Scientia Iranica D (2013) 20 (3), 765–770

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
www.sciencedirect.com

Improved polynomial approximations for the solution of nonlinear integral equations

A. Darijani [a,*], M. Mohseni-Moghadam [b]

[a] *Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, 76169-14111, Iran*
[b] *Islamic Azad University of Kerman, Kerman, Iran*

Received 3 March 2012; revised 20 July 2012; accepted 15 October 2012

KEYWORDS
Nonlinear integral equations;
Minimization;
Residual function;
Initial point.

Abstract In this paper, the solutions of nonlinear integral equations, including Volterra, Fredholm, Volterra–Fredholm of first and second kinds, are approximated as a linear combination of some basic functions. The unknown parameters of an approximate solution are obtained based on minimization of the residual function. In addition, the existence and convergence of these approximate solutions are investigated. In order to use Newton's method for minimization of the residual function, a suitable initial point will be introduced. Moreover, to confirm the efficiency and accuracy of the proposed method, some numerical examples are presented. It is shown that there are considerable improvements in our results compared with the results of the existing methods. All numerical computations have been performed on a personal computer using Maple 12.

1. Introduction

The general form of a nonlinear integral equation is given as follows:

$$\alpha g(y(x)) + \beta \int_0^1 k_1(x,t,y(t))dt + \gamma \int_0^x k_2(x,t,y(t))dt = f(x), \quad (1)$$

where parameters α, β, γ, functions $g(y(x)), k_1(x,t,y(t))$, $k_2(x,t,y(t))$ and $f(x)$ are known and $y(x)$ is the unknown function to be determined.

Several numerical methods for approximating the solution of the Eq. (1) have been presented. Most of these methods are based on the appropriate linear combinations of some basic functions, such as Chebyshev polynomials, Legendre polynomials, Bernstein polynomials, Spline functions, Taylor polynomials, Block-pulse functions and wavelets. Many attempts have been made to solve Eq. (1) in the case of:

$$g(y(x)) = y(x), \qquad k_1(x,t,y(t)) = k_1(x,t)[y(t)]^p,$$
$$k_2(x,t,y(t)) = k_2(x,t)[y(t)]^q, \quad (2)$$

where p and q are non-negative integers. Many researchers such as Yalcinbas [1], Bildik and Inc [2], Maleknejad et al. [3,4], Ordokhani et al. [5,6], Yousefi and Razaghi [7], Babolian et al. [8], Maleknejad et al. [9], Mohsen and El-Gamel [10], Sloss and Blyth [11], Hashemizadeh et al. [12] and Marzban et al. [13], presented the approximate solution using Taylor series, the modified decomposition method, Chebyshev and Bernstein polynomials, rationalized Haar functions, Legendre wavelets, triangular functions, sinc bases, Walsh functions and Hybrid functions, respectively. A few of these methods are applicable for finding an approximate solution for Eq. (1) in the case that the corresponding kernels are not as Eqs. (2).

Note that, the kernels $k_1(x,t,y(t)), k_2(x,t,y(t))$ can be simplified by the Wavelet basics, so we may use them for determination of approximate solutions of Eq. (1). In order to increase the rate of the convergence for approximate solutions, we approximate the solution by some polynomials. The minimization of the residual function has been used in [14] for

* Corresponding author.
E-mail addresses: darijaniali@gmail.com (A. Darijani), mohseni@mail.uk.ac.ir (M. Mohseni-Moghadam).
Peer review under responsibility of Sharif University of Technology.

ELSEVIER Production and hosting by Elsevier

1026-3098
http://dx.doi.org/10.1016/j.scient.2012.10.042

linear integral equations. Recently, Chen and Jiang [15] have obtained the approximate solution for linear integral equations based on the minimization of the norm of the residual function. In this paper, the basic idea [14,15] has been developed and applied to nonlinear integral equations. In fact, the solution of Eq. (1) is approximated by a linear combination of some basic polynomials. The polynomial approximations are obtained based on the minimization of the norm of the residual function. Based on the proposed method, the problem of solving a nonlinear integral equation is converted to a minimization problem of unconstrained nonlinear programming. In the present work, a new approach is introduced to increase the precision of the approximate polynomial solutions of the integral equations. The accuracy and convergence rate of this method are compared with the other existing ones, which indicates the improvement in the results. So, the main advantage of this method is its applicability for solving some integral equations of first and second kinds, integro-differential integral equations, multi-dimensional integral equations, ordinary and partial differential equations.

This paper is organized as follows. In Section 2, the existence and convergence of the best approximate solutions for nonlinear integral equations are investigated. In Sections 3 and 4 computational methods and the choice of initial point for obtaining the approximate solution of the Eq. (1) are proposed. In Section 5, the results of some numerical experiments are presented and they are compared with the results of existing methods. Finally Section 6 concludes the paper.

2. Preliminaries and fundamental theorems

Let us consider the operator L corresponding to Eq. (1) as follows:

$$L[z](x) = \alpha g_z(x) + \beta k_1[z](x) + \gamma k_2[z](x) - f(x), \tag{3}$$

where, z is a function defined on $[0, 1]$ and

$$k_1[z](x) = \int_0^1 k_1(x, t, z(t))dt,$$
$$k_2[z](x) = \int_0^x k_2(x, t, z(t))dt,$$
$$g_z(x) = g(z(x)).$$

Remark 1. In this paper, it is assumed that g and partial derivatives, k_1, k_2, are continuous. Also, it is supposed that Eq. (1) has a unique continuous solution on $[0, 1]$ and a differentiable on $(0, 1)$, which is denoted by y^*.

Definition 1. $\bar{y}_n = \bar{a}_0\varphi_0 + \bar{a}_1\varphi_1 + \cdots + \bar{a}_n\varphi_n$ is called a polynomial of the best approximation of degree, at most n, to the function y^*, if:

$$\|\bar{y}_n - y^*\|_\infty = \inf_{y_n \in P_n} \|y_n - y^*\|_\infty, \tag{4}$$

where:

$$y_n = a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n.$$

Definition 2. Let M be a positive number and sufficiently large, such that $\|\bar{y}_0 - y^*\|_\infty < M$. Define:

$$S_n = \{(a_0, a_1, \ldots, a_n) |\, \|y_n - y^*\|_\infty \le M\}, \tag{5}$$

where $y_n = a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n$.

Definition 3. Suppose $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ is a basis for the set of all polynomials of degree, at most, $n \cdot y_n^* = a_0^*\varphi_0 + a_1^*\varphi_1 + \cdots + a_n^*\varphi_n$ is called the approximate solution for Eq. (1) if:

$$\|L[y_n^*]\|_p = \min_{(a_0, a_1, \ldots, a_n) \in S_n} \|L[y_n]\|_p, \tag{6}$$

where:

$$y_n = a_0\varphi_0 + a_1\varphi_1 + \cdots + a_n\varphi_n, \tag{7}$$

and:

$$\|L[y_n]\|_p = \begin{cases} \left(\int_0^1 |L[y_n](x)|^p dx\right)^{1/p}, & \text{if } 1 \le p < \infty \\ \max_{0\le x\le 1} |L[y_n](x)|, & \text{if } p = \infty. \end{cases} \tag{8}$$

Lemma 1 (*Existence*). *With the above notations, Eq. (1) has the approximate solution, y_n^*, on S_n in the sense that:*

$$\|L[y_n^*]\|_p = \min_{(a_0, a_1, \ldots, a_n) \in S_n} \|L[y_n]\|_p.$$

Proof. First of all, continuity of the nonlinear operator, L, results from:

$$\|L[y_n + \Delta_n] - L[y_n]\|_p \le |\alpha|\, \|g_{y_n+\Delta_n} - g_{y_n}\|_p + |\beta|\, \|k_1[y_n + \Delta_n] - k_1[y_n]\|_p + |\gamma|\, \|k_2[y_n + \Delta_n] - k_2[y_n]\|_p. \tag{9}$$

According to definition S_n, it is clear that $(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n) \in S_n$, where $\bar{y}_n = \bar{a}_0\varphi_0 + \bar{a}_1\varphi_1 + \cdots + \bar{a}_n\varphi_n$, is the best approximation for y^*. Therefore, S_n is a nonempty set. Also, S_n is closed and bounded. Thus, there exists $y_n^* = a_0^*\varphi_0 + a_1^*\varphi_1 + \cdots + a_n^*\varphi_n$, such that:

$$\|L[y_n^*]\|_p = \min_{(a_0, a_1, \ldots, a_n) \in S_n} \|L[y_n]\|_p. \quad \square$$

Theorem 1 (*Convergence*). *There exists a subsequence of $\{y_n^*\}$ which is convergent to the exact solution of Eq. (1).*

Proof. Without loss of generality, we may assume that y^* is not a polynomial. Therefore, for any $n \in \mathbf{N}$, we have:

$$\|L[y_n^*]\|_p > 0. \tag{10}$$

At first, we prove $\lim_{n\to\infty} \|L[y_n^*]\|_p = 0$.

$$\|L[y_n]\|_p = \|L[y_n] - L[y^*]\|_p \le |\alpha|\, \|g_{y_n} - g_{y^*}\|_p + |\beta|\, \|k_1[y_n] - k_1[y^*]\|_p + |\gamma|\, \|k_2[y_n] - k_2[y^*]\|_p. \tag{11}$$

Since L is continuous, for any given $\epsilon > 0$ there exists a positive δ, such that:

$$\text{if } \|y_n - y^*\|_\infty < \delta, \text{ then } \|L[y_n]\|_p < \epsilon. \tag{12}$$

Also:

$$\bar{y}_n \longrightarrow y^*. \tag{13}$$

Hence, there exists a number, $N_1 \in \mathbf{N}$, such that if $n > N_1$, then:

$$\|\bar{y}_n - y^*\|_\infty < \delta. \tag{14}$$

According to Relations (12) and (14), for $n > N_1$, we have:

$$\|L[\bar{y}_n]\|_p < \varepsilon. \tag{15}$$

Since S_n is nonempty, by Lemma 1, we have:

$$\min_{(a_0, a_1, \ldots, a_n) \in S_n} \|L[y_n]\|_p = \|L[y_n^*]\|_p \le \|L[\bar{y}_n]\|_p. \tag{16}$$

Thus:

$$\lim_{n\to\infty} \|L[y_n^*]\|_p = 0. \tag{17}$$

According to definition $\|L[y_n^*]\|_p$, it is clear that:

$$\|L[y_0^*]\|_p \geq \|L[y_1^*]\|_p \geq \cdots \geq \|L[y_n^*]\|_p \geq \|L[y_{n+1}^*]\|_p \geq \cdots. \tag{18}$$

Now, we show that there exists a subsequence of $\{\|L[y_n^*]\|_p\}$ which is strictly decreasing. To this end, put $n_1 = 0$, we should find $n_2 > n_1$, such that $\|L[y_{n_2}^*]\|_p < \|L[y_{n_1}^*]\|_p$. Otherwise, for any $n \in \mathbf{N}$, we have:

$$\|L[y_n^*]\|_p = \|L[y_0^*]\|_p > 0. \tag{19}$$

Therefore, $\lim_{n\to\infty} \|L[y_n^*]\|_p = \|L[y_0^*]\|_p > 0$. This contradicts Eq. (17).

Similarly, there exists $n_3 > n_2$, such that $\|L[y_{n_3}^*]\|_p < \|L[y_{n_2}^*]\|_p$, and so on. Define:

$$S = \{y_{n_1}^*, y_{n_2}^*, \ldots, y_{n_k}^*, y_{n_{k+1}}^*, \ldots\} \cup \{y^*\}. \tag{20}$$

It is clear that S is a bounded (see definition S_n) and closed set (the uniqueness of the solution of the integral equation). Also, set S consists of the holomorphic functions in interval (0, 1). Therefore, S is compact (see [16, p. 32]). Also the restriction of L on S is a one to one and continuous function. Hence, the inverse of the restriction of L on S, which is denoted by $L_{|_S}$, does exist. Therefore

$$\|L_{|_S}^{-1}(L[y_{n_k}^*]) - L_{|_S}^{-1}(L[y^*])\|_p, \tag{21}$$

trends to zero as n trends to infinity. Thus

$$\|y_{n_k}^* - y^*\|_\infty \longrightarrow 0. \tag{22}$$

The proof is completed. □

In practical work, $\{y_{n_k}^*\}$ and $\{y_n^*\}$ are exactly the same. Therefore, the rate of convergence is high.

3. Computational method

In this subsection, we illustrate the calculation method for the determination of the unknown parameters of the approximate solution. For simplicity, the approximate solution is presented as a linear combination of basic functions; $\varphi_i(x) = x^i$, $(i = 0, 1, \ldots, n)$. Without loss of generality, we may use the 2-norm. So that, for finding the approximate solution for Eq. (1), it is enough to solve the following mathematics programming problem:

$$\min_{(a_0,a_1,\ldots,a_n)} \|L[y_n]\|_2 = \min_{(a_0,a_1,\ldots,a_n)} \left(\int_0^1 (L[y_n](x))^2 dx\right)^{1/2}. \tag{23}$$

Generally, to handle this problem we use the following approximation;

$$(\|L[y_n]\|_2)^2 = \int_0^1 (L[y_n](x))^2 dx \cong \frac{1}{N}[(L[y_n](x_0))^2 + (L[y_n](x_1))^2 + \cdots + (L[y_n](x_N))^2], \tag{24}$$

and we will minimize the following expression:

$$(L[y_n](x_0))^2 + (L[y_n](x_1))^2 + \cdots + (L[y_n](x_N))^2. \tag{25}$$

Newton's method is used for minimizing of mathematical Expression (25). It is obvious that the rate of convergence of this method depends on a suitable initial point, which will be discussed in the next section.

3.1. Initial point

By continuity of y^* and the Weierstrass approximation theorem, for any $\epsilon > 0$, there exists $n = n(\epsilon) \in \mathbf{N}$, such that;

$$\left\| y^* - \sum_{k=0}^{n} p_{nk} y^* \left(\frac{k}{n}\right) \right\|_\infty \leq \epsilon, \tag{26}$$

where the Bernstein polynomials, $p_{nk}(x)$, are defined as the following:

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1]. \tag{27}$$

Since y^* is continuous and usually does not have large oscillations, there exists a nonnegative Δy^*, such that for any $n \in \mathbf{N}$, we have:

$$y^*\left(\frac{k}{n}\right) = y^*(0) + \varepsilon_k,$$
$$|\epsilon_k| \leq \Delta y^*, \quad k = 0, 1, 2, \ldots, n. \tag{28}$$

Using Relations (26) and (28), for any $\epsilon > 0$, we have:

$$\left\| y^* - y^*(0) \sum_{k=0}^{n} p_{nk} \right\|_\infty \leq \varepsilon + \Delta y^*. \tag{29}$$

Therefore, in our computational method, we will select the coefficients of $x^0, x^1, \ldots, x^n$ in polynomial $y^*(0) \sum_{k=0}^{n} p_{nk}(x)$ as the initial point. Assume $A^0 = (a_0^0, a_1^0, \ldots, a_n^0)$ is the initial point, since:

$$\sum_{k=0}^{n} p_{nk}(x) = 1. \tag{30}$$

Therefore, we choose:

$$a_j^0 = \begin{cases} y^*(0), & \text{if } j = 0, \\ 0, & \text{if } j = 1, 2, \ldots, n. \end{cases} \tag{31}$$

According to Theorem 1 and Relation (29), for any $\epsilon > 0$, there exists $m \in \mathbf{N}$, such that:

$$\left\| y_m^* - y^*(0) \right\|_\infty \leq \varepsilon + \Delta y^*. \tag{32}$$

Hence:

$$\left\| \left(a_0^*, a_1^*, \ldots, a_m^*\right) - \left(y^*(0), 0, \ldots, 0\right) \right\| \leq \varepsilon + \Delta y^*. \tag{33}$$

Therefore, the initial point $(y^*(0), 0, \ldots, 0)$ is near to $(a_0^*, a_1^*, \ldots, a_m^*)$ for sufficiently small Δy^* and large m.

4. Numerical solutions for nonlinear Hammerstein integral equations of the second kind

Even though the proposed method is applicable for all nonlinear integral equations of first and second kinds, the existence of nonlinear terms in the integrand of these equations will increase the computation time. In order to solve this problem for nonlinear integral equations of the second kind, in some special cases, we first linearize the integrand term using a change of variable and then solve the problem by our method.

Let us consider a special case of Eq. (1) as follows:

$$g(y(x)) + \beta \int_0^1 k_1(x, t) F_1(t, y(t)) dt + \gamma \int_0^x k_2(x, t) F_2(t, y(t)) dt = f(x). \tag{34}$$

Suppose that g^{-1} exists and rewrite Eq. (34) as follows:

$$y(x) = g^{-1}\left(f(x) - \beta\int_0^1 k_1(x,t)Z_1(t)dt - \gamma\int_0^x k_2(x,t)Z_2(t)dt\right), \quad (35)$$

where:

$$Z_1(x) = F_1(x, y(x)), \qquad Z_2(x) = F_2(x, y(x)). \quad (36)$$

Therefore, Eq. (36) can be rewritten in the following form:

$$\begin{cases} Z_1(x) = F_1\left(x, g^{-1}\left(f(x) - \beta\int_0^1 k_1(x,t)Z_1(t)dt - \gamma\int_0^x k_2(x,t)Z_2(t)dt\right)\right), \\ Z_2(x) = F_2\left(x, g^{-1}\left(f(x) - \beta\int_0^1 k_1(x,t)Z_1(t)dt - \gamma\int_0^x k_2(x,t)Z_2(t)dt\right)\right). \end{cases} \quad (37)$$

Lemma 2. *With the above notations, $y(x)$ presented in Eq. (35) is the solution of Eq. (34) if, and only if, Z_1 and Z_2 are the solutions of the system of Eqs. (37).*

Proof. It is trivial.

By Lemma 2, it is enough to determine an approximate solution for the system of Eqs. (37) by using our method as follows:

$$\min \sum_{i=1}^{m}(\|L_1[Z_{1n}, Z_{2n}](x_i)\|_2^2 + \|L_2[Z_{1n}, Z_{2n}](x_i)\|_2^2), \quad (38)$$

where $L_1[Z_1, Z_2] = Z_1 - F_1$ and $L_2[Z_1, Z_2] = Z_2 - F_2$. Also, Z_{1n} and Z_{2n} are polynomials of degree n.

Finally, the solution of Eq. (34) is approximated as the following:

$$\tilde{y}_n(x) = g^{-1}\left(f(x) - \beta\int_0^1 k_1(x,t)Z_{1n}^*(t)dt - \gamma\int_0^x k_2(x,t)Z_{2n}^*(t)dt\right), \quad (39)$$

where (Z_{1n}^*, Z_{2n}^*) is the optimum solution of unconstrained optimization problem (38). Similar to Section 2, it is clear that $(Z_{1n}^*, Z_{2n}^*) \longrightarrow (Z_1, Z_2)$. Thus, Lemma 2 implies that $\tilde{y}_n$ convergences to y. In the next section, we will employ our method to solve some examples. □

5. Results and discussion

In this section, we will present some numerical examples in order to investigate the performance and efficiency of the proposed method. For this purpose, we have compared our results with those of [3] as well as with the exact solutions. This comparison is done based on error functions presented as:

$$e_n(x) = |y_n^*(x) - y^*(x)|, \quad (40)$$

and:

$$\tilde{e}_n(x) = |\tilde{y}_n(x) - y^*(x)|, \quad (41)$$

Table 1: Comparison of the results of Example 1 with those of [3].

x_i	$\tilde{e}_n(x)$ in our method $n = 8$	$\tilde{e_n}(x)$ in [3] $n = 8$
0	2.09690E−12	3.8182E−9
0.1	2.05577E−12	5.2485E−9
0.2	2.21619E−12	7.6313E−9
0.3	2.43920E−12	9.0047E−9
0.4	2.54063E−12	3.4113E−9
0.5	3.01770E−12	5.6989E−9
0.6	3.63401E−12	8.0405E−9
0.7	4.13450E−12	2.0541E−9
0.8	4.95115E−12	3.0202E−9
0.9	5.98211E−12	5.2225E−9
1	7.09833E−12	1.4717E−8

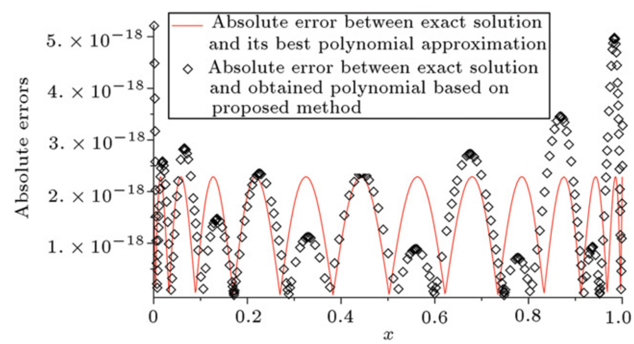

Figure 1: Comparison between absolute errors in two cases: (I) the exact solution and its best polynomial approximation, (II) the exact solution and obtained polynomial based on the proposed method for Example 1, $n = 12$.

with:

$$\|e_n\|_\infty = \max_{0<x<1} |e_n(x)|,$$
$$\|\tilde{e}_n\|_\infty = \max_{0<x<1} |\tilde{e}_n(x)|, \quad (42)$$

where y_n^* is the approximate solution obtained by our method in Section 2 and $\tilde{y}_n$ is the approximate solution presented in Section 4. Also, it is assumed that y_n^{**} is the best polynomial approximation of degree at most n for the exact solution of the integral equation determined using maple software.

Example 1. Consider the following nonlinear Volterra–Fredholm integral equation [3]:

$$y(x) = 2\cos(x) - 2 + 3\int_0^x \sin(x-t)y^2(t)dt + \frac{6}{7 - 6\cos(1)}\int_0^1 \cos^2(x)(1-t)(t + y(t))dt, \quad (43)$$

with the exact solution $y(x) = \cos(x)$. Table 1 shows that our method has smaller errors than those of the method introdused in [3]. So, Figure 1 indicates that y_n^{**} and y_n^* are almost the same and the obtained approximate solution is best.

Example 2. Consider the nonlinear Fredholm integral equation given in [3] by:

$$y(x) = 1 + x + \left(1 - \frac{3}{2}\ln(3) + \frac{\sqrt{3}}{6}\pi\right)x^2 + \int_0^1 2x^2 t \ln(y(t))dt, \quad (44)$$

Table 2: Comparison of the results of Example 2 with those of [3].

x_i	$\tilde{y}_n(x)$ in [3] $n=6$	$\tilde{y}_n(x)$ in our method $n=6$	$\tilde{y}_n(x)$ in [3] $n=6$	$\tilde{y}_n(x)$ in our method $n=6$	Exact solution
0	1.000000000000	1.000000000000	1.0000000000000000	1.000000000000000000	1
0.1	1.109999994993	1.110000000774	1.1099999999999905	1.109999999999999999	1.11
0.2	1.239999979975	1.240000003099	1.2399999999999620	1.239999999999999998	1.24
0.3	1.389999954945	1.390000006973	1.3899999999999142	1.389999999999999996	1.39
0.4	1.559999919902	1.560000012397	1.5599999999998473	1.559999999999999994	1.56
0.5	1.749999874847	1.750000019370	1.7499999999997610	1.749999999999999991	1.75
0.6	1.959999819780	1.960000027893	1.9599999999996562	1.959999999999999987	1.96
0.7	2.189999754701	2.190000037966	2.1899999999995320	2.189999999999999983	1.19
0.8	2.439999679609	2.440000049588	2.4399999999993890	2.439999999999999977	2.44
0.9	2.709999594506	2.710000062760	2.7099999999992264	2.709999999999999976	2.71
1	2.999999499390	3.0000000774827	2.9999999999990443	2.999999999999999966	3

Table 3: Maximum absolute errors based on our method and the method introduced in [3] for Example 3.

n	$\|\tilde{e}_n\|_\infty$ in [3]	$\|\tilde{e}_n\|_\infty$ in our method
8	2.09548E−10	6.531195E−12
12	5.06382E−14	6.804417E−16
16	1.47668E−16	8.231327E−20

with the exact solution $y(x) = 1+x+x^2$. Table 2 shows that our method has smaller errors than those of the method introduced in [3].

Example 3. Consider differential equation $y''(x) - \exp(y(x)) = 0$, with boundary conditions $y(0) = y(1) = 0$, which is of great interest in hydrodynamics [17]. This equation can be reformulated as the following nonlinear Fredholm–Hammerstein integral equation:

$$y(x) = \int_0^1 k(x,t)e^{y(t)}dt. \tag{45}$$

The exact solution of Eq. (45) is $y(x) = -\ln(2) + 2\ln\left(c \sec \frac{c(x-\frac{1}{2})}{2}\right)$, where c is the root of equation

$$\sqrt{2} = c \sec\left(\frac{c}{4}\right),$$

and:

$$k(x,t) = \begin{cases} -t(1-x), & t \le x, \\ -x(1-t), & x < t. \end{cases}$$

In this example, maximum absolute errors for the approximate solutions are shown in Table 3. Table 3 shows that our method has smaller errors than those of the method introduced in [3]. Figure 2 shows that in our method the bigger n results in the higher-precision.

Example 4. Consider the following nonlinear Volterra integral equation of the first kind:

$$\int_0^x (xy^2(t)+1)\ln(y(t))dt = \frac{-x}{4}(1-e^{-2x}-2xe^{-2x}) - \frac{x^2}{2}, \tag{46}$$

with the exact solution $y(x) = e^{-x}$.

Maximum absolute error by our method in Section 2, is shown in Table 4. Also, Table 4 indicates that the results of our method having a rapid rate of convergence.

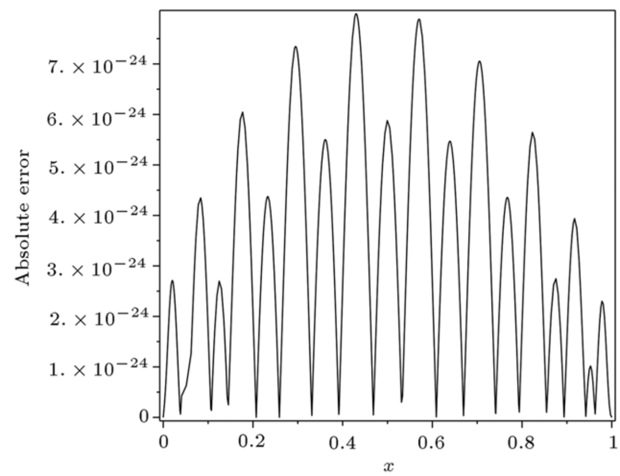

Figure 2: The absolute error for Example 3, $n = 20$.

Table 4: Maximum absolute errors based on our method for Example 4.

n	$\|y_n^* - y\|_\infty$
3	5.272004E−4
5	1.680973E−6
7	2.622387E−9
10	6.964135E−11

6. Conclusion

In this paper, the approximate solutions of nonlinear integral equations, including Volterra, Fredholm, Volterra–Fredholm of first and second kinds, are presented as a linear combinations of some basic polynomials. The unknown coefficients are calculated based on the minimization of norm-2 of the residual function. In addition, the existence and convergence of a subsequence of approximate solutions are investigated. A great advantage of our method, from a computational point of view, is the simplicity and quick reduction of an integral equation to a nonlinear optimization problem. This problem was solved by using the approximated norm-2 and Newton's method with a suitable initial point. In order to investigate the accuracy of our method, several numerical examples are presented. It was observed that the approximate solutions based on the proposed method in comparison with other existing solutions are more accurate.

References

[1] Yalcinbas, S. "Taylor polynomial solutions of nonlinear Volterra–Fredholm integral equations", *Appl. Math. Comput.*, 127, pp. 195–206 (2002).

[2] Bildik, N. and Inc, M. "Modified decomposition method for nonlinear Volterra–Fredholm integral equations", *Chaos Solitons Fractals*, 33, pp. 308–313 (2007).
[3] Maleknejad, K., Hashemizadeh, E. and Basirat, B. "Computational method based on Bernstein operational matrices for nonlinear Volterra–Fredholm–Hammerstein integral equations", *Commun. Nonlinear Sci. Numer. Simul.*, 17, pp. 52–61 (2012).
[4] Maleknejad, K., Sohrabi, S. and Rostami, Y. "Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials", *Appl. Math. Comput.*, 188, pp. 123–128 (2007).
[5] Ordokhani, Y. and Razzaghi, M. "Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via a collocation method and rationalized Haar functions", *Appl. Math. Lett.*, 21, pp. 4–9 (2008).
[6] Ordokhani, Y. "Solution of nonlinear Volterra–Fredholm–Hammerstein integral equations via rationalized Haar functions", *Appl. Math. Comput.*, 180, pp. 436–443 (2006).
[7] Yousefi, S. and Razzaghi, M. "Legendre wavelets method for the nonlinear Volterra–Fredholm integral equations", *Math. Comput. Simul.*, 70, pp. 1–8 (2005).
[8] Babolian, E., Mansouri, Z. and Hatamzadeh-Varmazyar, S. "Numerical solution of nonlinear Volterra–Fredholm integro-differential equations via direct method using triangular functions", *Comput. Math. Appl.*, 58, pp. 239–247 (2009).
[9] Maleknejad, K., Almasieh, H. and Roodaki, M. "Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations", *Commun. Nonlinear Sci. Numer. Simul.*, 15, pp. 3293–3298 (2010).
[10] Mohsen, A. and El-Gamel, M. "On the numerical solution of linear and nonlinear Volterra integral and integro-differential equations", *Appl. Math. Comput.*, 217, pp. 3330–3337 (2010).
[11] Sloss, B. and Blyth, W. "An Walsh function method for a non-linear Volterra integral equation", *J. Franklin Inst.*, 340, pp. 25–41 (2003).
[12] Hashemizadeh, E., Maleknejad, K. and Basirat, B. "Hybrid functions approach for the nonlinear Volterra–Fredholm integral equations", *Procedia Comput. Sci.*, 3, pp. 1189–1194 (2011).
[13] Marzban, H.R., Tabrizidooz, H.R. and Razzaghi, M. "A composite collocation method for the nonlinear mixed Volterra–Fredholm–Hammerstein integral equations", *Commun. Nonlinear Sci. Numer. Simul.*, 16, pp. 1186–1194 (2011).
[14] Delves, L.M. and Mohamad, J.L., *Computational Methods for Integral Equations*, Cambridge Univ. Press, Cambridge (1985).
[15] Chen, Z. and Jiang, W. "Approximate solution for mixed linear Volterra Fredholm integral equation", *Appl. Math. Lett.*, 25, pp. 1131–1134 (2012).
[16] Rudin, W., *Functional Analysis*, McGraw-Hill, 13th Reprint (1989).
[17] Kalman, R.E. and Kalaba, R.E., *Quasi-Linearization and Nonlinear Boundary-Value Problems*, Elsevier, New York (1969).

Ali Darijani received his B.S. and M.S. degrees in Applied Mathematics from Shahid Bahonar University of Kerman, Kerman, Iran, where he is now a Ph.D. student in the Mathematics Department.

Mahmoud Mohseni-Moghadam is a Professor of Mathematics at Shahid Bahonar University of Kerman, Kerman, Iran, where he teaches undergraduate and graduate levelcourses in the Applied Mathematics Department. His teaching focuses on: numerical analysis, integral equations, differential equations and numerical linear algebra. His current research programs include: integral equations and numerical linear algebra. Professor Mohseni-Moghadam has authored five books and more than a hundred articles for publication in international journals and conferences proceedings.