Existence and scattering theory for Boussinesq type equations with singular data

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ARTICLE INFO

Article history:
Received 16 June 2010
Revised 17 November 2010
Available online 26 November 2010

Keywords:
Generalized Boussinesq equation
Global and local solutions
Asymptotic behavior
Lorentz spaces

ABSTRACT

We study the initial value problem for the generalized Boussinesq equation and prove existence of local and global solutions with singular initial data in weak-$L^p$ spaces. Our class of initial data for global existence is larger than that of Cho and Ozawa (2007) [7]. Long time behavior results are obtained and a scattering theory is proved in that framework. With more structure, we show Sobolev $H^1$ and Lorentz-type $L^{(p,q)}$ regularity properties for the obtained solutions. The approach employed is unified for all dimensions $n \geq 1$.

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1. Introduction

We consider the initial value problem (IVP) for the generalized Boussinesq equation

$$\begin{align*}
\partial_t u - \Delta u + \Delta^2 u + \Delta f(u) &= 0, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n, \\
\partial_t u(x, 0) &= \phi(x) = \Delta(v_0), \quad x \in \mathbb{R}^n,
\end{align*}$$

(1.1)

where the unknown $u$ is a real function on $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(a) - f(b)| \leq C_f |a - b|(|a|^\rho - 1 + |b|^\rho - 1),$$

(1.2)

$\rho$.

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1 Supported by FAPESP and CNPq, Brazil.

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doi:10.1016/j.jde.2010.11.013
for some $1 < \rho < \infty$, $f(0) = 0$ and the constant $C_f > 0$ is independent of $a, b \in \mathbb{R}$. The initial values $u_0, v_0 : \mathbb{R}^n \to \mathbb{R}$ are given functions. Eq. (1.1) was derived by Boussinesq in [2] and it is of fundamental physical interest, because it occurs in a large range of phenomena, including the description of shallow water waves, dynamics of stretched string and another physical systems (see e.g. [2,7,15]).

The IVP (1.1) is equivalent to the system of equations

$$
\begin{align*}
&u_t = \Delta v, \\
&v_t = u - \Delta u - f(u), \\
&u(x, 0) = u_0(x), \\
&v(x, 0) = v_0(x),
\end{align*}
$$

(1.3)

and with this formulation, by Duhamel’s principle, the Cauchy problem (1.1) is formally equivalent to the integral equation

$$
[u(t), v(t)] = G(t)[u_0, v_0] - \int_0^t G(t - s)[0, f(u(s))] ds,
$$

(1.4)

where $G(t)$ is the group determined by the linear system associated with (1.3). For initial data $[u_0, v_0]$ and $t \in \mathbb{R}$, the linear operator $G(t)$ is defined by

$$
G(t)[u_0, v_0] = \int_{\mathbb{R}^n} e^{it\xi} \left[ \begin{array}{cc}
\cos(t|\xi|^2(\xi)) & -|\xi|^{-1} \sin(t|\xi|^2(\xi)) \\
\langle \xi \rangle |\xi|^{-2} \sin(t|\xi|^2(\xi)) & \cos(t|\xi|^2(\xi))
\end{array} \right] \hat{u}_0(\xi) \hat{v}_0(\xi) d\xi,
$$

(1.5)

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. In another notation,

$$
G(t)[u_0, v_0] = \left[ g_1(t)u_0 + g_2(t)v_0, g_3(t)u_0 + g_4(t)v_0 \right]
$$

(1.6)

where $g_1(t) = g_4(t)$ and $g_1(t), g_2(t), g_3(t)$ are the multiplier operators with symbols $\cos(t|\xi|^2(\xi))$, $-|\xi|^{-1} \sin(t|\xi|^2(\xi))$ and $\langle \xi \rangle |\xi|^{-2} \sin(t|\xi|^2(\xi))$, respectively. Throughout this manuscript, solutions of the integral system (1.4) will be called mild solutions for the IVP (1.3).

In the present paper we prove new results concerning local and global existence of mild solutions for (1.3), with initial data in a class of infinite energy spaces, namely in weak-$L^p$ spaces (i.e. $L^{(p, \infty)}$-spaces, see Theorems 1.1 and 1.2). Under further conditions, the regularity of the mild solutions, in $H^1$ and $L^{(p,0)}$-spaces (see (2.1) for the definition) is also studied in Remark 1.5 and in item (ii) of Theorems 1.1 and 1.2, respectively. The weak-$L^p$ norms of the global-in-time solutions present a polynomial-time decay as $|t| \to \infty$, which, with more structure, is refined by the asymptotic stability result stated in Theorem 1.4. Moreover, we show two scattering results for the global solutions in that infinite energy framework (see Theorem 1.3 and Corollary 1.5).

Several authors have addressed the model (1.1) from the point of view of local and global existence, and long time behavior. Firstly let us review some works about existence of solutions. The paper [1] proved the local well-posedness with initial data $u_0 \in H^{s+2}(\mathbb{R})$ and $\phi = (v_0)_x \in H^s(\mathbb{R}), v_0 \in H^{s+1}(\mathbb{R})$, $s > 1/2$, and $f \in C^{\infty}(\mathbb{R})$. For that matter, the authors of [1] converted (1.1) in the following equivalent system:

$$
\begin{align*}
&u_t = z_x \quad \text{and} \quad z_t = (u - u_{xx} - f(u))_x, \\
&u(x, 0) = u_0 \quad \text{and} \quad z(x, 0) = v_0,
\end{align*}
$$

where $x \in \mathbb{R}$ and $t > 0$. In the case $1 < \rho < 5$, they also show that the solution is global in time when the data $[u_0, v_0]$ lies close enough to a stable solitary wave. Later on, by transforming (1.1) in a system of nonlinear Schrödinger equations, existence results were obtained in [16] for $f(u) = |u|^\rho u$ and initial data satisfying $u_0 \in H^1(\mathbb{R})$ and $\phi = (v_0)_{xx}, v_0 \in H^1(\mathbb{R})$. In [12], still for $f(u) = |u|^\rho u$, the
author proved that (1.1) is locally well-posedness with either \([u_0, \phi] = [u_0, (v_0)_x] \in L^2(\mathbb{R}) \times \dot{H}^{-1}(\mathbb{R})\) and \(1 < \rho < 5\), or \([u_0, \phi] = [u_0, (v_0)_x] \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) and \(1 < \rho < \infty\). Also, in the latter case and by using energy conservation law, he showed that in fact the solutions exist in \(H^1(\mathbb{R})\) for all \(t > 0\) provided that the initial data is small enough in \(H^1(\mathbb{R}) \times L^2(\mathbb{R})\). Those local-in-time results of [12] were extended for high dimensions in [10]. For the particular nonlinearity \(f(u) = u^2\) and \(s > -1/4\), the same author of [10] obtained in [9] a local well-posedness result with data \([u_0, \phi] = [u_0, (v_0)_x] \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\) by means of the Bourgain Fourier restriction norm approach.

Concerning asymptotic behavior, we would like to mention the papers [8,7] for \(n > 1\) and [13] for \(n = 1\). The scattering of global small amplitude solutions were studied by [13], where the author considered \(n = 1\) and \(\rho > 2 + \sqrt{7}\), and works in the framework of \(H^s_p\)-spaces. Posteriorly, by employing Besov \(B^s_{p, 2}\) with \(1 < p < 2\) and positive regularity (that is \(s > 0\)), some results of [13] were improved by [7]. There, the authors also extend some previous results (and those of them) for the case \(n > 1\). From another point of view, the formulation (1.3) of the IVP (1.1) was used in [8] to study the reciprocal problem of the scattering theory, by constructing a solution with a given scattering state \(G(t)[\varphi, \psi]\). The results of [8] were tackled through the framework \(Y^1 = H^1 \times \dot{H}^1\) for \(n = 1\) and either \(H^s_{p+1} \times H^s_{p+1}\) \((s > 0)\) or \(Y^s = H^s \times D^{-1}H^{s-1}\) \((s > 1)\) when \(n > 1\), where \(D^{-1} = (-\Delta)^{-1/2}\).

So far, to the best of our knowledge, the larger initial data classes are those of [9] [for \(n = 1\) and \(f(u) = u^2\)] and [7]. Let us recall the embedding relations \(B^s_{p, 1} \subset B^s_{p, q} \subset B^s_{p, \infty}\), where \(1 < q \leq \infty\) and \(s \in \mathbb{R}\), and \(B^s_{p, 1} \subset H^s_p \subset B^s_{p, \infty} \subset L^p(\mathbb{R}^n, \mathbb{R})\) for \(s \geq 0\) and \(1/p = 1/p_1 = 1/p_2 = 1/s\), where the same inclusions still hold true for the homogeneous versions of those spaces (see [4]). Also, \(H^s_p\)-spaces with arbitrary \(s \in \mathbb{R}\) (in particular \(s > -1/4\)), homogeneous or not, does not contain any weak-\(L^p\) spaces. Compared to the previous works, we present a new singular class for initial data, for local and global-in-time theory (covering arbitrary \(n \geq 1\)), and obtain scattering results in that singular framework. In fact, for \(s > 0\), \(\delta = 1 - \frac{2}{p+1}\) and \(\Omega(\hat{\psi}) = |\hat{\xi}|(\hat{\xi})\hat{\psi}\), the data \([u_0, \phi]\) of [7] belongs to a subset of \(B^s_{p, q} \times \Omega(B^{s+\delta}_{p, q})\), which is contained in our initial data class (1.10) for global existence (see Remark 1.4). Moreover, our results cover large data in \(H^s_p\) for some values of \(s, p\) that seem to have not been treated in the previous literature (see Remark 1.5).

A good strategy for finding suitable Banach spaces to prove global existence is to use scaling arguments. Unfortunately the system (1.3), or equivalently (1.1), does not have a scaling relation. We overcome this problem by employing something like an intrinsic scaling for (1.3), which is induced by the group estimate (2.3) proved in Section 2. Precisely, the intrinsic scaling of (1.3) is defined by

\[
[u, v] \rightarrow [u_\lambda, v_\lambda] := \lambda^{-2} [u(\lambda x, \lambda^2 t), v(\lambda x, \lambda^2 t)] \quad \text{for} \quad \lambda > 0. \tag{1.7}
\]

Let \(D^s = (-\Delta)^{s/2}\) and \(J^s = (1 - \Delta)^{s/2}\), and consider the Banach space

\[
X_{(p, d)} = L^{(p, d)} \times D^{-1}H^{-1}_{(p, d)}
\]

endowed with the norm

\[
\|\cdot, \cdot\|_{X_{(p, d)}} = \max\{\|\cdot\|_{(p, d)}, \|\cdot\|_{D^{-1}H^{-1}_{(p, d)}}\},
\]

where \(H^{-1}_{(p, d)} = J(L^{(p, d)})\) and \((L^{(p, d)}, \|\cdot\|_{(p, d)})\) stands for a Lorentz space (see (2.1) in Section 2). For \(0 < T < \infty\), denote by \(E^T_{\beta}\) and \(E_\alpha\) the respective spaces of all Bochner measurable pairs \([u, v] : (-T, T) \rightarrow X_{(p+1, \infty)}\) and \([u, v] : (-\infty, \infty) \rightarrow X_{(p+1, \infty)}\) with norms

\[
\| [u, v] \|_{E^T_{\beta}} = \sup_{-T < t < T} \| t^\beta [u(t, \cdot), v(t, \cdot)] \|_{X_{(p+1, \infty)}} \tag{1.8}
\]
and

$$\| (u, v) \|_{E_\alpha} = \sup_{-\infty < t < \infty} |t|^\alpha \| [u(t, \cdot), v(t, \cdot)] \|_{X_{(\rho+1, \infty)}},$$

(1.9)

where \( \beta := \frac{n(\rho-1)}{2(\rho+1)}, \) and the exponent \( \alpha := \frac{1}{\beta} - \frac{n}{2(\rho+1)} \) is the unique one such that the norm (1.9) becomes invariant by intrinsically scaling (1.7). We also define the initial data space \( \mathcal{I}_0 \) as the set of all pairs \( [u_0, v_0] \in S'(\mathbb{R}^d) \times S'(\mathbb{R}^d) \) such that

$$\| [u_0, v_0] \|_{\mathcal{I}_0} = \sup_{-\infty < t < \infty} |t|^\alpha \| G(t)[u_0, v_0] \|_{X_{(\rho+1, \infty)}} < \infty.$$  

(1.10)

Let us mention that the previous works [5,6] study respectively asymptotic self-similarity and scattering for the nonlinear Schrödinger equation by constructing solutions with initial data \( u_0 \) belonging to a class in the spirit of (1.10), namely \( \sup_{t \geq 0} \| e^{it\Delta}u_0 \|_{L^{p+1}} < \infty. \)

Throughout this paper we stand for \( \rho_0(n) = \frac{n+2+\sqrt{n^2+12n+4}}{2} > 1 \) the positive root of the equation \( np^2 -(n+2)\rho -2 = 0 \) and the parameter \( \gamma(n) = \infty \) if \( n = 1, 2 \) and \( \gamma(n) = \frac{n+2}{n-2} \) in otherwise. In the sequel, we state our local-in-time existence result.

**Theorem 1.1 (Local-in-time solutions).** Assume that \( 1 < \rho < \rho_0(n) \) and \( 1 \leq d \leq \infty. \)

(i) **(Existence)** If \( [u_0, v_0] \in L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)}, \) then there exists \( 0 < T < \infty, \) such that the IVP (1.3) has a mild solution \( [u, v] \in \mathcal{E}_\beta^T \) satisfying \( [u(t), v(t)][u_0, v_0] \) in \( S'(\mathbb{R}^d) \) as \( t \rightarrow 0^+. \) The solution \( [u, v] \) is unique in a given ball of \( \mathcal{E}_\beta^T, \) and the data-solution map \( [u_0, v_0] \rightarrow [u, v] \) from \( L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)} \) to \( \mathcal{E}_\beta^T \) is locally Lipschitz.

(ii) **(L^p,q-Regularity)** If \( [u_0, v_0] \in L^{(\frac{\rho+1}{\rho}, d)} \times L^{(\frac{\rho+1}{\rho}, d)} \) then the previous solution satisfies

$$\sup_{-T < t < T} |t|^{\beta} \| u(t, \cdot) \|_{(\rho+1, d)} < \infty \quad \text{and} \quad \sup_{-T < t < T} |t|^{\beta} \| v(t, \cdot) \|_{\partial t^{-1} H^{\rho-1}_{(\rho+1, d)}} < \infty.$$  

(1.11)

**Remark 1.1 (Infinite energy data).** The initial data class \( L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)} \) of Theorem 1.1 covers singular data \( [u_0, v_0] \) such that \( u_0, v_0 \notin L^2; \) for instance

$$u_0 = \Sigma_{j=1}^{\infty} v_j P_k(x-x_j)[x-x_j]^{-k-n_0 \rho+1} \quad \text{and} \quad v_0 = \Sigma_{j=1}^{\infty} \eta_j Q_{k,j}(x-\tilde{x}_j)[x-\tilde{x}_j]^{-s-n_0 \rho+1},$$

(1.12)

where \( x_j, \tilde{x}_j \in \mathbb{R}^d, v_j, \eta_j \) are constant and \( P_k(x), Q_{k,j}(x) \) are homogeneous polynomials with respective degree \( k \) and \( s. \) In fact, the data (1.12) have not even finite local-energy, that is \( u_0 \notin L^2_{\text{loc}} \) and \( v_0 \notin L^2_{\text{loc}}. \) In particular, \( \phi = \Delta(v_0) \notin H^{-2}. \)

**Remark 1.2.** The proof of Theorem 1.1 actually shows us that the existence assertion within Theorem 1.1(i) still works if we replace the condition \( [u_0, v_0] \in L^{(\frac{\rho+1}{\rho}, \infty)} \times L^{(\frac{\rho+1}{\rho}, \infty)} \) by

$$\sup_{-T < t < T} |t|^{\beta} \| G(t)[u_0, v_0] \|_{X_{(\rho+1, \infty)}} < \infty \quad \text{(see (3.6)).}$$

(1.13)

Our global-in-time existence result is the following:
Theorem 1.2 (Global-in-time solutions). Let \( \rho_0(n) < \rho < \gamma(n) \) and \([u_0, v_0] \in \mathcal{I}_0\).

1. (Existence) There is \( \varepsilon > 0 \) such that if \(||[u_0, v_0]||_{\mathcal{I}_0} \leq \varepsilon \) then the IVP (1.3) has a unique global-in-time mild solution \([u, v] \in E_\alpha \) satisfying \(||[u, v]||_{E_\alpha} \leq 2\varepsilon \). The solution \([u, v] \) satisfies \([u(t), v(t)] \to [u_0, v_0] \) in \(S'(\mathbb{R}^n)\) as \(t \to 0^+\). Moreover, the data-solution map is locally Lipschitz.

2. \(L^p,q\)-Regularity Let \(1 \leq d \leq \infty\), \(0 \leq h < 1 - \alpha \rho\) and assume that
   \[
   \Gamma_{1,h} := \sup_{-\infty < t < \infty} |t|^{\alpha + h} \|G(t)[u_0, v_0]\|_{X_{1(\rho+1,d)}} < \infty. \tag{1.14}
   
   \]
   There exists \(0 < \varepsilon_0 \leq \varepsilon\) such that if \(\Gamma_{1,0} \leq \varepsilon_0\), then the previous solution satisfies
   \[
   \sup_{-\infty < t < \infty} |t|^{\alpha + h} \|u(t, \cdot)\|_{(\rho+1,d)} < \infty \quad \text{and} \quad \sup_{-\infty < t < \infty} |t|^{\alpha + h} \|v(t, \cdot)\|_{D^{-1}H_{(\rho+1,d)}^1} < \infty. \tag{1.15}
   
   \]

Remark 1.3. The parameters \(\gamma(n)\) and \(\rho_0(n)\) seem to have a universal trait. Indeed, in previous works, they appear related to the Sobolev critical exponent and optimal time-decay rate for norm \(\|\cdot\|_{L^{\rho+1}}\), respectively (see e.g. [11,14]).

Remark 1.4. Let us define the operators \((\Omega(\psi))^\wedge = |\xi| |\hat{\psi}| \mathcal{F}\) and \(\mathcal{L} = (\Delta^{-1}\Omega)^{\frac{1}{2}} - 1\), and the space \(\mathcal{D}_{p,q} = M_{p,q} \times \Omega(M_{p,q})\), where \(M_{p,q} = (\mathcal{L}^{\frac{-1}{2}}B_{p,q}^n) \cap B_{p,q}^{n+\delta_p}\), \(\delta_p = 1 - 2(p - 1)/p\) and \(B_{p,q}^n\) stands for a Besov space. In [7] the authors assume the initial data \((u_0, \phi) \in \mathcal{D}_{\frac{p+1}{p}}^{\frac{2}{p+1}}\) with \(s > 0\) for the IVP (1.1), while in Theorem 1.2 we consider \((u_0, \phi) = (u_0, \Delta v_0)\) with \([u_0, v_0] \in \mathcal{I}_0\). Our initial data class is larger than that of [7], that is \((u_0, \Delta^{-1} \psi) \in \mathcal{I}_0\) for all \((u_0, \phi) \in \mathcal{D}_{\frac{p+1}{p}}^{\frac{2}{p+1}}\). To see this, first observe that it follows from [7, Lemma 2.4] that
   \[
   \|l_1(t)\|_{B_{p,q}^s} \leq C(1 + |t|)^{-\frac{n}{2}(\frac{1}{r} - 1)} \left(\|\mathcal{L}^\frac{\delta_p}{2} \Omega^{-1} \psi\|_{B_{r,2}^\frac{s}{r}} + \|\Omega^{-1} \psi\|_{B_{r,2}^{s+\delta_p}}\right) 
   \]
   \[
   = C(1 + |t|)^{-\frac{n}{2}(\frac{1}{r} - 1)} \|\Omega^{-1} \psi\|_{M_{r,2}^s}, \tag{1.16}
   
   \]
   where \(s > 0\), \(1 < r \leq 2\) and, \(l_1(t)\) and \(l_2(t)\) are multiplier operators with symbols \(\cos(t|\xi|)\) and \(\sin(t|\xi|)\), respectively. Since \(\Delta^{-1}\Omega = D^{-1} J\) and \(B_{\rho+1,2}^{s+\delta} \subset B_{\rho+1,\infty}^s \subset L^{(\rho+1,2)}\) (by Sobolev-type embedding) for \(\delta = 1 - 2/(\rho + 1)\), we obtain
   \[
   E = B_{\rho+1,2}^{s+\delta} \times \Delta^{-1}\Omega \left(B_{\rho+1,2}^{s+\delta}\right) \subset X_{(\rho+1,\infty)}. \tag{1.17}
   
   \]
   Noting that \(\alpha < \beta\) when \(\rho_0(n) < \rho < \gamma(n)\), estimates (1.16)-(1.17) yield
   \[
   \left\|[u_0, v_0]\right\|_{\mathcal{I}_0} = \sup_{-\infty < t < \infty} |t|^\alpha \|G(t)[u_0, v_0]\|_{X_{(\rho+1,\infty)}} \leq C \sup_{-\infty < t < \infty} |t|^\alpha \|G(t)[u_0, v_0]\|_{E} 
   \]
   \[
   \leq C \left(\sup_{-\infty < t < \infty} |t|^\alpha (1 + |t|)^{-\beta}\right) \left(\|u_0\|_{M_{\rho+1,2}^{s+\delta}} + \|\Omega^{-1} v_0\|_{M_{\rho+1,2}^{s+\delta}}\right) 
   \]
   \[
   \leq C \left\|[u_0, v_0]\right\|_{\mathcal{D}_{\frac{p+1}{p}}^{\frac{2}{p+1}}}.
   
   \]
   In the next remark we comment as a slight modification of our results covers large initial data \([u_0, v_0] \in H_p^s \times H_p^s\) for some values of \(s, p\).
Remark 1.5.

(a) The proof of Theorem 1.2 shows us that it still holds true if one replaces \( \sup_{-\infty < t < \infty} \) by \( \sup_{-T < t < T} \) throughout its statement. In other words a local-in-time version of Theorem 1.2 is valid. Let \( s > 0 \) and \( 1 < p < \rho + 1 \leq \frac{np}{n - sp} \) (\( < \infty \) if \( n < sp \)). Sobolev embedding yields \( H^s_p \subset L^p \cap L^\infty \subset L^{(\rho + 1, \infty)} \), and then

\[
\sup_{-T < t < T} t^\varepsilon \| G(t)[u_0, v_0] \|_{X_{s+1, \infty}} \leq C \sup_{-T < t < T} t^\varepsilon \| G(t)[u_0, v_0] \|_{H^s_p \times D^{-1}H^{s-1}_p} \leq C T^\varepsilon \| [u_0, v_0] \|_{H^s_p \times H^s_p},
\]

(1.18)

for \( \varepsilon > 0 \), since \( G(t) \) is continuous from \( H^s_p \times H^s_p \) to \( H^s_p \times D^{-1}H^{s-1}_p \). In particular, for \( [u_0, v_0] \in H^s_p \times H^s_p \), \( |\varepsilon| = \alpha \) and \( 0 < T \leq (\varepsilon C^{-1} \|[u_0, v_0]^{-1/2}_{H^s_p \times H^s_p})^{1/\alpha} \), where \( \varepsilon > 0 \) is as in Theorem 1.2, it follows that \( \sup_{-T < t < T} t^\varepsilon \| G(t)[u_0, v_0] \|_{X_{s+1, \infty}} \leq \varepsilon \). Therefore, without assume any smallness condition on the initial data, the local-in-time version of Theorem 1.2 (commented more above) provides a solution \([u(t), v(t)]\) on \((-T, T)\), satisfying \( \sup_{-T < t < T} t^\varepsilon \| [u, v] \|_{X_{s+1, \infty}} < \infty \), when \( \rho_0(n) < \rho < \gamma(n) \). Also, taking \( \varepsilon = \beta \) in (1.18) and \( [u_0, v_0] \in H^s_p \times H^s_p \), by Remark 1.2 and Theorem 1.1, one can obtain a local-in-time mild solution in the range \( 1 < \rho < \rho_0(n) \).

(b) \((H^1\text{-Regularity})\) For \( s \geq 1 \) and \( p = 2 \), the previous item of this remark assures the existence of a unique local-in-time solution with data \([u_0, v_0] \in H^2 \times H^2 \). In that case, by [12,10], there exists a solution \([\tilde{u}, \tilde{v}] \in C([-T_0, T_0]; H^2 \times H^2)\) for some small \( T_0 > 0 \). Since \( H^s \subset H^2 \cap L^\infty \subset L^{(\rho + 1, \infty)} \subset D^{-1}H^{s-1}_{(\rho + 1, \infty)} \), we have \([\tilde{u}, \tilde{v}] \in E^T_0 \) and \( \| [\tilde{u}, \tilde{v}] \|_{E^T_0} \to 0 \) as \( T_0 \to 0^+ \). Thus, taking \( T_0 \) small enough, by uniqueness in \( E^T_0 \) stated in Theorem 1.1, the integral solution \([u(t), v(t)]\) coincides with \([\tilde{u}(t), \tilde{v}(t)]\) on \([-T_0, T_0]\) and therefore it belongs to \( C([-T_0, T_0]; H^2 \times H^2) \).

We prove the following scattering result:

**Theorem 1.3 (Scattering).** Under the hypotheses of Theorem 1.2. Let \([u, v]\) be a solution of (1.3) provided by Theorem 1.2 with initial data \([u_0, v_0]\). Then there exist \([u_0^+, v_0^+] \) and \([u_0^-, v_0^-] \) \in \( I_0 \) such that

\[
\| [u(t) - u^+(t), v(t) - v^+(t)] \|_{X_{s+1, d}} = O(t^{-(\alpha + \rho h)}), \quad \text{as } t \to +\infty,
\]

(1.19)

\[
\| [u(t) - u^-(t), v(t) - v^-(t)] \|_{X_{s+1, d}} = O(t^{-(\alpha + \rho h)}), \quad \text{as } t \to -\infty,
\]

(1.20)

where \([u^+(t), v^+(t)]\) and \([u^-(t), v^-(t)]\) are the unique solutions of the associated linear problem (1.5) with initial data \([u_0^+, v_0^+]\) and \([u_0^-, v_0^-] \), respectively.

Notice that item (i) of Theorem 1.2 says us that \( \| [u(t), v(t)] \|_{X_{s+1, d}} = O(|t|^{-\alpha}) \) provided that \( \| G(t)[u_0, v_0] \|_{X_{s+1, d}} = O(|t|^{-\alpha}) \) as \( |t| \to \infty \). With more structure, in particular this last fact will be refined in the next theorem.

**Theorem 1.4 (Asymptotic stability).** Under the hypotheses of Theorem 1.2. Let \([u, v]\) and \([\tilde{u}, \tilde{v}] \in \mathcal{E}_d\) be two solutions of (1.3) obtained through Theorem 1.2 with initial data \([u_0, v_0]\) and \([\tilde{u}_0, \tilde{v}_0]\), respectively. We have that

\[
\lim_{|t| \to \infty} |t|^\alpha h \| G(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0] \|_{X_{s+1, d}} = 0
\]

(1.21)
if only if
\[
\lim_{|t| \to \infty} |t|^{\alpha + h} \left\| [u(t) - \tilde{u}(t), \nu(t) - \tilde{\nu}(t)] \right\|_{X(p+1,d)} = 0. \tag{1.22}
\]

In particular, the condition (1.21) holds true when \( u_0, \nu_0 \in L^{(p+1,d)} \).

By employing Theorem 1.4, we can improve the scattering decays (1.19)–(1.20).

**Corollary 1.5.** Assume the hypotheses of Theorem 1.2 and let \([u^\pm(t), \nu^\pm(t)]\) be as in Theorem 1.3. If
\[
\lim_{t \to \pm \infty} |t|^{\alpha + h} \left\| G(t)[u_0, \nu_0] \right\|_{X(p+1,d)} = 0, \tag{1.23}
\]
then
\[
\left\| [u(t) - u^\pm(t), \nu(t) - \nu^\pm(t)] \right\|_{X(p+1,d)} = o(|t|^{-(\alpha + ph)}), \quad \text{as} \quad t \to \pm \infty. \tag{1.24}
\]

This manuscript is organized as follows: in the next section we give some preliminaries about Lorentz spaces, and provide linear estimates. The proofs of the results are given in Section 3.

## 2. Lorentz spaces and linear estimate

We begin with some preliminaries about Lorentz spaces. For a deeper discussion about them we refer the reader to [4,3]. The Lorentz spaces \( L^{(p,q)} \) is the set all measurable functions such that the norm \( \| \cdot \|_{(p,q)} \) is finite, where

\[
\| f \|_{(p,q)} = \begin{cases} 
\left( \frac{p}{q} \int_0^\infty \left[ t^\frac{1}{p} f^{**}(t) \right]^q \frac{dt}{t} \right)^\frac{1}{q}, & \text{if } 1 < p < \infty, \ 1 \leq q < \infty, \\
\sup_{t>0} t^\frac{1}{p} f^{**}(t), & \text{if } 1 < p \leq \infty, \ q = \infty,
\end{cases} \tag{2.1}
\]

and

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \quad \text{and} \quad f^*(t) = \inf \left\{ s > 0 : m(\{ x \in \mathbb{R}^n : |f(x)| > s \}) \leq t \right\},
\]

with \( m \) denoting the Lebesgue measure. Those spaces can be constructed by means of interpolation, namely \( L^{(p,q)} = (L^1, L^\infty)_{\frac{1}{p}, \frac{1}{q}} \) with \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \). Moreover, reiteration theorem in the interpolation theory yields

\[ (L^{(p_0,q_0)}, L^{(p_1,q_1)})_{\theta,q} = L^{(p,q)}, \]

provided that \( 1 < p_0 < p_1 < \infty, \ 0 < \theta < 1, \ \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \) and \( 1 \leq q_0, q_1, q \leq \infty \). In special case \( q = \infty \), \( L^{(p,\infty)} \) is also called weak-\( L^p \) spaces. This space is the largest one in the family of Lorentz spaces \( L^{(p,q)} \), since the inclusions \( L^{(p,1)} \subset L^{(p,q_1)} \subset L^{(p,p)} = L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)} \) hold true for all \( 1 \leq q_1 \leq p \leq q_2 \leq \infty \). Hölder-type inequalities work well in \( L^{(p,q)} \)-spaces, precisely
\[ \|fg\|(r,s) \leq C(r) \|f\|(p_1,q_1) \|g\|(p_2,q_2), \]  
(2.2)

where \(1 < p_1, p_2 < \infty\), \(\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}\), \(s \geq 1\) and \(\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}\).

In the next lemma we state some linear estimates for the group \(G(t)\) on the Lorentz spaces. Below, we stand for \(\|[\varphi, \psi]\|_{(p,q)} = \max\{\|\varphi\|_{(p,q)}, \|\psi\|_{(p,q)}\}\) the norm in \(L^{(p,q)} \times L^{(p,q)}\).

**Lemma 2.1.** Let \(1 \leq q \leq \infty\), \(1 < p \leq 2\) and \(p'\) such that \(\frac{1}{p} + \frac{1}{p'} = 1\). There is a positive constant \(C_G = C(n, p)\) such that

\[ \|G(t)[\varphi, \psi]\|_{X_{(p',q)}} \leq C_G |t|^{-\frac{n}{2}(\frac{2}{p'q} - 1)}\|[\varphi, \psi]\|_{(p,q)},\]  
(2.3)

for all measurable \(\varphi, \psi\).

**Proof.** Let \(l_1(t)\) and \(l_2(t)\) be as in Remark 1.4 and recall the \(L' - L''\) estimate

\[ \|l_i(t)\varphi\|_{L'} \leq C|t|^{-\frac{n}{2}(\frac{2}{p} - 1)\max\{\|\varphi\|_{L^p}, \|\psi\|_{L^p}\}} \text{ for all } t \neq 0, \]  
(2.4)

We write \(\tilde{z} = [\varphi, \psi]\) and \(G(t)\tilde{z} = (\pi_1G(t)\tilde{z}, \pi_2G(t)\tilde{z})\), where \(\pi_i\) is the \(i\)-th projection. It follows from (1.6) and (2.4) that

\[ \|\pi_1G(t)\tilde{z}\|_{L^{p'}} \leq C|t|^{-\frac{n}{2}(\frac{2}{p} - 1)\max\{\|\varphi\|_{L^p}, \|\psi\|_{L^p}\}} \text{ for all } t \neq 0, \]  
(2.5)

where \(1 < p \leq 2\) and \(C > 0\) depends only on \(n, p\). Next, since \(DJ^{-1}g_3(t) = l_2(t)\) and \(g_4(t) = l_1(t)\) (see (1.6)), we have

\[ DJ^{-1}\pi_2G(t)\tilde{z} = DJ^{-1}g_3(t)\varphi + DJ^{-1}g_4(t)\psi = l_2(t)\varphi + DJ^{-1}l_1(t)\psi. \]

By \(L'\)-continuity of the operator \(DJ^{-1}\) and (2.4), it follows that

\[ \|\pi_2G(t)\tilde{z}\|_{D^{-1}H^1_{p'}} = \|DJ^{-1}\pi_2G(t)\tilde{z}\|_{L^{p'}} \leq \|l_2(t)\varphi\|_{L^{p'}} + \|DJ^{-1}l_1(t)\psi\|_{L^{p'}} \leq \|l_2(t)\varphi\|_{L^{p'}} + \|l_1(t)\psi\|_{L^{p'}} \leq C|t|^{-\frac{n}{2}(\frac{2}{p} - 1)\max\{\|\varphi\|_{L^p}, \|\psi\|_{L^p}\}}. \]  
(2.6)

Now, the estimates (2.5) and (2.6), and a real interpolation argument, yield

\[ \|\pi_1G(t)\tilde{z}\|_{(p',q)} \leq C|t|^{-\frac{n}{2}(\frac{2}{p'} - 1)\max\{\|\varphi\|_{(p,q)}, \|\psi\|_{(p,q)}\}}, \]  
(2.7)

\[ \|\pi_2G(t)\tilde{z}\|_{D^{-1}H^{-1}_{(p',q)}} \leq C|t|^{-\frac{n}{2}(\frac{2}{p'} - 1)\max\{\|\varphi\|_{(p,q)}, \|\psi\|_{(p,q)}\}}. \]  
(2.8)

for all \(t \neq 0\) and \([\varphi, \psi] \in L^{(p,q)} \times L^{(p,q)}\). The inequalities (2.7) and (2.8) together are equivalent to (2.3). \(\square\)
3. Proof of theorems

3.1. Nonlinear estimates

First we remind the reader of the Beta function $B(\nu, \eta) = \int_{0}^{1} (1 - s)^{\nu-1} s^{\eta-1} ds$, which is finite for all $\nu > 0$ and $\eta > 0$. For $k_1, k_2 < 1$ and $t > 0$, the change of variable $s \rightarrow st$ yields

$$\int_{0}^{t} (t - s)^{-k_1} s^{-k_2} ds = t^{1-k_1-k_2} \int_{0}^{1} (1 - s)^{-k_1} s^{-k_2} ds = t^{1-k_1-k_2} B(1-k_1, 1-k_2) < \infty. \quad (3.1)$$

Denote by $\mathcal{F}(u) = - \int_{0}^{t} G(t - s)[0, f(u(s))] ds$ the nonlinear part of the integral equation (1.4). In order to apply a fixed point argument for finding a solution of (1.4), we need the following estimates for $\mathcal{F}(\cdot)$:

**Proposition 3.1.**

(i) Let $1 < \rho < \rho_0(n), \beta = \frac{n(\rho - 1)}{2(\rho + 1)},$ and $1 \leq d \leq \infty$. There is a constant $K_{\beta} > 0$ such that

$$\sup_{-T < t < T} |t|^\beta \| \mathcal{F}(u) - \mathcal{F}(\tilde{u}) \|_{X(\rho+1,d)}$$

$$\leq K_{\beta} T^{1-\rho \beta} \sup_{-T < t < T} |t|^\beta \| u - \tilde{u} \|_{(\rho+1,d)} \times \sup_{-T < t < T} \left( |t|^\beta (\rho-1) \| u \|_{(\rho+1,d)} + |t|^\beta (\rho-1) \| \tilde{u} \|_{(\rho+1,d)} \right), \quad (3.2)$$

for all measurable $u$ and $\tilde{u}$.

(ii) Let $\rho_0(n) < \rho < \frac{n+2}{n-2} = \gamma(n), \alpha = \frac{1}{\rho-1} - \frac{n}{2(\rho+1)},$ and $1 \leq d \leq \infty$, and $0 \leq h < 1 - \alpha \rho$. There is a constant $K_{\alpha, h} > 0$ such that

$$\sup_{-\infty < t < \infty} |t|^{\alpha + h} \| \mathcal{F}(u) - \mathcal{F}(\tilde{u}) \|_{X(\rho+1,d)}$$

$$\leq K_{\alpha, h} \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| u - \tilde{u} \|_{(\rho+1,d)} \times \sup_{-\infty < t < \infty} \left( |t|^{\alpha (\rho-1)} \| u \|_{(\rho+1,d)} + |t|^{\alpha (\rho-1)} \| \tilde{u} \|_{(\rho+1,d)} \right), \quad (3.3)$$

for all measurable $u$ and $\tilde{u}$.

**Proof.** There is no loss of generality in assuming $t > 0$. By applying Lemma 2.1 with $p = \frac{\rho+1}{\rho}$ and $q = d$, and afterwards Hölder inequality (2.2), we obtain

$$\| \mathcal{F}(u) - \mathcal{F}(\tilde{u}) \|_{X(\rho+1,d)} \leq \int_{0}^{t} \| G(t - s)[0, f(u(s)) - f(\tilde{u}(s))] \|_{X(\rho+1,d)} ds$$

$$\leq C \int_{0}^{t} (t - s)^{-\frac{2}{\rho}} (\rho-1)^{-1} \left( |u - \tilde{u}| \right) \left( |u|^{\rho-1} + |\tilde{u}|^{\rho-1} \right) ds$$
\[
\leq C \int_0^t (t-s)^{-\beta} \|u - \tilde{u}\|_{(\rho+1,d)} \left(\|u\|_{(\rho+1,d)}^{\rho-1} + \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1}\right) ds. 
\]

Notice that \(-\beta = -\frac{n(\rho-1)}{2(\rho+1)} > -1\) and \(-\beta \rho > -1\) when \(1 < \rho < \rho_0(n)\). Thus, by using (3.1), the r.h.s. of (3.4) can be bounded by

\[
\leq C \left( \sup_{0 < t < T} t^\beta \|u - \tilde{u}\|_{(\rho+1,d)} \sup_{0 < t < T} \left( t^\beta (\rho-1) \|u\|_{(\rho+1,d)}^{\rho-1} + t^\beta (\rho-1) \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1} \right) \right) 
\times \int_0^t (t-s)^{-\beta} s^{-\beta \rho} ds 
\]

\[= \mathcal{K}_t t^{-\beta} t^{-\rho} \left( \sup_{0 < t < T} t^\beta \|u - \tilde{u}\|_{(\rho+1,d)} \sup_{0 < t < T} \left( t^\beta (\rho-1) \|u\|_{(\rho+1,d)}^{\rho-1} + t^\beta (\rho-1) \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1} \right) \right), \]

which implies (3.2). In the sequel we deal with (3.3). Again from Lemma 2.1, Hölder inequality (2.2) and the property (3.1), we have

\[
\|\mathcal{F}(u) - \mathcal{F}(\tilde{u})\|_{X_{(\rho+1,d)}} 
\leq C \int_0^t (t-s)^{-\frac{3}{2}} (\rho+1)^{-1} \|u - \tilde{u}\|_{(\rho+1,d)} \left(\|u\|_{(\rho+1,d)}^{\rho-1} + \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1}\right) ds 
\leq C \left( \sup_{t > 0} t^{\alpha + h} \|u - \tilde{u}\|_{(\rho+1,d)} \sup_{t > 0} \left( t^{\alpha (\rho-1)} \|u\|_{(\rho+1,d)}^{\rho-1} + t^{\alpha (\rho-1)} \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1} \right) \right) 
\times \int_0^t (t-s)^{-\beta} s^{-\alpha \rho - h} ds 
\]

\[= C^{1 - \beta - \rho} \left( \sup_{t > 0} t^{\alpha + h} \|u - \tilde{u}\|_{(\rho+1,d)} \sup_{t > 0} \left( t^{\alpha (\rho-1)} \|u\|_{(\rho+1,d)}^{\rho-1} + t^{\alpha (\rho-1)} \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1} \right) \right) 
\]

\[= K_\alpha t^{-\alpha - h} \left( \sup_{t > 0} t^{\alpha + h} \|u - \tilde{u}\|_{(\rho+1,d)} \sup_{t > 0} \left( t^{\alpha (\rho-1)} \|u\|_{(\rho+1,d)}^{\rho-1} + t^{\alpha (\rho-1)} \|\tilde{u}\|_{(\rho+1,d)}^{\rho-1} \right) \right), \]

since \(\alpha \rho + h < 1\) and \(\beta + (\rho - 1)\alpha = 1\). \(\square\)

3.2. Proof of Theorem 1.1

Local existence. Consider the ball \(M_{2\delta} = \{[u, v] \in \mathcal{C}_T^\rho; \|[u, v]\|_{\mathcal{E}_\rho^T} \leq 2\delta\}\) endowed with the complete metric \(\mathcal{Z}(\cdot, \cdot)\) defined by \(\mathcal{Z}([u, v], [\tilde{u}, \tilde{v}]) = \|[u - \tilde{u}, v - \tilde{v}]\|_{\mathcal{E}_\rho^T}\). For some \(\delta > 0\), we wish to show that the map

\[
\Phi([u, v]) := G(t)[u_0, v_0] - \int_0^t G(t-s)[0, f(u(s))] ds 
\]

\[:= G(t)[u_0, v_0] + \mathcal{F}(u) \quad (3.5)\]
is a contraction on \((M_{2\delta}, Z)\). Recall (1.8) and that \(\|\cdot, \cdot\|_{L^{(\rho+1, \infty)}}\) stands for the norm in \(L^{(\rho+1, \infty)}\) and \(L^{(\rho, \infty)}\). Applying Lemma 2.1 with \(p = \rho+1\) and \(d = \infty\), we have

\[
\|G(t)[u_0, v_0]\|_{L^p} = \sup_{-T < \tau < T} |t|^{\rho} \|G(t)[u_0, v_0]\|_{X_{(\rho+1, \infty)}} \leq C_G \|u_0, v_0\|_{L^{(\rho+1, \infty)}} < \infty. \tag{3.6}
\]

Now we take \(\delta = C_G \|u_0, v_0\|_{L^{(\rho+1, \infty)}}\) and \(T > 0\) such that \(2^\rho \delta^\rho T^{1-\rho} < 1\). Proposition 3.1(i) with \(\bar{u} = 0\) and \(d = \infty\) yields

\[
\|\Phi([u, v])\|_{E^T} \leq \|G(t)[u_0, v_0]\|_{L^p} + \|\mathcal{F}(u)\|_{E^T} \leq C_G \|u_0, v_0\|_{L^{(\rho+1, \infty)}} + K_\beta T^{1-\rho^2} \|[u, v]\|_{L^p} \leq \delta + K_\beta T^{1-\rho^2} 2^\rho \delta^\rho < \delta + 2\delta, \tag{3.7}
\]

for all \([u, v] \in M_{2\delta}\) and thus \(\Phi(M_{2\delta}) \subset M_{2\delta}\). From Proposition 3.1(i), we also have

\[
\|\Phi([u, v]) - \Phi([\bar{u}, \bar{v}])\|_{E^T} \leq \|\mathcal{F}(u) - \mathcal{F}(\bar{u})\|_{E^T} \leq K_\beta T^{1-\rho^2} \|[u, v] - [\bar{u}, \bar{v}]\|_{E^T} \leq K_\beta T^{1-\rho^2} 2^\rho \delta^\rho \|[u, v] - [\bar{u}, \bar{v}]\|_{E^T}, \tag{3.8}
\]

for all \([u, v], [\bar{u}, \bar{v}] \in M_{2\delta}\). Since \(K_\beta T^{1-\rho^2} 2^\rho \delta^\rho \|[u, v] - [\bar{u}, \bar{v}]\|_{E^T} < 1\), the map \(\Phi\) is a contraction in \(M_{2\delta}\) and then, the Banach fixed point theorem assures existence of a solution \([u, v] \in E^T\) for \((1.4)\). The proof of the weak convergence \([u, v] \rightarrow [u_0, v_0]\) follows from standard arguments, and so we omit it. On the other hand if \([u, v], [\bar{u}, \bar{v}] \in M_{2\delta}\) are two solutions with respective initial data \([u_0, v_0], [\bar{u}_0, \bar{v}_0]\), then, similarly in deriving (3.7) one obtains

\[
\|[u, v] - [\bar{u}, \bar{v}]\|_{E^T} = \|G(t)[u_0 - \bar{u}_0, v_0 - \bar{v}_0]\|_{E^T} + \|\mathcal{F}(u) - \mathcal{F}(\bar{u})\|_{E^T} \leq C_G \|[u_0, v_0] - [\bar{u}_0, \bar{v}_0]\|_{L^{(\rho+1, \infty)}} + K_\beta T^{1-\rho^2} 2^\rho \delta^\rho \|[u, v] - [\bar{u}, \bar{v}]\|_{E^T},
\]

which, since \(K_\beta T^{1-\rho^2} 2^\rho \delta^\rho \|[u, v] - [\bar{u}, \bar{v}]\|_{E^T} < 1\), implies the Lipschitz continuity of the data-solution map. \(\square\)

**L^p,q-regularity.** From fixed-point argument employed above, we know that the previous solution \([u, v]\) is the limit in \(M_{2\delta}\) of the Picard sequence \([u_m, v_m]\) \(m \in \mathbb{N}\):

\[
[u_1, v_1] = G(t)[u_0, v_0] \quad \text{and} \quad [u_{m+1}, v_{m+1}] = G(t)[u_0, v_0] - \int_0^t G(t-s)[0, f(u_m(s))] \, ds. \tag{3.9}
\]

In the sequel we shall show the following bounds

\[
\sup_{-T < t < T} |t|^\rho \|u_m(\cdot, t)\|_{L^{(\rho+1, d)}} \leq C < \infty \quad \text{and} \quad \sup_{-T < t < T} |t|^\rho \|v_m(\cdot, t)\|_{L^{D^{-1}H^{\rho+1}_d}} \leq C < \infty, \tag{3.10}
\]

for all \(m \in \mathbb{N}\). Notice that the uniqueness of limit in \(\mathcal{S'}(\mathbb{R}^d)\) together with (3.10) imply (1.11).
In order to obtain (3.9), we employ Lemma 2.1 (with \( p = \frac{\rho + 1}{\rho} \) and \( q = d \)) and Proposition 3.1 (with \( \tilde{u} = 0 \)) to estimate (3.8) as
\[
\sup_{-T < t < T} |t|^{\beta} \| [u_{m+1}, v_{m+1}] \|_{X_{(\rho,1,d)}} \\
\leq C_G \| [u_0, v_0] \|_{(\rho,1,d)} + K_{\beta} T^{1-\rho \beta} \left( \sup_{-T < t < T} |t|^{\beta} \| u_m(\cdot, t) \|_{(\rho,1,d)} \right)^\rho.
\]
(3.10)

We rewrite (3.10) as
\[
L_{m+1} \leq L_1 + K_{\beta} T^{1-\rho \beta} (L_m)^\rho,
\]
(3.11)
where \( L_1 = C_G \| [u_0, v_0] \|_{(\rho,1,d)} < \infty \) and
\[
L_m = \sup_{-T < t < T} |t|^{\beta} \max \{ \| u_m(\cdot, t) \|_{(\rho,1,d)}, \| v_m(\cdot, t) \|_{D^{-1}H_{(\rho,1,d)}} \} = \sup_{-T < t < T} |t|^{\beta} \| [u_m, v_m] \|_{X_{(\rho,1,d)}}.
\]
for \( m \geq 2 \). Taking \( T > 0 \) in such a way that \( 2^\rho K_{\beta} T^{1-\rho \beta} (L_1)^{\rho-1} < 1 \), by means of an induction argument one can obtain that \( L_m \leq C = 2L_1 \), for all \( m \in \mathbb{N} \), and therefore the proof is complete. \( \square \)

3.3. Proof of Theorem 1.2

Part (i) - Global existence. Consider the map \( \Phi \) as in (3.5) and the ball \( M_{2\varepsilon} = \{ [u, v] \in \mathcal{E}_\alpha; \| [u, v] \|_\alpha \leq 2\varepsilon \} \) in \( \mathcal{E}_\alpha \). Applying Proposition 3.1(ii) with \( h = 0 \) and \( d = \infty \) it follows that
\[
\| \Phi([u, v]) - \Phi([\tilde{u}, \tilde{v}]) \|_{\mathcal{E}_\alpha} \\
= \| \mathcal{F}(u) - \mathcal{F}(\tilde{u}) \|_{\mathcal{E}_\alpha} \\
\leq K_{\alpha,0} \| [u, v] - [\tilde{u}, \tilde{v}] \|_{\mathcal{E}_\alpha} \left( \| [u, v] \|_{\mathcal{E}_\alpha}^{\rho-1} + \| [\tilde{u}, \tilde{v}] \|_{\mathcal{E}_\alpha}^{\rho-1} \right) \\
\leq 2^\rho \varepsilon^{\rho-1} K_{\alpha,0} \| [u, v] - [\tilde{u}, \tilde{v}] \|_{\mathcal{E}_\alpha}, \quad \text{for all } [u, v], [\tilde{u}, \tilde{v}] \in M_{2\varepsilon}.
\]
(3.12)

By hypotheses
\[
\| G(t)[u_0, v_0] \|_{\mathcal{E}_\alpha} = \| [u_0, v_0] \|_{\mathcal{I}_0} \leq \varepsilon,
\]
and then, recalling \( \mathcal{F}(0) = 0 \), the inequality (3.12) with \( [\tilde{u}, \tilde{v}] = 0 \) yields
\[
\| \Phi([u, v]) \|_{\mathcal{E}_\alpha} \leq \| G(t)[u_0, v_0] \|_{\mathcal{E}_\alpha} + \| \mathcal{F}(u) \|_{\mathcal{E}_\alpha} \\
\leq \| G(t)[u_0, v_0] \|_{\mathcal{E}_\alpha} + \| [u, v] \|_{\mathcal{E}_\alpha}^{\rho} \\
\leq \varepsilon + 2^\rho \varepsilon^{\rho} K_{\alpha,0} \leq 2\varepsilon,
\]
(3.13)
provided that \( 2^\rho \varepsilon^{\rho-1} K_{\alpha,0} < 1 \) and \( [u, v] \in M_{2\varepsilon} \). Unlike Theorem 1.1, in this time \( K_{\alpha,0} \) does not depend on \( T \), and we will get \( \Phi \) a contraction by making the initial data, that is, \( \varepsilon > 0 \) small enough. In fact, by (3.12) and (3.13), if \( 0 < \varepsilon < (2^\rho K_{\alpha,0}^{-1})^{1/(\rho-1)} \) then the map \( \Phi \) is a contraction and have a fixed point in \( M_{2\varepsilon} \), which is the unique solution \( [u, v] \) of the integral equation (1.4) satisfying \( \| [u, v] \|_{\mathcal{E}_\alpha} \leq 2\varepsilon \).

Next, let \( [u, v], [\tilde{u}, \tilde{v}] \in M_{2\varepsilon} \) satisfy (1.4) with respective data \( [u_0, v_0], [\tilde{u}_0, \tilde{v}_0] \in \mathcal{I}_0 \). Since \( 2^\rho \varepsilon^{\rho-1} K_{\alpha,0} < 1 \), the Lipschitz continuity of the data-solution map follows at once from
\[ \| [u, v] - [\tilde{u}, \tilde{v}] \|_{E_\alpha} = \| G(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0] \|_{E_\alpha} + \| \mathcal{F}(u) - \mathcal{F}(\tilde{u}) \|_{E_\alpha} \leq \| [u_0 - \tilde{u}_0, v_0 - \tilde{v}_0] \|_{I_0} + 2^\rho \varepsilon \rho^{-1} K_{\alpha, 0} \| [u, v] - [\tilde{u}, \tilde{v}] \|_{E_\alpha}. \]

**Part (ii) – --regularity.** The previous global solution also can be approximated by Picard sequence (3.8), but in the present case, the limit is taken in space \( E_\alpha \) instead of \( E^0_\alpha \). We claim that the sequence (3.8) is uniformly bounded in the norm \( \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| \cdot \|_{X_{(\rho + 1, d)}} \). To see this, denote

\[ \Gamma_{m, h} = \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| [u_m, v_m] \|_{X_{(\rho + 1, d)}}. \]  

(3.14)

First we deal with the case \( h = 0 \). Computing the norm \( \sup_{-\infty < t < \infty} |t|^{\alpha} \| \cdot \|_{X_{(\rho + 1, d)}} \) in (3.8), an application of Proposition 3.1(ii) with \( h = 0 \) and \( \tilde{u} = 0 \) yields

\[ \Gamma_{m+1, 0} = \sup_{-\infty < t < \infty} |t|^{\alpha} \| [u_{m+1}, v_{m+1}] \|_{X_{(\rho + 1, d)}} \leq \sup_{-\infty < t < \infty} |t|^{\alpha} \| G(t)[u_0, v_0] \|_{X_{(\rho + 1, d)}} + K_{\alpha, 0} \left( \sup_{-\infty < t < \infty} |t|^{\alpha} \| u_m \|_{X_{(\rho + 1, d)}} \right)^\rho \leq \Gamma_{1, 0} + K_{\alpha, 0} \Gamma_{m, 0} \leq 2 \Gamma_{1, 0}, \]  

(3.15)

provided that

\[ \Gamma_{1, 0} \leq \varepsilon_0 \]  

and \( 2^\rho K_{\alpha, 0} (\varepsilon_0)^{\rho-1} < 1. \)  

(3.16)

Applying now Proposition 3.1(ii) (with \( h \neq 0 \) and \( \tilde{u} = 0 \)) it follows that

\[ \Gamma_{m+1, h} = \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| [u_{m+1}, v_{m+1}] \|_{X_{(\rho + 1, d)}} \leq \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| G(t)[u_0, v_0] \|_{X_{(\rho + 1, d)}} + K_{\alpha, h} \sup_{-\infty < t < \infty} |t|^{\alpha + h} \| u_m \|_{X_{(\rho + 1, d)}} \left( \sup_{-\infty < t < \infty} |t|^{\alpha} \| u_m \|_{X_{(\rho + 1, d)}} \right)^{\rho - 1} \leq \Gamma_{1, h} + 2^{\rho - 1} \varepsilon_0^{\rho - 1} K_{\alpha, h} \Gamma_{m, h} \text{ for all } m \in \mathbb{N} \text{ (by (3.16)–(3.15))}. \]  

(3.17)

Taking \( \varepsilon_0 \leq \varepsilon \) satisfying (3.16) and \( 2^{\rho - 1} \varepsilon_0^{\rho - 1} K_{\alpha, h} < 1 \), the r.h.s. of (3.17) can be bounded by \( 2 \Gamma_{1, h} \) \((< \infty \text{ by hypothesis})\), and then it follows the desired claim. Finally, the uniqueness of the limit in sense of distributions yields (1.15). \( \square \)

### 3.4. Proof of Theorem 1.3

We only prove (1.19) because the case \( t \to -\infty \) it follows from analogous argument. Define

\[ [u_0^+, v_0^+] = [u_0, v_0] - \int_0^\infty G(-s)[0, f(u(s))] \, ds \in I_0. \]

Notice that the solution \([u^+, v^+] = G(t)[u_0^+, v_0^+]\) of the linear problem associated to (1.3), and with \( [u_0^+, v_0^+] \), can be written as
\[ u^{+}, v^{+} = G(t)[u_0, v_0] - \int_{0}^{\infty} G(t-s)[0, f(u(s))] ds. \]  

(3.18)

Subtracting (3.18) from (1.4), and recalling (3.14), we have

\[
\| [u - u^{+}, v - v^{+}] \|_{X(\rho+1,d)} = \left\| \int_{t}^{\infty} G(t-s)[0, f(u(s))] ds \right\|_{X(\rho+1,d)} \\
\leq \int_{t}^{\infty} (s-t)^{-\frac{n}{2(\rho+1)} - 1} s^{-\rho} ds \left( \sup_{t>0} \| u(t) \|_{(\rho+1,d)} \right)^{\rho} \\
\leq 2^{\rho} \int_{1,\rho}^{\infty} t^{-\alpha - \rho h} \left( s-1 \right)^{-\frac{n}{2(\rho+1)} - 1} s^{-\rho} ds \leq Ct^{-\alpha - \rho h},
\]

since \( \frac{n}{2} (\frac{2\rho}{\rho+1} - 1) + \alpha \rho = 1 + \alpha \) and

\[
\int_{1}^{\infty} (s-1)^{-\frac{n}{2(\rho+1)} - 1} s^{-\rho} ds \\
\leq \int_{1}^{2} (s-1)^{-\frac{n}{2(\rho+1)} - 1} ds + \int_{2}^{\infty} (s-1)^{-\frac{n}{2(\rho+1)} - 1 - \rho h} ds < \infty.
\]

3.5. Proof of Theorem 1.4

Let \([u, v]\) and \([\tilde{u}, \tilde{v}]\) be two solutions for (1.4) with initial data \([u_0, v_0]\) and \([\tilde{u}_0, \tilde{v}_0]\), respectively. Without loss of generality assume \(t > 0\). We compute the difference between their integral equations and take the norm \( t^{\alpha + h} \cdot \| u(t) \|_{X(\rho+1,\infty)} \) to obtain

\[
t^{\alpha + h} \left\| [u(t), v(t)] - [\tilde{u}(t), \tilde{v}(t)] \right\|_{X(\rho+1,\infty)} \\
\leq t^{\alpha + h} \left\| G(t)([u_0, v_0] - [\tilde{u}_0, \tilde{v}_0]) \right\|_{X(\rho+1,\infty)} \\
+ t^{\alpha + h} \left\| \int_{0}^{t} G(t-s)[0, f(u) - f(\tilde{u})] ds \right\|_{X(\rho+1,\infty)}. \tag{3.19}
\]

Take \( \varepsilon > 0 \) such that \( 2^{\rho} \varepsilon^{\rho-1} K_{\alpha, h} < 1 \) where \( K_{\alpha, h} \) is as in Proposition 3.1. From Theorem 1.2 we know that \( \| [u, v] \|_{E_{\alpha}} \| [\tilde{u}, \tilde{v}] \|_{E_{\alpha}} \leq 2\varepsilon \), provided that \( \| [u_0, v_0] \|_{Z_0} \leq \varepsilon \). We estimate (3.19) in the following way:

\[
t^{\alpha + h} \left\| \int_{0}^{t} G(t-s)[0, f(u) - f(\tilde{u})] ds \right\|_{X(\rho+1,\infty)} \\
\leq Ct^{\alpha + h} \int_{0}^{t} (t-s)^{-\frac{n}{2(\rho+1)} - 1} s^{-\alpha \rho - h} ds \left\| [u(s), v(s)] - [\tilde{u}(s), \tilde{v}(s)] \right\|_{X(\rho+1,\infty)} ds
\]
\[ \times \left( \| [u, v] \|_{E_d}^{\rho - 1} + \| [\tilde{u}, \tilde{v}] \|_{E_d}^{\rho - 1} \right) \]
\[ \leq C 2^\rho e^{\rho - 1} t^{\alpha + h} \int_0^t (t - s)^{-\frac{\rho}{2} \frac{2}{p + 1} - 1} s^{-\alpha \rho - h} s^{\alpha + h} \]
\[ \times \| [u(s), v(s)] - [\tilde{u}(s), \tilde{v}(s)] \|_{X_{\rho + 1, \infty}} ds. \] (3.20)

Notice that, through the change of variable \( s \mapsto ts \), (3.20) becomes
\[ \leq C 2^\rho e^{\rho - 1} \int_0^1 (1 - s)^{-\frac{\rho}{2} \frac{2}{p + 1} - 1} s^{-\alpha \rho - h} (ts)^{\alpha + h} \]
\[ \times \| [u(ts), v(ts)] - [\tilde{u}(ts), \tilde{v}(ts)] \|_{X_{\rho + 1, \infty}} ds, \] (3.21)

since \( 1 - \frac{n}{2} \left( \frac{2p}{p + 1} - 1 \right) - (\rho - 1)\alpha = 0 \). From Theorem 1.2 we have that
\[ M := \limsup_{t \to \infty} t^{\alpha + h} \| [u(t), v(t)] - [\tilde{u}(t), \tilde{v}(t)] \|_{X_{\rho + 1, \infty}} < \infty. \] (3.22)

Computing \( \limsup_{t \to \infty} \) in (3.19), the condition (1.21), the converge dominated theorem and (3.19)–(3.21) together imply
\[ M \leq \left( C 2^\rho e^{\rho - 1} \int_0^1 (1 - s)^{-\frac{\rho}{2} \frac{2}{p + 1} - 1} s^{-\alpha \rho - h} ds \right) M \]
\[ = 2^\rho e^{\rho - 1} K_{\alpha, h} M. \]

Therefore \( M = 0 \) (because \( 2^\rho e^{\rho - 1} K_{\alpha, h} < 1 \)) and it follows (1.22) in the case \( d = \infty \). The proof for \( d \neq \infty \) is similar to the previous one, but in this case, one needs to use \( \sup_{t > 0} t^{\alpha + h} \| [u, v] \|_{X_{\rho + 1, d}} \leq 2 F_{1, h} \leq 2 \varepsilon \) (according to (3.17) and (1.15)) instead of \( \| [u, v] \|_{E_d} \leq 2 \varepsilon \).

Next, we deal with the reciprocal assertion. For that matter, we denote
\[ G = \sup_{t > 0} t^{\alpha(\rho - 1)} \left\| [u(t), v(t)] \right\|_{X_{\rho + 1, d}}^{\rho - 1} + \sup_{t > 0} t^{\alpha(\rho - 1)} \left\| [\tilde{u}(t), \tilde{v}(t)] \right\|_{X_{\rho + 1, d}}^{\rho - 1} < \infty \] (by Theorem 1.2).

Recall the notation \( M \) in (3.22) and estimate
\[ \limsup_{t \to \infty} t^{\alpha + h} \left\| G(t) [u_0 - \tilde{u}_0, v_0 - \tilde{v}_0] \right\|_{X_{\rho + 1, d}} \]
\[ \leq \limsup_{t \to \infty} t^{\alpha + h} \left\| [u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)] \right\|_{X_{\rho + 1, d}} + \left\| t^{\alpha + h} \right\| \left\| G(t - s) \left[ 0, f(u) - f(\tilde{u}) \right] \right\|_{X_{\rho + 1, d}} \]
\[ \leq 0 + \left( C G \int_0^1 (1 - s)^{-\frac{\rho}{2} \frac{2}{p + 1} - 1} s^{-\alpha \rho - h} ds \right) M = 0, \]

since \( M = 0 \) by hypothesis. Finally, the last assertion in Theorem 1.2 follows from
\[
\lim_{t \to \infty} \|r^{\alpha + h} G(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0]\|_{X^{(\rho + 1, d)}_{\infty}} \\
\leq \lim_{t \to \infty} \|r^{\alpha + h - \frac{2}{p+1}} (\rho - 1)\|_{X^{(\rho + 1, d)}_{\infty}} (\rho + 1, d) \text{ (by Lemma 2.1)} \\
= \lim_{t \to \infty} \|t^{\alpha + h - \frac{1}{\rho + 1} - \frac{2}{p+1}}\|_{X^{(\rho + 1, d)}_{\infty}} = 0, \quad \text{because } \rho_0(n) < \rho < \rho(n). \quad \square
\]

3.6. Proof of Corollary 1.5

Since (1.23) holds true, an application of Theorem 1.4 with \([\tilde{u}_0, \tilde{v}_0] = 0\) and \([\tilde{u}, \tilde{v}] = 0\) yields
\[
\lim_{t \to \infty} t^{\alpha + h} \|[u(t), v(t)]\|_{X^{(\rho + 1, d)}_{\infty}} = 0. \quad (3.23)
\]

From (3.18) and (1.4), Lemma 2.1 and the change of variable \(s \to st\), we have
\[
\|[u(t) - u^+(t), v(t) - v^+(t)]\|_{X^{(\rho + 1, d)}_{\infty}} \\
= \left\| \int_G (t - s)[0, f(u(s))] ds \right\|_{X^{(\rho + 1, d)}_{\infty}} \quad (3.24)
\]

Next we multiply (3.24)–(3.25) by \(t^{\alpha + \rho h}\) and afterwards compute \(\limsup_{t \to \infty}\) in the result to obtain
\[
0 \leq \limsup_{t \to \infty} t^{\alpha + \rho h} \|[u(t) - u^+(t), v(t) - v^+(t)]\|_{X^{(\rho + 1, d)}_{\infty}} \\
\leq 2^{\rho - 1} \int_{1,h} (s - 1) - \frac{2}{p + 1} s^{\alpha + h} ds \limsup_{t \to \infty} t^{\alpha + h} \|[u(t), v(t)]\|_{X^{(\rho + 1, d)}_{\infty}} = 0 \quad \text{by (3.23)},
\]

since \(\int_{1}^{\infty} (s - 1) - \frac{2}{p + 1} s^{\alpha + h} ds < \infty\). This implies (1.24) as \(t \to \infty\). The case \(t \to -\infty\) follows from entirely parallel arguments. \(\square\)

References