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A controlling cohomology of the deformation theory of Lie triple systems

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Abstract

The deformation theory of Lie triple systems is developed. We shall present that the Yamaguti cohomology plays a crucial role in this theory.

Keywords: Lie triple system; Algebraic deformation; Yamaguti cohomology

Introduction

Lie triple system

The Lie triple system was first formulated in terms of the identities by Jacobson [11, 12]. Examing these identities and eliminating two from them, Yamaguti [14] established the present formulation of a Lie triple system.

Let k be a field of characteristic zero. A *Lie triple system* (Lts) is a vector space T over k with a trilinear multiplication $[a\,b\,c]$ satisfying

$$[a\,a\,b] = 0, \tag{1}$$

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$$[a\,b\,c] + [b\,c\,a] + [c\,a\,b] = 0, \tag{2}$$

$$\big[a\,b\,[c\,d\,e]\big] = \big[[a\,b\,c]\,d\,e\big] + \big[c\,[a\,b\,d]\,e\big] + \big[c\,d\,[a\,b\,e]\big] \tag{3}$$

for $a, b, c, d, e \in T$. Equation (3) says that $D(a,b) : T \to T$ defined by $D(a,b)(z) := [a\,b\,z]$ is a derivation of the trilinear composition $[- - -]$.

Let L be a Lie algebra with product $[a, b]$, then the ternary composition $[a\,b\,c] = [\![a, b], c]$ satisfies the above identities, hence L is a Lts. Conversely, any Lts T can be considered as a subspace of a Lie algebra as follows (Bertram [1], Jacobson [12]): Let $H := \mathrm{Der}(T) \subset \mathrm{End}(T)$ be the space of derivations of the Lie triple product on T. Then the direct sum $T \oplus \mathrm{Der}(T)$ turns out to be a Lie algebra with a bilinear product $[- -]$ given by $[a, b] := D(a, b)$, $[a, f] := -f(a)$, $[f, a] := -[a, f]$, $[f, g] := f \circ g - g \circ f$, for $a, b \in T$, $f, g \in \mathrm{Der}(T)$. The standard-imbedding of T is the subalgebra $L(T) := T \oplus [T, T]$ of the Lie algebra $T \oplus \mathrm{Der}(T)$. The Lts T is realized as the -1-eigenspace of the involution $\sigma : L(T) \to L(T)$ defined by $\sigma((a, f)) := (-a, f)$. This observation will enable us to apply the well developed theory for Lie algebras to study Lts's. In fact, Lister [13] constructed a structure theory of Lts's, and Harris [10] developed a cohomology theory of Lts's on this line.

On the other hand, Yamaguti's approach [14] to a cohomology theory of Lts's was intrinsic. His cohomology theory is discussed without going out of a Lts into an enveloping Lie algebra, so that the Yamaguti coboundary is defined in terms of only elements of a Lts.

Deformation theory

The deformation theory of algebras was introduced by Gerstenhaber in a series of papers [3–7]. It has subsequently been extended to covariant functors from a small category to algebras [8] and to algebraic systems, bialgebras, Hopf algebras [9] by Gerstenhaber and Schack, also to Leibniz pairs and Poisson algebras [2] by Flato, Gerstenhaber and Voronov, etc.

In this paper we shall develop a deformation theory of Lie triple systems, keeping in mind the following aspects due to Gerstenhaber [4]:

Aspects of deformation theory

(1) A definition of the class of objects within which *deformation* takes place and identification of the *infinitesimals* of a given object with the elements of a suitable *cohomology group*.
(2) A theory of the *obstructions* to the *integration* of an infinitesimal deformation.
(3) A *parameterization* of the set of objects.
(4) A determination of the natural automorphisms of the parameter space and the determination of the *rigid* objects.

Let us recall that a suitable cohomology group for the deformation theory of associative algebras, Lie algebras and noncommutative Poisson algebras is the Hochschild cohomology, the Chevalley–Eilenberg cohomology and the total cohomology of them, respectively.

1. Deformation of a Lie triple system

Let T be a Lts over k with a ternary composition $\alpha(-,-,-) = [- - -] : T \times T \times T \to T$. A *d*eformation of T is a formal power series $\alpha_t : T[\![t]\!] \times T[\![t]\!] \times T[\![t]\!] \to T[\![t]\!]$ (t: deformation parameter) of the form

$$\alpha_t(a,b,c) := \alpha_0(a,b,c) + t\alpha_1(a,b,c) + t^2\alpha_2(a,b,c) + \cdots,$$

where each $\alpha_i : T \times T \times T \to T$ is a k-trilinear map and extended to that of the power series ring $k[\![t]\!]$ and $\alpha_0 = \alpha$. Furthermore $T[\![t]\!]$ is the set of all formal power series in t whose coefficients are elements of T and a module over $k[\![t]\!]$. Then α_t defines the trilinear multiplication on $T[\![t]\!]$ and such a system is denoted by

$$T_t := \big(T[\![t]\!], \alpha_t\big).$$

According to the aspect (1) of the deformation theory, T_t is required to be the same kind as T, that is, a $k[\![t]\!]$-Lts. Thus the conditions, corresponding to (1)–(3),

$$\alpha_t(a,a,b) = 0, \tag{4}$$

$$\alpha_t(a,b,c) + \alpha_t(b,c,a) + \alpha_t(c,a,b) = 0, \tag{5}$$

$$\alpha_t\big(a,b,\alpha_t(c,d,e)\big) = \alpha_t\big(\alpha_t(a,b,c),d,e\big) + \alpha_t\big(c,\alpha_t(a,b,d),e\big) + \alpha_t\big(c,d,\alpha_t(a,b,e)\big) \tag{6}$$

must be satisfied. The conditions (4) and (5) lead to the obvious equations

$$\alpha_i(a,a,b) = 0, \tag{7}$$

$$\alpha_i(a,b,c) + \alpha_i(b,c,a) + \alpha_i(c,a,b) = 0 \tag{8}$$

for $i = 0, 1, 2, \ldots$. The condition (6) is expressed as

$$\sum t^{i+j}\alpha_i\big(a,b,\alpha_j(c,d,e)\big) = \sum t^{i+j}\big\{\alpha_i\big(\alpha_j(a,b,c),d,e\big) + \alpha_i\big(c,\alpha_j(a,b,d),e\big) + \alpha_i\big(c,d,\alpha_j(a,b,e)\big)\big\}.$$

Then we have

$$\sum_{i+j=n}\big\{\alpha_i\big(\alpha_j(a,b,c),d,e\big) + \alpha_i\big(c,\alpha_j(a,b,d),e\big) + \alpha_i\big(c,d,\alpha_j(a,b,e)\big) - \alpha_i\big(a,b,\alpha_j(c,d,e)\big)\big\} = 0$$

for $n = 0, 1, 2, \ldots$. If we put

$$\alpha\circ\beta(x_1,x_2,x_3,x_4,x_5)=\alpha\big(\beta(x_1,x_2,x_3),x_4,x_5\big)+\alpha\big(x_3,\beta(x_1,x_2,x_4),x_5\big)$$
$$+\alpha\big(x_3,x_4,\beta(x_1,x_2,x_5)\big)-\alpha\big(x_1,x_2,\beta(x_3,x_4,x_5)\big),$$

then they can be written in the forms

$$\alpha_0\circ\alpha_1+\alpha_1\circ\alpha_0=0, \tag{9}$$

and for $n\geqslant 2$,

$$-\alpha_0\circ\alpha_n-\alpha_n\circ\alpha_0=\alpha_1\circ\alpha_{n-1}+\alpha_2\circ\alpha_{n-2}+\cdots+\alpha_{n-1}\circ\alpha_1. \tag{10}$$

We call these the *deformation equations* for a Lie triple system.

Let α_t,α'_t be deformations of a Lts T with $\alpha'_t:=\alpha+t\alpha'_1+t^2\alpha'_2+\cdots$. The deformation α'_t is *equivalent* to α_t, denoted by $\alpha'_t\sim\alpha_t$, if there exists a $k[\![t]\!]$-module isomorphism $f_t:T'_t\to T_t$ of the form

$$f_t=1_T+tf_1+t^2f_2+\cdots,$$

where each f_i is a k-linear map $T\to T$ extended to be $k[\![t]\!]$-linear such that

$$\alpha'_t(a,b,c)=f_t^{-1}\alpha_t\big(f_t(a),f_t(b),f_t(c)\big):=\alpha_t*f_t(a,b,c) \tag{11}$$

$(a,b,c\in T)$. If we write $[a\,b\,c]'_t=\alpha'_t(a,b,c)$ and $[a\,b\,c]_t=\alpha_t(a,b,c)$, then the above equality (11) means $f_t([a\,b\,c]'_t)=[f_t(a)\,f_t(b)\,f_t(c)]_t$, in other words, that f_t is a $k[\![t]\!]$-Lts isomorphism.

When $\alpha_1=\alpha_2=\cdots=0$, we say that α_t is the *null deformation* and write $T_0=(T[\![t]\!],\alpha_t)$. The null deformation is a just "formal power series triple system" $T[\![t]\!]$. A deformation α_t is said to be the *trivial deformation* when $T_t\sim T_0$.

2. Yamaguti cohomology

We first recall the definition of a Lts-module given by Yamaguti [14]. Let T be a Lts and V be a vector space over k. V is called a *T-module* if there exists a bilinear map $\theta:(a,b)\to\theta(a,b)$ of $T\times T$ into the associative algebra of the linear transformations of V satisfying the following conditions:

$$\theta(c,d)\theta(a,b)-\theta(b,d)\theta(a,c)-\theta\big(a,[b\,c\,d]\big)+D(b,c)\theta(a,d)=0, \tag{12}$$
$$\theta(c,d)D(a,b)-D(a,b)\theta(c,d)+\theta\big([a\,b\,c],d\big)+\theta\big(c,[a\,b\,d]\big)=0, \tag{13}$$

where $D(a,b)=\theta(b,a)-\theta(a,b)$. The Lts T itself can be considered as a T-module by the action

$$\theta(a,b)(v)=[v\,a\,b].$$

Remark (*An interpretation of* (12), (13)). Suppose that the vector space direct sum $E = T \oplus V$ is itself the Lts such that

(1) T is a subsystem,
(2) $[a\,b\,c]$ lies in V if any one of a, b, c is in V, and
(3) $[a\,b\,c] = 0$ if any two of a, b, c are in V.

This is a general idea of defining a module. Substituting $\theta(a, b)(v)$ for $[v\,a\,b]$ in the defining identity (3), we have the conditions (12) and (13).

Let V be the T-module defined by a bilinear map θ. For each $n \geqslant 0$ we define a k-vector space $C^{2n+1}(T, V)$ of $(2n+1)$-cochains of T with coefficients in V as follow: A cochain $f \in C^{2n+1}(T, V)$ is a k-multilinear function of $2n+1$ variables $f : T \times \cdots \times T \to V$ satisfying

$$f(x_1, \ldots, x_{2n-2}, x, x, y) = 0$$

and

$$f(x_1, \ldots, x_{2n-2}, x, y, z) + f(x_1, \ldots, x_{2n-2}, y, z, x) + f(x_1, \ldots, x_{2n-2}, z, x, y) = 0$$

for $x_i, x, y, z \in T$, and we understand $\mathrm{Hom}_k(T, V)$ by $C^1(T, V)$.

The *Yamaguti coboundary* is a k-linear map $\delta^{2n-1} : C^{2n-1}(T, V) \to C^{2n+1}(T, V)$ defined by

$$\begin{aligned}
&\delta^{2n-1} f(x_1, \ldots, x_{2n+1}) \\
&\quad = \theta(x_{2n}, x_{2n+1}) f(x_1, \ldots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1}) f(x_1, \ldots, x_{2n-2}, x_{2n}) \\
&\qquad + \sum_{k=1}^{n} (-1)^{n+k} D(x_{2k-1}, x_{2k}) f\bigl(x_1, \ldots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \ldots, x_{2n+1}\bigr) \\
&\qquad + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f\bigl(x_1, \ldots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \ldots, [x_{2k-1}\,x_{2k}\,x_j], \ldots, x_{2n+1}\bigr)
\end{aligned}$$

where $\widehat{\ }$ denotes omission. With this coboundary the Yamaguti cochain forms a complex

$$C^1(T, V) \xrightarrow{\delta^1} C^3(T, V) \xrightarrow{\delta^3} C^5(T, V) \longrightarrow \cdots,$$

and $\delta^{2n+1}\delta^{2n-1} = 0$ for $n = 1, 2, \ldots$ (consult with Yamaguti [14] on the proof). The cocycles and coboundaries are denoted by $Z^\bullet(T, V)$ and $B^\bullet(T, V)$ respectively, while the *Yamaguti cohomology* is $H^\bullet(T, V) = Z^\bullet(T, V)/B^\bullet(T, V)$.

In the case that V is the Lts T itself, the operator δ is written down for lower orders as

$$\begin{aligned}
&\delta^1 f(x_1, x_2, x_3) \\
&\quad = \bigl[f(x_1)\,x_2\,x_3\bigr] - \bigl[f(x_2)\,x_1\,x_3\bigr] + \bigl[x_1\,x_2\,f(x_3)\bigr] - f\bigl([x_1\,x_2\,x_3]\bigr), \\
&\delta^3 f(x_1, x_2, x_3, x_4, x_5) \\
&\quad = \bigl[f(x_1, x_2, x_3)\,x_4\,x_5\bigr] - \bigl[f(x_1, x_2, x_4)\,x_3\,x_5\bigr] - \bigl[x_1\,x_2\,f(x_3, x_4, x_5)\bigr] \\
&\qquad + \bigl[x_3\,x_4\,f(x_1, x_2, x_5)\bigr] + f\bigl([x_1\,x_2\,x_3], x_4, x_5\bigr) + f\bigl(x_3, [x_1\,x_2\,x_4], x_5\bigr) \\
&\qquad + f\bigl(x_3, x_4, [x_1\,x_2\,x_5]\bigr) - f\bigl(x_1, x_2, [x_3\,x_4\,x_5]\bigr)
\end{aligned}$$

for $x_1, \ldots, x_5 \in T$.

3. Infinitesimal of deformation

Let us return to the deformation equations (9), (10) and Eqs. (7), (8). It follows from (7) and (8) that each α_i can be viewed as an elements of the 3-Yamaguti cochain space $C^3(T, T)$. One can easily see that $\alpha_0 \circ \alpha_n + \alpha_n \circ \alpha_0 = \delta\alpha_n$ for $n \geqslant 1$. Hence the deformation equations (9), (10) can be expressed as

$$\delta\alpha_1 = 0, \tag{14}$$

$$\alpha_1 \circ \alpha_{n-1} + \alpha_2 \circ \alpha_{n-2} + \cdots + \alpha_{n-1} \circ \alpha_1 = -\delta\alpha_n \tag{15}$$

in terms of the 3-Yamaguti coboundary δ. The *infinitesimal* of the deformation α_t is α_1. Since α_1 is a 3-cocycle by (14), this concept suits the aspects of the Gerstenhaber's deformation theory.

Assume that deformations $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ and $\alpha'_t = \alpha + t\alpha'_1 + t^2\alpha'_2 + \cdots$ are equivalent under $f_t = 1_T + tf_1 + t^2 f_2 + \cdots$. The defining equation $\alpha'_t = \alpha_t * f_t$, i.e., $f_t(\alpha'_t(a, b, c)) = \alpha_t(f_t(a), f_t(b), f_t(c))$ is equivalent to

$$f_n\alpha + f_{n-1}\alpha'_1 + \cdots + f_0\alpha'_n = \alpha F_n + \alpha_1 F_{n-1} + \cdots + \alpha_n F_0$$

or

$$\alpha'_n = \alpha_n + \alpha F_n - f_n\alpha + \sum_{i=1}^{n-1}\bigl(\alpha_i F_{n-i} - f_{n-i}\alpha'_i\bigr), \tag{16}$$

where $f_0 = 1_T$ and

$$\alpha_i F_j(a, b, c) = \sum_{k+l+m=j} \alpha_i\bigl(f_k(a), f_l(b), f_m(c)\bigr). \tag{17}$$

For $n = 1$ one has $\alpha'_1 = \alpha_1 + \delta f_1$. Thus we have

Theorem 1 (Infinitesimal). *Let α_t, α'_t be equivalent deformations of a Lts (T, α), then the first-order terms of them belong to the same cohomology class in the third Yamaguti cohomology group $H^3(T, T)$.*

4. Rigidity

A Lts T is *analytically rigid* if every deformation T_t is equivalent to the null deformation T_0. As the deformation theory of algebras with *binary* products, such as associative algebras and Lie algebras, we have a fundamental theorem.

Theorem 2 (Rigidity). *If T is a Lts with $H^3(T, T) = 0$, then T is analytically rigid.*

Proof. Let α_t is a deformation of a Lts (T, α) and write $\alpha_t = \alpha + t^r \alpha_r + t^{r+1}\alpha_{r+1} + \cdots$. It follows from (15) that $\delta\alpha_r = 0$, i.e., $\alpha_r \in Z^3(T, T)$. By our assumption $H^3(T, T) = 0$ we can find $f^{(r)} \in C^1(T, T)$ such that $\alpha_r = \delta f^{(r)}$. Now consider the deformation $\alpha'_t = \alpha_t * (1_T - t^r f^{(r)})$. In this case Eq. (16) is

$$\alpha'_r = \alpha_r + \alpha F_r - \left(-f^{(r)}\right)\alpha = \alpha_r - \delta f^{(r)} = 0,$$

observing (17): $\alpha F_r(a, b, c) = \alpha(-f^{(r)}(a), b, c) + \alpha(a, -f^{(r)}(b), c) + \alpha(a, b, -f^{(r)}(c))$. Hence $\alpha'_t = \alpha + t^{r+1}\alpha'_{r+1} + \cdots$. Repeating this procedure, one can remove an increasing number of terms of any deformation $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$:

$$\left(\cdots\left(\left(\alpha_t * \left(1_T - t^1 f^{(1)}\right)\right) * \left(1_T - t^2 f^{(2)}\right)\right) * \cdots\right) = \alpha,$$

which implies $\alpha_t \sim \alpha$. □

5. Obstruction, integration

A 3-cocycle $\alpha_1 \in Z^3(T, T)$ is said to be *integrable* if there exists a one parameter family α_t whose first-order term is α_1, so that, $\alpha_t = \alpha + t\underline{\alpha_1} + t^2\alpha_2 + \cdots$. Now let us go back to the deformation equation (15):

$$\alpha_1 \circ \alpha_{n-1} + \alpha_2 \circ \alpha_{n-2} + \cdots + \alpha_{n-1} \circ \alpha_1 = -\delta\alpha_n.$$

Suppose that we have already had $\alpha_1, \ldots, \alpha_{n-1}$. We want to find α_n satisfying (15). But there is an *obstruction* to do so. The left-hand side of (15) define a 5-cocycle and we shall call this an obstruction cocycle. The verification of this fact sets us a long computation, and as a result convinces us that the Yamaguti cohomology is a suitable one for the deformation theory of Lts's.

Lemma. *If α and β are 3-cocycles, then $\alpha \circ \alpha$ and $\alpha \circ \beta + \beta \circ \alpha$ are 5-cocycles.*

Proof. Lemma follows from the following formula:

$$\begin{aligned}
&\delta(f\circ g)(x_1,x_2,x_3,x_4,x_5,x_6,x_7)\\
&\quad=\delta f\bigl(g(x_1,x_2,x_3),x_4,x_5,x_6,x_7\bigr)+\delta f\bigl(x_3,g(x_1,x_2,x_4),x_5,x_6,x_7\bigr)\\
&\qquad+\delta f\bigl(x_3,x_4,g(x_1,x_2,x_5),x_6,x_7\bigr)+\delta f\bigl(x_3,x_4,x_5,g(x_1,x_2,x_6),x_7\bigr)\\
&\qquad+\delta f\bigl(x_3,x_4,x_5,x_6,g(x_1,x_2,x_7)\bigr)-\delta f\bigl(x_1,x_2,g(x_3,x_4,x_5),x_6,x_7\bigr)\\
&\qquad-\delta f\bigl(x_1,x_2,x_5,g(x_3,x_4,x_6),x_7\bigr)-\delta f\bigl(x_1,x_2,x_5,x_6,g(x_3,x_4,x_7)\bigr)\\
&\qquad+\delta f\bigl(x_1,x_2,x_3,x_4,g(x_5,x_6,x_7)\bigr)-f\bigl(\delta g(x_1,x_2,x_3,x_4,x_5),x_6,x_7\bigr)\\
&\qquad-f\bigl(x_5,\delta g(x_1,x_2,x_3,x_4,x_6),x_7\bigr)-f\bigl(x_5,x_6,\delta g(x_1,x_2,x_3,x_4,x_7)\bigr)\\
&\qquad-f\bigl(x_1,x_2,\delta g(x_3,x_4,x_5,x_6,x_7)\bigr)+f\bigl(x_3,x_4,\delta g(x_1,x_2,x_5,x_6,x_7)\bigr)\\
&\qquad-\bigl[f(x_1,x_2,x_3)\,x_4\,g(x_5,x_6,x_7)\bigr]+\bigl[g(x_1,x_2,x_3)\,x_4\,f(x_5,x_6,x_7)\bigr]\\
&\qquad-\bigl[x_3\,f(x_1,x_2,x_4)\,g(x_5,x_6,x_7)\bigr]+\bigl[x_3\,g(x_1,x_2,x_4)\,f(x_5,x_6,x_7)\bigr]\\
&\qquad+\bigl[f(x_1,x_2,x_5)\,x_6\,g(x_3,x_4,x_7)\bigr]-\bigl[g(x_1,x_2,x_5)\,x_6\,f(x_3,x_4,x_7)\bigr]\\
&\qquad+\bigl[f(x_1,x_2,x_5)\,g(x_3,x_4,x_6)\,x_7\bigr]-\bigl[g(x_1,x_2,x_5)\,f(x_3,x_4,x_6)\,x_7\bigr]\\
&\qquad-\bigl[f(x_1,x_2,x_6)\,x_5\,g(x_3,x_4,x_7)\bigr]+\bigl[g(x_1,x_2,x_6)\,x_5\,f(x_3,x_4,x_7)\bigr]\\
&\qquad-\bigl[f(x_1,x_2,x_6)\,g(x_3,x_4,x_5)\,x_7\bigr]+\bigl[g(x_1,x_2,x_6)\,f(x_3,x_4,x_5)\,x_7\bigr]\\
&\qquad-\bigl[f(x_3,x_4,x_5)\,x_6\,g(x_1,x_2,x_7)\bigr]+\bigl[g(x_3,x_4,x_5)\,x_6\,f(x_1,x_2,x_7)\bigr]\\
&\qquad-\bigl[x_5\,f(x_3,x_4,x_6)\,g(x_1,x_2,x_7)\bigr]+\bigl[x_5\,g(x_3,x_4,x_6)\,f(x_1,x_2,x_7)\bigr]. \qquad \square
\end{aligned}$$

This lemma immediately leads to the following result.

Proposition. *Let $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ be a deformation of a Lts (T,α). Then*

$$\alpha_1\circ\alpha_{n-1}+\alpha_2\circ\alpha_{n-2}+\cdots+\alpha_{n-1}\circ\alpha_1\in Z^5(T,T).$$

Now we can state the third fundamental theorem.

Theorem 3 (Integration). *If T is a Lts with $H^5(T,T)=0$, then every* 3*-cocycle is integrable.*

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