant 3$ and $\operatorname{depth}_A(C) = 0$, it follows that $\operatorname{depth}_A(K) = 1$. We will make use of these calculations shortly.

Let $R \in \mathfrak{N}$, with maximal ideal $\mathfrak{n}$. Then $\operatorname{Ext}^1_A(\mathfrak{a}, M)$ has finite length over R and $H^1_{\mathfrak{n}}(C) = 0$. From $0 \to K \to \overline{M} \to C \to 0$, we obtain the exact sequence $H^1_{\mathfrak{n}}(C) \to H^2_{\mathfrak{n}}(K) \to H^2_{\mathfrak{n}}(\overline{M})$, where $H^2_{\mathfrak{n}}(\overline{M}) = 0$ as well. As a result, $H^2_{\mathfrak{n}}(K) = 0$. Similarly, we obtain the exact sequence:

$$H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M)) \xrightarrow{\cdot f} H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M)) \to H^2_{\mathfrak{n}}(K) = 0.$$

Since the map $\cdot f$ is surjective, if $H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M))$ is finitely generated, then it equals zero. We claim that this is the case. Since $\operatorname{Hom}_A(\mathfrak{a}, M)$ is finitely generated over R, $H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M))$ satisfies the descending chain condition. By Matlis and local duality:

$$H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M)) \cong H^2_{\mathfrak{n}}(\operatorname{Hom}_A(\mathfrak{a}, M))^{\vee\vee} \cong \operatorname{Ext}^{d-2}_R(\operatorname{Hom}_A(\mathfrak{a}, M), R)^{\vee},$$

where $(-)^{\vee} = \operatorname{Hom}_R(-, E_R(k))$. $\operatorname{Ext}^{d-2}_R(\operatorname{Hom}_A(\mathfrak{a}, M), R)$ has finite length as an R-module. Consequently, $\operatorname{Ext}^{d-2}_R(\operatorname{Hom}_A(\mathfrak{a}, M), R)^{\vee}$ satisfies the ascending chain condition, as desired. Thus, since $H^2_{\mathfrak{m}}(\operatorname{Hom}_A(\mathfrak{a}, M)) = 0, \operatorname{depth}_A(\operatorname{Hom}_A(\mathfrak{a}, M))$ must be strictly greater than two, recalling our previous calculations. But this contradicts the depths as computed from the short exact sequence $0 \to \operatorname{Hom}_A(\mathfrak{a}, M) \xrightarrow{\cdot f} \operatorname{Hom}_A(\mathfrak{a}, M) \to K \to 0$, which proves the claim. □

This means that $\operatorname{Hom}_A(\mathfrak{a}, M)$ is a lift of $\overline{M}$. In other words, we have the short exact sequence:

$$0 \longrightarrow \operatorname{Hom}_A(\mathfrak{a}, M) \xrightarrow{\cdot f} \operatorname{Hom}_A(\mathfrak{a}, M) \longrightarrow \overline{M} \longrightarrow 0.$$

Thus, if $\operatorname{Ker}(\operatorname{Cl}(A) \to \operatorname{Cl}(B))$ is non-trivial, there are infinitely many lifts of $\overline{M}$. More specifically, by Griffith and Weston [13, Corollary 1.3], the kernel is torsion free; so $[\mathfrak{a}^{(m)}] \neq [\mathfrak{a}^{(n)}]$ for all $m, n > 0$. By (2.1), $[\operatorname{Hom}_A(\mathfrak{a}^{(m)}, M)] \neq [\operatorname{Hom}_A(\mathfrak{a}^{(n)}, M)]$. Thus, $\operatorname{Hom}_A(\mathfrak{a}^{(m)}, M)$ and $\operatorname{Hom}_A(\mathfrak{a}^{(n)}, M)$ are non-isomorphic lifts of $\overline{M}$ for all $m, n > 0$, which provides the contradiction, and thus proves the theorem. □

Remark 4.2. It should be noted that the above proof requires f to be in $\mathfrak{N}_A$, rather than in $\mathfrak{m}^N$. However, the integer N obtained gives a lower bound for injectivity of the map on divisor class groups.

Remark 4.3. This result gives rise to a couple of questions. Is Theorem 4.1 true without a small Cohen–Macaulay module? In other words, is a small Cohen–Macaulay module really necessary? Note that the assumption of a big Cohen–Macaulay module M will not suffice since one cannot argue that $[\mathrm{Hom}_A(\mathfrak{a}, M)]$ is non-trivial. This is just one of many places in the proof where finite generation is needed. Secondly, is the theorem true in characteristic $p > 0$ or mixed characteristic? In either case, there might be some p-torsion elements in the kernel of $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$. Finally, is there a hypersurface A/fA satisfying R_1, with $f \in \mathfrak{N}_A$, such that $\mathrm{Cl}(A) \to \mathrm{Cl}((A/fA)')$ is not injective? Such an A could not possess a small Cohen–Macaulay module, which would disprove the small Cohen–Macaulay conjecture. This remains an open question.

Example 4.1 (*Danilov* [7, p. 128]). Let $A = \mathbb{Q}[[X, Y, Z]]/(pX^3 + p^2Y^3 - aZ^3)$, where $a \in \{3, 4, 5, 10, 11, 14, 18, 21, \ldots\}$ is obtained from the study of Diophantine equations in [25, Table 4^g] and p is a prime that does not divide a. Then $j^*\colon\ \mathrm{Cl}(A[[T]]) \to \mathrm{Cl}(A)$ is not injective. Note that A represents an isolated singularity, but dim $A = 2$.

Example 4.2. Let $A = \mathbb{C}[X, Y, Z, W]/(XY - ZW)$. Then the domain $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ is a hypersurface of A since $B = A/(w - z)A$, where the lower case letters represent the images in the ring A. Hence, we obtain a map $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$. (Again, A represents an isolated singularity, but its dimension is three.) It can be shown that $\mathrm{Cl}(A) \cong \mathbb{Z}$ and $\mathrm{Cl}(B) \cong \mathbb{Z}_2$ by using the fact that both groups are generated by the ideal (x, z). The kernel of the map $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ is necessarily non-trivial. In fact, for any integer $n > 2$, if $B_n = \mathbb{C}[X, Y, Z]/(XY - Z^n)$, then $\mathrm{Cl}(B_n)$ is isomorphic to $\mathbb{Z}_n$. Therefore, the maps $\mathrm{Cl}(A) \to \mathrm{Cl}(B_n)$ all fail to be injective. However, these maps do not satisfy the hypotheses of Theorems 3.1 and 4.1 since the elements $f_n = XY - Z^n$ do not lie in higher and higher powers of the maximal ideal.

Remark 4.4. Non-trivial examples of the ring A described in Theorem 4.1 can be obtained by appealing to algebraic geometry. Let V be a non-singular variety over an algebraically closed field k of characteristic zero such that its homogeneous coordinate ring $S(V)$ has a small Cohen–Macaulay module M. It suffices to let V be any non-singular irreducible hypersurface in $\mathbb{P}^3_k$, like $V = Z(X_0^4 + X_1^4 + X_2^4 + X_3^4)$, which does not satisfy Chow's condition of *proper* [5, pp. 816–818]. "Enlarge" V by taking its product with $\mathbb{P}^1_k$. Call this product W. There is a commutative diagram, where the rows and columns are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z} \arrow[r, "\cong"] \arrow[d] & \mathbb{Z} \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathrm{Cl}(W) \arrow[r] \arrow[d] & \mathrm{Cl}(S(W)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Cl}(V) \arrow[r] \arrow[d] & \mathrm{Cl}(S(V)) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

One can see that the torsion-free rank of $\mathrm{Cl}(S(W))$, as an abelian group, grows from that of $\mathrm{Cl}(S(V))$ by a factor of $\mathbb{Z}$. This process can be iterated, so that at each step we obtain a non-Cohen–Macaulay ring whose dimension has grown by one and whose divisor class group has grown by $\mathbb{Z}$.

Finally, note that irreducible hypersurface sections satisfying R_1 are guaranteed by Bertini's Theorem [9, p. 10]. One can also generate irreducible hypersurface sections in the generic way described below.

Example 4.3. Let $(A, \mathfrak{m})$ be an excellent local normal domain that is an isolated singularity of dimension $d \geqslant 4$. Set $B = A[X_1, X_2, \ldots, X_d]$. Then $B_{\mathfrak{m}[\underline{X}]}$ retains the relevant properties of A, with $\mathrm{Cl}(B_{\mathfrak{m}[\underline{X}]}) \cong \mathrm{Cl}(A)$. The elements $f_n = \sum_{i=1}^{d} a_i^n X_i$, where $\{a_1, \ldots, a_d\}$ is a system of parameters for A, represent a sequence of elements $f_1, f_2, \ldots$ of $B_{\mathfrak{m}[\underline{X}]}$ as in Theorems 3.1 and 4.1.

Acknowledgements

I would like to extend a special thanks to my advisor, Dr. Phillip Griffith, for all of his help with this manuscript. Secondly, I would like to acknowledge Sean Sather–Wagstaff and the referee for their suggestions on earlier versions of this paper. In addition, I would like to thank N. Mohan Kumar, and more generally, the Department of Mathematics at Washington University in St. Louis, where this paper was written in part. Finally, I express my gratitude to Melvin Hochster and Doug Hanes for their assistance with some material in my dissertation since this paper is based on that thesis.

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