Castelnuovo's Lemma and h-vectors of Cohen-Macaulay homogeneous domains

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Abstract

This paper gives a new postulation of the Hilbert function of a Cohen-Macaulay homogeneous domain.

If \( A \) is a Cohen-Macaulay homogeneous algebra over a field \( k \), there are positive integers \( h_0, h_1, \ldots, h_s \) satisfying \( \sum_{i \geq 0} \dim_k A_i \lambda^i = (h_0 + h_1 \lambda + \cdots + h_s \lambda^s)/(1 - \lambda)^d \), where \( d \) is the Krull dimension of \( A \). We call the vector \((h_0, h_1, \ldots, h_s)\) the \( h \)-vector of \( A \).

Let \( A \) be a Cohen-Macaulay homogeneous domain over \( \mathbb{C} \) with the \( h \)-vector \((h_0, h_1, \ldots, h_s)\). It is well known that \( h_i \geq h_1 \) for all \( 2 \leq i \leq s - 1 \). We will show that if the equality holds for some \( 2 \leq i \leq s - 2 \) then \( h_1 = h_2 = \cdots = h_{i-1} \) and \( h_s \leq h_1 \) (when \( h_s \geq 2 \), the condition \( h_{s-1} = h_1 \) also implies the same assertion). To prove this result, we will modify Castelnuovo's argument in his study on curves of maximal genus.


0. Introduction

By a homogeneous algebra over a field \( k \), we mean here a noetherian commutative \( k \)-algebra \( A \) with identity, together with a vector space direct sum decomposition \( A = \bigoplus_{i \geq 0} A_i \), such that: (a) \( A_0 = k \), (b) \( A_i A_j \subseteq A_{i+j} \) and (c) \( A \) is generated as a \( k \) algebra by \( A_1 \). The Hilbert function \( H(A, \cdot) \) of \( A \) is defined by \( H(A, i) = \dim_k A_i \), for \( i \geq 0 \), while the Hilbert series is given by

\[
F(A, \lambda) = \sum_{i \geq 0} H(A, i) \lambda^i = \frac{h_0 + h_1 \lambda + \cdots + h_s \lambda^s}{(1 - \lambda)^d},
\]

where \( d \) is the Krull dimension of \( A \) and \( h_0, h_1, \ldots, h_s \) are certain integers satisfying \( h_s \neq 0 \). We call the vector \((h_0, h_1, \ldots, h_s)\) the \( h \)-vector of \( A \). If \( A \) is Cohen-Macaulay, we

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have that $h_i > 0$ for all $0 \leq i \leq s$. It is clear that the $h$-vector of $A$ together with its Krull dimension determines the Hilbert function of $A$ and conversely.

A famous theorem of Macaulay gives a characterization of a numerical functions which occur as the Hilbert function $H(A, n)$ of a homogeneous $k$-algebra $A$. One can easily find a similar characterization for the $h$-vectors of Cohen–Macaulay homogeneous $k$-algebras (see [10] for further information).

The problem becomes quite difficult if one deal with Cohen–Macaulay homogeneous domain, while it is conjectured that the $h$-vector of a Cohen–Macaulay homogeneous domain is under much stronger restrictions than that of a general Cohen–Macaulay homogeneous $k$-algebra. The complete characterizations are obtained in a few special cases: (a) when $h_1 \leq 1$ (trivial), (b) when $h_1 = 2$ (see [6]; also [5]) and (c) when $h_1 = 3$ and $A$ is Gorenstein [1]. But, when $h_1 \geq 4$, very little is known (it seems that Stanley's inequality [11, Proposition 3.4] is one of the most important results in this area).

The main result of this paper gives a new restriction on the $h$-vectors of Cohen–Macaulay homogeneous domains:

**Theorem 3.2.** Let $A$ be a homogeneous Cohen–Macaulay domain over an algebraically closed field of characteristic 0. Let $(h_0, h_1, \ldots, h_s)$ be the $h$-vector of $A$. If $h_s \geq 2$, and $h_i = h_1$ (equivalently, $h_i \leq h_1$) for some $2 \leq i \leq s - 1$, then we have $h_1 = h_2 = \cdots = h_{s-1}$ and $h_s \leq h_1$. In the case $h_s = 1$, if there is an integer $2 \leq i \leq s - 2$ such that $h_i = h_1$, then $h_1 = h_2 = \cdots = h_{s-1}$.

When $h_s = 1$ and $s \geq 4$, the condition $h_{s-1} = h_1$ dose not imply $h_1 = h_2$. In fact, we can easily construct a Gorenstein homogeneous domain whose $h$-vector $(h_0, h_1, \ldots, h_s)$ forms $s \geq 4$, $h_s = 1$, $h_{s-1} = h_1$ and $h_1 \neq h_2$.

To prove the theorem, we use techniques of Eisenbud and Harris [7] which are related to Castelnuovo theory on projective curves. More precisely, we use uniform position lemma and generalize a classical result of Castelnuovo which concerns a finite set of points in a projective space.

**Theorem 2.1.** (Generalized Castelnuovo's lemma). Let $X \subset \mathbb{P}^r$ be a not necessarily reduced zero-dimensional subscheme in uniform position. Denote the $h$-vector of the projective coordinate ring of $X$ by $(h_0, h_1, \ldots, h_s)$ with $h_s \neq 0$. If there exists some $2 \leq i \leq s - 2$ such that $h_i \leq h_1 + d - 1$, then there is a $d$-dimensional rational normal scroll containing $X$. If $h_s \geq 2$ and $s \geq 3$, then $h_{s-1} \leq h_1 + d - 1$ also implies the same assertion.

When $i = 2$, the above theorem is nothing other than the main result of the author's paper [12]. When $i = 2$ and $d = 1$, the above theorem was first proved by Castelnuovo (resp. Eisenbud and Harris [3, 4]) in the reduced case (resp. in the general case).
1. Zero-dimensional schemes

We work over an algebraically closed field $k$ of arbitrary characteristic unless otherwise specified. By $\mathbb{P}^r$, we denote the projective $r$ space over $k$. Let $S := k[X_0, \ldots, X_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$.

Given a subscheme $V \subset \mathbb{P}^r$, we denote by $I_V$ the saturated homogeneous ideal of $V$. We say that a subscheme $V \subset \mathbb{P}^r$ is nondegenerate, if no hyperplane contains $V$.

Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme and $R := S/I_X$ be the homogeneous coordinate ring of $X$. Unless otherwise specified, $X$ and $R$ are used in this meaning throughout this paper. $R$ is a (not necessarily reduced) $1$-dimensional Cohen–Macaulay homogeneous algebra.

The Hilbert function of $X$ is denoted by $H_X: \mathbb{Z} \to \mathbb{N}(n \mapsto \dim_k R_n)$, while the degree of $X$ is given by $\deg X = \sup \{H_X(n) | n \geq 0\}$. If $X$ is reduced, then $\deg X$ is equal to the number of points contained in $X$.

Since $R$ is $1$-dimensional, the $h$-vector $(h_0, h_1, \ldots, h_s)$ of $R$ is given by $s = \min \{n | H_X(n) = \deg X\}$ and $h_i = H_X(i) - H_X(i - 1)$. We have that $h_0 = 1$, $h_i > 0$ for all $1 \leq i \leq s$, $h_0 + h_1 + \cdots + h_s = \deg X$, and

$$F(R, \lambda) := \sum_{i \geq 0} H_X(i) \lambda^i = \frac{h_0 + h_1 \lambda + \cdots + h_s \lambda^s}{1 - \lambda}.$$  

Definition 1.1. The graded $R$-module $\omega_R := \text{Ext}_S^1(R, S(-r-1))$ is called the canonical module of $R$.

The next fact is very important.

Proposition 1.2. (Stanley [10]).

$$F(\omega_R, \lambda) := \sum_{i \in \mathbb{Z}} \dim_k (\omega_R)_i \lambda^i = \frac{\lambda^{-s+1}(h_s + h_{s-1} \lambda + \cdots + h_0 \lambda^s)}{1 - \lambda}.$$  

We now recall a few well-known geometric conditions on zero-dimensional schemes.

Definition 1.3. We say that $X$ is in linearly general position, if every proper subspace $L \subset \mathbb{P}^r$ satisfies $\deg(L \cap X) < 1 + \dim L$, or equivalently, if every subscheme $Y \subset X$ satisfies $H_Y(1) = \min \{\deg Y, r + 1\}$.

Definition 1.4. Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme. We say that $X$ is in uniform position, if $X$ is in linearly general position and every subscheme $Y \subset X$ satisfies $H_Y(n) = \min \{H_X(n), \deg Y\}$ for all $n \in \mathbb{Z}$.

The next result is due to Kreuzer. We shall say that a map of $k$-vector spaces $\phi: U \otimes V \to W$ is $1$-generic, if $\phi(u \otimes v) \neq 0$ whenever $u, v \neq 0$, and nondegenerate, if $u \in U$ and $\phi(u \otimes v) = 0$ (resp. $v \in V$ and $\phi(U \otimes v) = 0$) imply $u = 0$ (resp. $v = 0$).
Proposition 1.5 (Kreuzer [9, Theorems 3.2 and 2.6]). Let $X \subseteq \mathbb{P}^{r}$ be a zero-dimensional subscheme in uniform position. Set $s = \min \{n | H_{X}(n) = \deg X\}$. If $s \geq 2$, then the multiplication map $S_{1} \otimes (\omega_{R})_{-s+1} \rightarrow (\omega_{R})_{-s+2}$ is 1-generic, and the multiplication map $R_{n} \otimes (\omega_{R})_{-s+1} \rightarrow (\omega_{R})_{-s+n+1}$ is non-degenerate for all $n \geq 0$.

Remark 1.6. (a) Let $X \subseteq \mathbb{P}^{r}$ be a zero-dimensional subscheme in linearly general position (e.g. uniform position) with the $h$-vector $(h_{0}, h_{1}, \ldots, h_{s})$. Then we have $h_{i} \geq h_{1} = r$ for all $1 \leq i \leq s - 1$. Hence we have $H_{X}(n) \geq \min \{1 + nr, \deg X\}$ for all $n \geq 0$ (see for example [9]).

(b) For any non-reduced point $x \in X$, the local artinian ring $\mathcal{O}_{X,x}$ has a nonzero socle. Since we have that $\dim k \mathcal{O}_{X,x}/(a) = \dim k \mathcal{O}_{X,x} - 1$ where $0 \neq a \in \text{soc}(\mathcal{O}_{X,x})$, there is a subscheme $Y \subset X$ such that $\deg Y = \deg X - 1$ (if $X$ is reduced, $Y = X \setminus \{x\}$ for some point $x \in X$). Moreover for each integer $1 \leq n \leq \deg X$, there is a subscheme $Y \subset X$ such that $\deg Y = n$.

On the other hand, if $X$ is in uniform position and has the $h$-vector $(h_{0}, h_{1}, \ldots, h_{s})$, then every subscheme $Y \subset X$ is in uniform position again, and has the $h$-vector $(h_{0}, h_{1}, h_{2}, \ldots, h_{i-1}, h_{i})$ for some $i \leq s$ and $h_{i} \leq h_{i}$.

We also need the following result from Eisenbud's "1-generic matrix" theory.

Proposition 1.7. Let $\phi: U \otimes V \rightarrow W$ be a linear map of $k$-vector spaces and $M$ be the linear form matrix with entries in $W$ which corresponds to $\phi$ (the correspondence between a bilinear map and a linear form matrix is given in the introduction of [2]). If $\phi$ is 1-generic and $\dim k V = 2$, then $M$ is equivalent to a unique scrollar matrix $M(a_{1}, \ldots, a_{d})$ with $1 \leq a_{1} \leq \cdots \leq a_{d}, \sum_{1}^{d} a_{i} = \dim k U$. That is,

$$M \cong M(a_{1}, \ldots, a_{d}) = \begin{pmatrix} x_{1,0} & x_{1,1} & \cdots & x_{1,a_{1}-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,a_{2}} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,a_{1}} & x_{1,a_{2}} & \cdots & x_{1,a_{d}} \end{pmatrix}$$

Proof. From a well-known formula on determinantal ideals, the assumption of Theorem 5.1 of [2] is satisfied automatically in this case. \(\square\)

For further information of a 1-generic matrix and the definition of a rational normal scroll, see [2] or [7].

2. Generalized Castelnuovo's lemma

Theorem 2.1 (Generalized Castelnuovo's lemma). Let $X \subseteq \mathbb{P}^{r}, r \geq 2$ be a not necessarily reduced zero-dimensional subscheme in uniform position. Denote the $h$-vector of the projective coordinate ring of $X$ by $(h_{0}, h_{1}, \ldots, h_{s})$ with $h_{s} \neq 0$. If there exists some $2 \leq i \leq s - 2$ such that $h_{i} \leq h_{1} + d - 1$, then there is a $d$-dimensional rational normal
scroll containing $X$. If $h_s \geq 2$ and $s \geq 3$, then $h_{s-1} \leq h_1 + d - 1$ also implies the same assertion.

**Proof.** If the assumption of the theorem is satisfied, we have $h_{i+1} \geq 2$ (see Remark 1.6 (a)). Hence we can find a subscheme $Y \subset X$ whose $h$-vector is $(h_0, h_1, \ldots, h_i, 2)$ by the argument in Remark 1.6 (b). Since $(I_Y)_2 = (I_X)_2$ and the defining ideal of a rational normal scroll is generated by quadrics, we can replace $X$ by $Y$. So we may assume that $h_s = 2, s \geq 3$ and $h_{s-1} \leq h_1 + d - 1 = r + d - 1$. Then we have $\dim_k (\omega_R)_{-s+1} = 2$ by Proposition 1.2.

Let $M$ be the matrix with entries in $(\omega_R)_{-s+2}$ which corresponds to the multiplication map $S_1 \otimes (\omega_R)_{-s+1} \to (\omega_R)_{-s+2}$. Since this map is 1-generic by Proposition 1.5, $M$ is equivalent to a scrollar matrix $M(a_1, \ldots, a_{d'})$ with $1 \leq a_1 \leq \cdots \leq a_{d'}$ and $\sum_{i=1}^{d'} a_i = r + 1$ by Proposition 1.7. We have $d' \leq d$, since $\dim_k M \leq \dim_k (\omega_R)_{-s+2} = 2 + h_{s-1} \leq r + 1 + d$ where $\dim_k M$ means the dimension of the linear span of the entries of $M$. Put $l:= \min \{i \mid a_i \geq 2\}$ and $b_j:= \sum_{i=1}^{j} a_i$ for each $j \geq 0$.

We can find a basis $x_0, \ldots, x_r$ of $S_1$ and $x_0, x_1$ of $(\omega_R)_{-s+1}$ respectively such that $x_i x_0 = x_{i-1} x_1$ for all $b_j < i < b_j$ and $l \leq j \leq d'$.

Set

$$M' = \begin{pmatrix} x_{b_1} & x_{b_1+1} & \cdots & x_{b_1-2} \\ x_{b_1+1} & x_{b_1+2} & \cdots & x_{b_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{b_{d'}} & x_{b_{d'}+2} & \cdots & x_{b_{d'}-1} \end{pmatrix}$$

the scrollar matrix of type $M(a_l - 1, a_{l+1} - 1, \ldots, a_{d'} - 1)$ with entries in $S_1$. An explicit calculation shows that $I_2(M') \cdot (\omega)_R = 0$. In fact, we have

$$(x_s x_{r-1} - x_{r-1} x_s) \cdot x_0 = x_s x_{r-1} x_0 - x_{r-1} (x_s x_0)$$

$$= x_s x_{r-1} x_0 - x_{r-1} (x_s x_1)$$

$$= x_s x_{r-1} x_0 - x_{r-1} (x_{r-1} x_1)$$

$$= x_s x_{r-1} x_0 - x_{r-1} (x_s x_0) = 0,$$

$$(b_{i-1} < \forall s < b_i, h_{j-1} < \forall t < h_{j+1}, l \leq \forall i, j \leq d')$$

similarly,

$$(x_s x_{r-1} - x_{r-1} x_s) \cdot x_1 = 0.$$

But, the multiplication map $R_2 \otimes (\omega_R)_{-s+1} \to (\omega_R)_{-s+3}$ is nondegenerate by Proposition 1.5. So we have $I_2(M') \subset I_X$. That is, the $d'$-dimensional rational normal scroll defined by $I_2(M')$ contains $X$. Since $d \geq d'$, there is a $d$-dimensional rational normal scroll containing $X$. □

The case $i = 2$ of Theorem 2.1 has been proved in the author's paper [12] using the essentially same idea.
Even in the reduced case, our theorem improves some classical results in projective geometry. In fact, classical results in this direction concerns only the number of conditions on quadrics \( (h_2, \text{in our context}) \).

When \( i = 2, d = 1 \) and \( X \) is reduced, Theorem 2.1 is nothing other than the well-known result called “Castelnuovo’s lemma” (cf. [7]). Recently, using a geometric method, Eisenbud and Harris [3], [4] proved the following theorem which says that Castelnuovo’s Lemma remains valid in the context of schemes. We can give a new proof of their result.

**Corollary 2.2.** (Eisenbud and Harris [3, Theorem 2.1; 4, Theorem 2.2]). Suppose that \( X \) is a not necessarily reduced zero-dimensional subscheme of \( \mathbb{P}^r \) in linearly general position.

(a) If \( \deg X = r + 3 \), then there is a unique rational normal curve containing \( X \).

(b) If \( \deg X \geq 2r + 3 \) but \( X \) imposes only \( 2r + 1 \) conditions on quadrics, then there is a unique rational normal curve containing \( X \).

**Proof.** (a) The uniqueness part is easy (cf. [3]).

We can prove the existence of the rational normal curve containing \( X \) by the same arguments in our proof of Theorem 2.1, since \( X \) is in uniform position and has the \( h \)-vector \((1, r, 2)\).

(b) Let \( Y \) be a subscheme of \( X \) with \( \deg Y = 2r + 3 \). Then \( Y \) is in uniform position and has the \( h \)-vector \((1, r, r, 2)\). Since \((I_X)_2 = (I_Y)_2\), the statement follows from Theorem 2.1 immediately. \( \square \)

**Corollary 2.3.** Let \( X \subset \mathbb{P}^r \) be a zero-dimensional subscheme in uniform position and \((h_0, h_1, \ldots, h_s)\) be the \( h \)-vector of the projective coordinate ring of \( X \). If \( h_s \geq 2 \) and \( s \geq 3 \), then the following are equivalent.

(i) \( h_1 = h_i \) for some \( 2 \leq i \leq s - 1 \).

(ii) \( h_1 = h_2 = \cdots = h_{s-1} \) and \( h_s \leq h_1 \).

(iii) There is a unique rational normal curve which contains \( X \).

**Proof.** (iii) \( \Rightarrow \) (ii): Well-known. (ii) \( \Rightarrow \) (i): Obvious. (i) \( \Rightarrow \) (iii): From Theorem 2.1. \( \square \)

**Remark 2.4.** Even in the case \( h_s = 1 \), if there is some \( 2 \leq i \leq s - 2 \) such that \( h_i = h_1 \), then \( h_1 = h_2 = \cdots = h_{s-1} \) and \( h_s \leq h_1 \).

Of course, the assumption \( h_s \geq 2 \) of Corollary 2.3 is necessary. There are many examples of a zero-dimensional subscheme in uniform position whose \( h \)-vector satisfies \( s \geq 4, h_s = 1, h_1 = h_{s-1} \) and \( h_2 \neq h_1 \). In fact, almost all zero-dimensional complete intersections have such \( h \)-vectors.
3. \(h\)-vectors of Cohen–Macaulay homogeneous domains

In this section, we assume that \(k\) is an algebraically closed field with \(\text{char } k = 0\). The next theorem is J. Harris' famous result.

**Uniform Position Lemma** (Harris [7, Corollary 3.4]). Let \(C \subset \mathbb{P}^r_k\) be a reduced, irreducible and non-degenerate curve. Then a general hyperplane section \(C \cap H\) is a set of points in uniform position.

In virtue of this result, we can use our results on zero-dimensional schemes to study the Hilbert function of a homogeneous domain. The next fact is a key lemma.

**Lemma 3.1.** Let \(A\) be a homogeneous Cohen–Macaulay domain over \(k\) of dimension \(d\). Then the \(h\)-vector of \(A\) is the \(h\)-vector of the projective coordinate ring of a set of points in uniform position.

**Proof.** By Bertini's theorem and the uniform position lemma, there is a linear regular sequence \(x = x_1, \ldots, x_{d-1} \in A_1\) for which \(A/(x)\) is the projective coordinate ring of a set of points in uniform position. It is well-known that \(A\) and \(A/(x)\) have the same \(h\)-vector (see [10]). \(\square\)

**Theorem 3.2.** Let \(A\) be a homogeneous Cohen–Macaulay domain over an algebraically closed field of characteristic 0. Let \((h_0, h_1, \ldots, h_s)\) be the \(h\)-vector of \(A\).

(a) If \(h_s \geq 2, s \geq 3\) and \(h_1 = h_s\) for some \(2 \leq i \leq s - 1\), then we have \(h_1 = h_2 = \cdots = h_{s-1}\) and \(h_s \leq h_1\). In the case \(h_s = 1\), if there is an integer \(2 \leq i \leq s - 2\) such that \(h_i = h_1\), then \(h_1 = h_2 = \cdots = h_{s-1}\).

(b) If there exists some \(2 \leq i \leq s - 2\) such that \(h_i \leq h_1 + d - 1\), then \(h_2 \leq \binom{h_1 + 1}{2} - \binom{h_1 + d - 1}{2}\). If \(h_2 \geq 2\) and \(s \geq 3\), then \(h_{s-1} \leq h_1 + d - 1\) also implies the same assertion.

**Proof.** (a) follows by using Lemma 3.1, Corollary 2.3 and Remark 2.4. (b) follows by using Lemma 3.1 and Theorem 2.1. \(\square\)

**Remark 3.3.** For a given sequence \(h = (h_0, h_1, \ldots, h_s)\) satisfying \(h_0 = 1, h_1 = h_2 = \cdots = h_{s-1}\) and \(h_s \leq h_1\), there exists a Cohen–Macaulay homogeneous domain whose \(h\)-vector coincides with \(h\). If \(s \geq 5\) (or \(s = 4\) and \(h_s \geq 2\)) and \(\dim A = 2\), then \(A\) is the projective coordinate ring of a curve of maximal genus in \(\mathbb{P}^{h_1 + 1}\). The structure of these curves is studied by Castelnuovo (cf. [7]).

Next, we study "level rings", which is the intermediate concept between Cohen–Macaulay and Gorenstein.

Let \(A = \bigoplus_{n \geq 0} A_n\) be a homogeneous ring over \(k\) with Krull dimension \(d\), \(h\)-vector \((h_0, h_1, \ldots, h_s)\). Let \(\omega_A\) be the canonical module of \(A\). It is easy to see that \(-s + d = \min\{n | (\omega_A)_n \neq 0\}\).
Definition 3.4. \( A \) is called level if \( \omega_A \) is generated by \((\omega_A)_{-s+d}\) as an \( A \)-module.

Contrary to the Gorenstein property, it is impossible to determine a Cohen–Macaulay homogeneous domain to be level or not by its \( h \)-vector only (cf. [8, 5]). But we can prove the following.

Theorem 3.5. Let \( A \) be a Cohen–Macaulay homogeneous domain over \( k \) with \( h \)-vector \((h_0, h_1, \ldots, h_s)\). If \( h_s \geq 2 \) and \( h_i \leq h_1 + 1 \) for all \( 2 \leq i \leq s - 1 \), then \( A \) is a level ring.

Proof. If \( x \in A_1 \) is a nonzero divisor and \( A/(x) \) is level, then \( A \) itself is level. By an argument similar to that in Lemma 3.1, it is sufficient to prove the following proposition.

Proposition 3.6. Let \( X \subset \mathbb{P}^r \), \( r \geq 2 \) be a set of points in uniform position with \( h \)-vector \((h_0, h_1, \ldots, h_s)\). If \( h_s \geq 2 \) and \( h_i \leq h_1 + 1 \) for all \( 2 \leq i \leq s - 1 \), then the coordinate ring of \( X \) is level.

Remark 3.7. There exist many examples of a finite set of points in uniform position \( X \subset \mathbb{P}^r \) whose \( h \)-vector satisfies \( h_s \geq 2 \) and \( h_i = h_1 + 1 \) for all \( 2 \leq i \leq s - 1 \). When \( r \geq 3 \) and \( \deg X \geq 2r + 5 \), \( X \) is always contained in an elliptic normal curve (i.e., an arithmetically Gorenstein curve with degree \( r + 1 \)). See [7] for details. Moreover, for any finite set of points in uniform position in \( \mathbb{P}^r \), \( r \geq 3 \), Eisenbud and Harris [7] proved that \( h_2 = h_1 + 1 \) implies \( h_i \leq h_1 + 1 \) for all \( i \).

To prove the above proposition, we need a few lemmas.

Lemma 3.8. Let \( X \) be as in Proposition 3.6. Suppose that \( Y \subset X \) is a subset with \( \deg Y = \deg X - 1 \). Denote the coordinate ring of \( X \) (resp. \( Y \)) by \( R \) (resp. \( R' \)). Then, there exists the following exact sequence,

\[
0 \to \omega_R \to \omega_R \to (S/P)(s - 1) \to 0,
\]

where \( S := \text{Sym}_k R_1 \) is the coordinate ring of \( \mathbb{P}^r \) and \( P \subset S \) is a height \( r \) homogeneous prime ideal.

Proof. Consider the natural exact sequence

\[
0 \to I \to R \to R' \to 0. \quad (*)
\]

By the definition of "uniform position", we have that

\[
\dim_k I_n = \begin{cases} 
1 & \text{if } n \geq s, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( I \) is a 1-dimensional Cohen–Macaulay module, \( I \) is a cyclic \( S \)-module and isomorphic to \((S/P)(-s)\) where \( P \subset S \) is a height \( r \) homogeneous prime ideal. We note that \( \text{Ext}_S^r((S/P)(-s), S(-r - 1)) \cong (S/P)(s - 1) \). Applying \( \text{Ext}_S^r(-, S(-r - 1)) \) to the sequence \((*)\), we obtain the desired exact sequence. \( \square \)
Lemma 3.9. Let the notation be as in Lemma 3.8. For any \( i \geq -s + 2 \) with \( (\omega_R \otimes k)_i \neq 0 \), we have \( (\omega_R \otimes k)_i \neq 0 \).

Proof. The assertion follows from the exact sequence of Lemma 3.8. \( \square \)

Proof of Proposition 3.6. It is easy to see that \( (\omega_R \otimes k)_i = 0 \) for all \( i \geq 1 \). So it suffices to prove that \( (\omega_R \otimes k)_i = 0 \) for each \( -s + 2 \leq i \leq 0 \).

Assume the contrary (i.e., \( (\omega_R \otimes k)_i \neq 0 \) for some \( -s + 2 \leq i \leq 0 \)). Deleting a certain number of points from \( X \), we may assume that \( s_i = 2 \) and \( (\omega_R \otimes k)_{-s + 2} \neq 0 \) by Lemma 3.9.

Case I: Suppose that \( s_{i-1} = h_1 \) (equivalently \( s_{i-1} \leq h_1 \)). The multiplication map \( S_1 \otimes (\omega_R)_{-s + 1} \to (\omega_R)_{-s + 2} \) is 1-generic by Proposition 1.5, and we have that

\[
\dim_k S_1 \cdot (\omega_R)_{-s + 1} = \dim_k (\omega_R)_{-s + 2} - (\omega_R \otimes k)_{-s + 2}
\]

\[
= \dim_k (\omega_R)_{-s + 1} + h_{s-1} - \dim_k (\omega_R \otimes k)_{-s + 2}
\]

\[
< \dim_k (\omega_R)_{-s + 1} + \dim S_1 - 1.
\]

by the assumption. It contradicts the fact that any 1-generic map \( \phi: U \otimes V \to W \) satisfies \( \dim_k W \geq \dim_k U + \dim_k V - 1 \) (see [2]).

Case II: Suppose that \( s_{i-1} = h_1 + 1 \). Since \( (\omega_R \otimes k)_{-s + 2} \neq 0 \), the 1-generic map \( S_1 \otimes (\omega_R)_{-s + 1} \to (\omega_R)_{-s + 2} \) is not surjective. Hence we have \( \dim_k (S_1 \cdot (\omega_R)_{-s + 1}) = r + 2 \). Since \( \dim_k (\omega_R)_{-s + 1} = 2 \), the above multiplication map corresponds to a scrollar matrix of type \( M(r + 1) \) by Proposition 1.7.

By the same trick as in proof of Theorem 2.1, \( X \) is contained in a rational normal curve. So we have that \( s_{i-1} = h_1 \) by Corollary 2.3. It contradicts the assumption \( s_{i-1} = h_1 + 1 \). \( \square \)

Remark 3.10. (a) Our assumption that \( A \) is a domain cannot be removed in Theorem 3.5. For instance, let \( S = k[a, b, c, \ldots, j] \) be a ten-dimensional polynomial ring. Set

\[
I = I_3 \begin{pmatrix} a & d & 0 & 0 \\ b^3 & e^3 & g & i \\ c^2 & f^2 & h & j \end{pmatrix}.
\]

It is easy to see that \( A := S/I \) is a homogeneous Cohen–Macaulay ring with h-vector \((1, 2, 3, 2)\), but \( A \) is not level (of course, \( A \) is not a domain).

(b) There exists a non level Cohen–Macaulay homogeneous domain with h-vector \((1, 2, 3, 4, 2)\) (cf. [5]). Hence, the assumption of Theorem 3.5 that \( h_i \leq h_1 + 1 \) cannot be weakened to \( h_i \leq h_1 + 2 \).

Corollary 3.11 (Hibi [8]). Let \( A \) be a Cohen–Macaulay homogeneous domain over an algebraically closed field of characteristic 0. If the h-vector of \( A \) is of the form \((h_0, h_1, h_2)\), then \( A \) is level.
Proof. If $h_2 = 1$, then $A$ is Gorenstein by [10, Theorem 4.4]. If $h_2 \geq 2$, then the assumption of Theorem 3.5 is automatically satisfied. \hfill \Box

References