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Fractional visco-elastic Euler–Bernoulli beam

M. Di Paola [a], R. Heuer [b], A. Pirrotta [a,*]

[a] Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM), Università di Palermo, Viale delle Scienze, Palermo, Italy
[b] Vienna University of Technology, Center of Mechanics and Structural Dynamics, Vienna, Austria

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ABSTRACT

Aim of this paper is the response evaluation of fractional visco-elastic Euler–Bernoulli beam under quasi-static and dynamic loads. Starting from the local fractional visco-elastic relationship between axial stress and axial strain, it is shown that bending moment, curvature, shear, and the gradient of curvature involve fractional operators. Solution of particular example problems are studied in detail providing a correct position of mechanical boundary conditions. Moreover, it is shown that, for homogeneous beam both correspondence principles also hold in the case of Euler–Bernoulli beam with fractional constitutive law. Virtual work principle is also derived and applied to some case studies.

1. Introduction

In the past the “classical” models as Maxwell and Kelvin–Voigt one or more complex combinations of such units composed by springs and dashpots have been used to capture visco-elastic phenomena like relaxation and/or creep (Flügge, 1967; Pipkin, 1972; Christensen, 1982). Such elementary models show some inconsistencies: (i) experimental relaxation and creep functions are more or less well fitted by different composition of springs and dashpots. This is a very serious problem since the inverse of the constitutive law $\sigma = [L\varepsilon]$ may not be written as $\varepsilon = L^{-1}[\sigma]$ where L is a linear differential operator, (ii) whatever the number and combinations of elementary units are, the kernel of hereditary integrals is of exponential type and then for a constant load (creep-test) for $t \to \infty$ the strain takes an asymptotic value. Such a behavior is not observed in real experiments that show an increasing trend as $t \to \infty$.

From these observations we may state that visco-elastic models based upon combinations of spring and dashpots may capture the real behavior only for short observation time.

A more realistic description of creep and/or relaxation is given by a power law function with real order exponent, Nutting (1921) and Gemant (1936), confirming experimental data, Di Paola et al. (2011).

As soon as we assume a power law function for creep the constitutive law relating deformation and stress is ruled by a Riemann Liouville fractional integral with order equal to that of the power law, and viceversa, starting from the relaxation function, the $\sigma - \varepsilon$ constitutive law is ruled by its inverse operator that is the Caputo’s fractional derivative.

Moreover also the behavior for $t \to \infty$ is captured with a power law fractional constitutive law. Such a model is called fractional hereditary model since fractional operators are involved and readers are referred to Samko et al. (1993), Podlubny (1999) and Hilfer (2000).

For these reasons in the second part of the last century a lot of researches have been carried out enforcing the knowledge of fractional hereditary materials (Caputo and Mainardi, 1971; Gonsovski and Rossikhin, 1973; Stiassnie, 1979; Bagley and Torvik, 1983, 1986; Schmidt and Gaul, 2002; Mainardi and Gorenflo, 2007; Mainardi, 2010; Evangelatos and Spanos, 2011).

Once the local visco-elastic behavior is written in local form Euler–Bernoulli or Timoshenko beam may be treated in a very simple way. Applications by using the classical models have been studied in the past (Flügge, 1967; Wang et al., 1997) often by using Laplace transformations. Very recently Yao et al. (2011) proposed the quasi-static analysis of beams described by fractional Kelvin visco-elastic model using Laplace transformations. Even though the derivations are correct no physical implication of the hereditary model based upon fractional hereditary materials comes out.

In this paper the problem of fractional Euler–Bernoulli beam based upon the simplest model is treated operating in time domain, in order to highlight a lot of observations that remain hidden in Laplace domain. First of all, it is shown that for a simple homogeneous beam (statically determined or not) both correspondence principles (see Flügge (1967)) also hold for fractional beams.

As regards, in detail, it will be shown that: in a fractional visco-elastic beam subjected to loads which are applied simultaneously at initial time and then held constant, the stresses are the

* Corresponding author. Tel.: +39 3204395957.
E-mail address: antonina.pirrotta@unipa.it (A. Pirrotta).

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same as those in the purely elastic case, while strains and displacements depend on time and may be calculated from the purely elastic case by simply replacing the elastic modulus with the inverse of the creep function. Additionally the second version of the aforementioned principle remains valid for a fractional visco-elastic beam under imposed displacements that remain constant leading to the same displacements and strains according to the elastic case, while stresses may be derived from the purely elastic case by simply replacing the elastic modulus with the relaxation function.

Moreover the virtual work principle for fractional visco-elastic material is proposed, opening the way for numerical analysis of frames, Timoshenko beams, and more complex structures.

2. Fractional constitutive law

Let $E(t)$ and $D(t)$ the relaxation and the creep function, respectively. $E(t)$ can be interpreted as the stress history for a unit strain $\varepsilon(t) = U(t)$, and $D(t)$ represents the strain history for a unit stress $\sigma(t) = U(t)$, where $U(t)$ is the unit step function.

At the beginning of the last century, Nutting (1921) observed that $E(t)$ is well suited by a power law decay

$$E(t) = \frac{E_\beta}{\Gamma(1-\beta)} t^{-\beta} 0 < \beta < 1 \tag{1}$$

where $\Gamma(\cdot)$ is the Euler–Gamma function, $E_\beta/\Gamma(1-\beta)$ and β are characteristic coefficients depending on the material at hand. Once $E(t)$ is determined in the form expressed in Eq. (1) the function $D(t)$ is given as

$$D(t) = \frac{1}{E_\beta \Gamma(1+\beta)} t^{\beta} \quad 0 < \beta < 1 \tag{2}$$

The result of Eq. (2) is obtained simply taking into account that $E(s)D(s) = s^{-2}$ where $E(s)$ and $D(s)$ are the Laplace transform of $E(t)$ and $D(t)$, respectively, and s denotes the Laplace parameter.

Due to Boltzman superposition principle (compare e.g. Flügge (1967), Pipkin (1972)), the stress history, for an assigned strain history $\varepsilon(t)$ may be easily derived in the form

$$\sigma(t) = \int_0^t E(t-\bar{t})\dot{\varepsilon}(\bar{t})d\bar{t} \tag{3}$$

Conversely the strain history, for an assigned stress history σ is given as

$$\varepsilon(t) = \int_0^t D(t-\bar{t})\dot{\sigma}(\bar{t})d\bar{t} \tag{4}$$

Eqs. (3) and (4) are valid if the system starts at rest at $t = 0$, otherwise $E(t)\varepsilon(0)$ and $D(t)\sigma(0)$ have to be added in Eq. (3) and in Eq. (4), respectively.

As soon as we assume that the kernel in the convolution integrals (3) and (4) are given as in Eq. (1), respectively, the fractional constitutive law of the visco-elastic material results in the form

$$\sigma(t) = E_\beta \left({}_C D_{0^+}^{\beta} \varepsilon\right)(t) \tag{5}$$

and

$$\varepsilon(t) = \frac{1}{E_\beta}\left(D_{0^+}^{-\beta}\sigma\right)(t) \tag{6}$$

where the symbol $({}_C D_{0^+}^{\beta} \varepsilon)(t)$ is the Caputo's fractional derivative defined as

$$\left({}_C D_{0^+}^{\beta} \varepsilon\right)(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{\varepsilon}(\bar{t})}{(t-\bar{t})^{\beta}} d\bar{t} \tag{7}$$

While $(D_{0^+}^{-\beta}\sigma)(t)$ is the Riemann–Liouville fractional integral defined as

$$\left(D_{0^+}^{-\beta}\sigma\right)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\sigma(\bar{t})}{(t-\bar{t})^{1-\beta}} d\bar{t} \tag{8}$$

Consider that the constitutive laws in Eqs. (7) and (8) interpolate the purely elastic behavior ($\beta = 0$)and the purely viscous behavior ($\beta = 1$), and is represented in literature by springpot element depicted in Fig. 1.

It is worth stressing that the Caputo's fractional derivative coincides with the Riemann–Liouville fractional derivative only for quiescent systems or for systems that operate from $t = -\infty$. In all other cases, results in terms of the Riemmann–Liouville or Caputo's fractional derivative are quite different to each another, and fractional differential equations involving Riemann–Liouville fractional derivative show some inconsistencies in terms of initial conditions, Samko et al. (1993), Podlubny (1999), Hilfer (2000) and Evangelatos and Spanos (2011). Contrary, such a problem disappears when working in terms of Caputo's fractional derivative. In the remainder of the paper fractional operators are performed only with respect to time, thus no distinction between partial fractional operators in time and space has to be considered.

3. Governing equation of fractional visco-elastic Euler–Bernoulli beam

Let us consider an isotropic homogeneous visco-elastic Euler–Bernoulli beam of length L, Fig. 2, and the local constitutive equations are expressed according to Eqs. (5) and (6). The beam is referred to the axes (x, y, z) with origin located at the centroid of the cross section, and (x, y) are principal axes of inertia of the cross section. All external spatially distributed loads, denoted as $q_y(z, t)$, are assumed to act in y-direction, thus orthogonally to the z-axis, and the analyzed transverse displacement, $v(z, t)$, is also oriented in y-direction.

Let $M_x(z, t)$ be the bending moment and $T_y(z, t)$ the shear at abscissa z and at time t.

The conservation of momentum and of moment of momentum applied to a beam element of length dz are written as

$$\frac{\partial T_y(z,t)}{\partial z} = \rho(z)\frac{\partial^2 v(z,t)}{\partial t^2} - q_y(z,t) \tag{9}$$

$$\frac{\partial M_x(z,t)}{\partial z} = T_y(z,t) \tag{10}$$

where $\rho(z)$ is the mass per unit length.

The axial strain, $\varepsilon(y, z; t)$, is related to the stress $\sigma(y, z; t)$ according to Eqs. (5) and (6) that, particularized for the underlying continuous beam problem, are rewritten as

$$\varepsilon(y,z;t) = \frac{1}{E_\beta}\left(D_{0^+}^{-\beta}\sigma\right)(y,z;t) \; 0 < \beta < 1 \tag{11}$$

$$\sigma(y,z;t) = E_\beta\left({}_C D_{0^+}^{\beta}\varepsilon\right)(y,z;t) \; 0 < \beta < 1 \tag{12}$$

In virtue of the Euler–Bernoulli hypothesis, the kinematic relation reads

$$\varepsilon(y,z;t) = -y\frac{\partial^2 v(z,t)}{\partial z^2} \tag{13}$$

Since the constitutive law expressed in Eq. (12) contains a linear operator and $\varepsilon(y, z; t)$ is also linear with respect to the coordinate y, the normal stress $\sigma(y, z; t)$ is as well linearly distributed with respect to y. Proper combination of the equilibrium condition in axial direction with the definition of the stress resultant $M_x(z, t)$ leads to

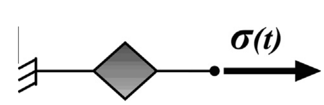

Fig. 1. Springpot element: fractional model.

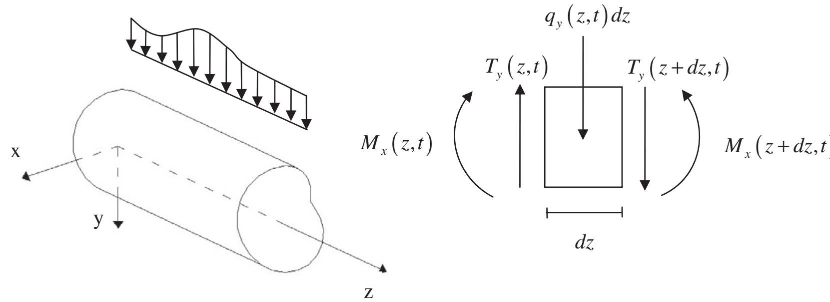

Fig. 2. Euler–Bernoulli beam; (a) layout of the beam; (b) free body diagram of a beam element.

$$\sigma(y,z;t) = \frac{M_x(z,t)}{I_x(z)} y \tag{14}$$

where $I_x(z)$ is the moment of inertia of the cross section with respect to the x-axis.

By inserting Eqs. (13) and (14) into Eq. (11) we obtain

$$-E_\beta I_x(z) \frac{\partial^2 v(z,t)}{\partial z^2} = \left(D_{0^+}^{-\beta} M_x\right)(z,t) \tag{15}$$

Spatial differentiation of Eq. (15) and taking into account Eq. (10) results in

$$-E_\beta \frac{\partial}{\partial z}\left[I_x(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] = (D_{0^+}^{-\beta} T_y)(z,t) \tag{16}$$

The inverse relationships between $M_x(z, t)$, $T_y(z, t)$ and $v(z, t)$ are given as

$$\begin{aligned} M_x(z,t) &= -E_\beta I_x(z) \frac{\partial^2}{\partial z^2}\left[\left({}_C D_{0^+}^{\beta}\, v\right)(z,t)\right] \\ &= -E_\beta I_x(z) {}_C D_{0^+}^{\beta}\left(\frac{\partial^2}{\partial z^2}[v(z,t)]\right) \end{aligned} \tag{17}$$

$$T_y(z,t) = -E_\beta \frac{\partial}{\partial z}\left[I_x(z)\frac{\partial^2}{\partial z^2}\left[\left({}_C D_{0^+}^{\beta}\, v\right)(z,t)\right]\right] \tag{18}$$

Finally by using Eqs. (9) and (16) we obtain

$$\rho(z)(D_{0^+}^{-\beta}\ddot{v})(z,t) + E_\beta \frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] = (D_{0^+}^{-\beta} q_y)(z,t) \tag{19}$$

where $\ddot{v}(z,t) = \partial^2 v(z,t)/\partial t^2$ is the acceleration. The inverse form of Eq. (19) reads

$$\rho(z)\ddot{v}(z,t) + E_\beta \frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2}{\partial z^2}\left[\left({}_C D_{0^+}^{\beta}\, v\right)(z,t)\right]\right] = q_y(z,t) \tag{20}$$

Equation (19) and (20) are the fractional differential equations for the visco-elastic Euler–Bernoulli beam. It has to be emphasized that mechanical boundary conditions will be obtained from Eqs. (17) and (18) particularized for $z = 0$ and $z = L$.

4. Quasi static case and the correspondence principles

Let us suppose that $q_y(z, t)$ varies in such a slow way in time that the inertial forces $\rho(z)\ddot{v}(z,t)$ may be neglected. In this case the equation of motion is simplified into

$$E_\beta \frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] = (D_{0^+}^{-\beta} q_y)(z,t) \tag{21}$$

Further, suppose that $q_y(z,t) = \bar{q}_y(z)\psi(t)$, then the forcing term in Eq. (21) simplifies into

$$\left(D_{0^+}^{-\beta} q_y\right)(z,t) = \bar{q}_y(z)\left(D_{0^+}^{-\beta}\psi\right)(t) \tag{22}$$

Since the system is linear, from this equation it may be recognized that, the visco-elastic Euler–Bernoulli beam in the quasi static case behaves like a classical beam in which the external load simply varies in time according to the Riemann–Liouville fractional integral of the load amplifier $\psi(t)$. It follows that $v(z,t) = \bar{v}(z)(D_{0^+}^{-\beta}\psi)(t)$ and then Eq. (21) may be rewritten in the form

$$E_\beta \frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2 \bar{v}(z)}{\partial z^2}\right] = \bar{q}_y(z) \tag{23}$$

Then the beam response functions may be easily derived since

$$v(z,t) = \bar{v}(z)\left(D_{0^+}^{-\beta}\psi\right)(t); \quad \frac{\partial v(z,t)}{\partial z} = \frac{d}{dz}\bar{v}(z)(D_{0^+}^{-\beta}\psi)(t) \tag{24a}$$

$$M_x(z,t) = -E_\beta I_x(z)\frac{d^2}{dz^2}\bar{v}(z) = M_x(z) \tag{24b}$$

$$T_y(z,t) = -E_\beta \frac{d}{dz}\left(I_x(z)\frac{d^2}{dz^2}\bar{v}(z)\right) = T_y(z) \tag{24c}$$

having taken into account Eqs. (17) and (18).

From Eq. (24) it may be recognized that $M_x(z, t)$ and $T_y(z, t)$ do not depend on t and then the boundary conditions read

$$v(0,t) = \bar{v}(0)(D_{0^+}^{-\beta}\psi)(t); \quad \frac{\partial v(0,t)}{\partial z} = \frac{d}{dz}\bar{v}(0)(D_{0^+}^{-\beta}\psi)(t) \tag{25a}$$

$$M_x(0) = -E_\beta I_x(0)\left(\frac{d^2}{dz^2}\bar{v}(z)\right)_{z=0} \tag{25b}$$

$$T_y(0) = -E_\beta \frac{d}{dz}\left(I_x(z)\frac{d^2}{dz^2}\bar{v}(z)\right)_{z=0} \tag{25c}$$

Analogous expressions are readily found in $z = L$.

Moreover, from Eq. (24) it is apparent that for a homogeneous beam the distribution of moments and shear may be computed considering the beam as purely elastic (with an elastic modulus $E = E_\beta$) and the corresponding displacements may be evaluated amplifying by $(D_{0^+}^{-\beta}\psi)(t)$ the function obtained integrating Eq. (23) like in the elastic case. Moreover, for more complex load history of the type

$$q_y(z,t) = \sum_{j=1}^{n} \bar{q}_{y_j}(z)\psi_j(t) \tag{26}$$

all previous considerations hold true and the linearity of the system allows us to affirm that the total response is simply the summation of the response at each single load $\bar{q}_{y_j}(z)\psi_j(t)$.

It is worth stressing that the above considerations lead to the confirmation of the first correspondence principle even for beam having a fractional constitutive law.

For the case of classical visco-elastic Euler–Bernoulli beam two correspondence principles have been stated (Flügge, 1967):

(I) If a visco-elastic beam is subjected to loads which are applied simultaneously at initial time and then held constant, the stresses are the same as those in the purely elastic case under the same load, while strains and displacements depend on time and are derived from the purely elastic case by simply replacing the elastic modulus with the inverse of the creep function.

(II) If a visco-elastic beam is subjected, in selected points, to imposed displacements, which are applied simultaneously at initial time and then held constant, the displacements of all points and all the strains are the same as in the corresponding elastic beam, while stresses may be derived from the purely elastic case by simply replacing the elastic modulus with the relaxation function.

It will be shown that the two aforementioned principles also hold considering the fractional constitutive laws. The first correspondence principle may be easily demonstrated since setting the loads constant it means that $\psi(t) = U(t)$ and consequentially the amplifier function $(D_{0^+}^{-\beta}\psi)(t)$ will result

$$\left(D_{0^+}^{-\beta}\psi\right)(t) = \left(D_{0^+}^{-\beta}U\right)(t) = \frac{t^\beta}{\Gamma(1+\beta)} = E_\beta D(t) \tag{27}$$

Inserting Eq. (27) into Eq. (24), reminding that $D(t)$ is the creep function one gets that stresses are the same as those in the purely elastic case, while strains and displacements depend on time and may be calculated from the purely elastic case by simply replacing the elastic modulus with the inverse of the creep function.

As regards the second correspondence principle remains valid too, in fact for a fractional visco-elastic beam under imposed displacements, ($v(z,t) = \bar{v}(z)U(t)$) since $q_y(z,t)$=0 and then Eq. (21) reverts into a homogeneous equation being the Riemmann–Liouville fractional integral equal to zero. Then displacement response function is $v(z,t) = \bar{v}(z)U(t)$ that means displacements and strains are the same of the elastic case, while inserting $v(z,t) = \bar{v}(z)U(t)$ into Eqs. (17) and (18) we get

$$M_x(z,t) = -E_\beta I_x(z)\frac{\partial^2}{\partial z^2}[\bar{v}(z)]({}_C D_{0^+}^{\beta}U)(t) = M_x(z)E(t) \tag{28a}$$

$$T_y(z,t) = -E_\beta \frac{\partial}{\partial z}\left[I_x(z)\frac{\partial^2}{\partial z^2}[\bar{v}(z)]\right]({}_C D_{0^+}^{\beta}U)(t) = T_y(z)E(t) \tag{28b}$$

being $({}_C D_{0^+}^{\beta}U)(t) = t^{-\beta}/\Gamma(1-\beta)$. The latter results mean that stresses may be derived from the purely elastic case by simply replacing the elastic modulus with the relaxation function $E(t)$. This is just the second version of correspondence principle extended to fractional visco-elastic Euler–Bernoulli beams.

5. Fractional Kelvin–Voigt constitutive law

Another case of relevant interest is related to the fact that a pure fractional constitutive law described by Eq. (5) leads to the undesired result that under a hydrostatic pressure at $t \to \infty$ the body will be concentrated in a point and this is in contrast with the real behavior of any materials. On the other hand if we assume that $\sigma_{ii} = k\varepsilon_{ii}$, that is the hydrostatic stress σ_{ii} is related to the volumetric component ε_{ii}, through a Bulk modulus k, the relation between the longitudinal stress and the corresponding longitudinal strain ε will be enriched of an elastic component $E_\infty\varepsilon(t)$, that is

$$\sigma(t) = E_\infty\varepsilon(t) + \int_0^t E(t-\bar{t})\dot{\varepsilon}(\bar{t})d\bar{t} \tag{29}$$

and then Eq. (5) reverts into

$$\sigma(t) = E_\infty\varepsilon(t) + E_\beta\left({}_C D_{0^+}^{\beta}\varepsilon\right)(t) \tag{30}$$

where E_∞ is the elastic modulus measured at $t \to \infty$ during the relaxation test. The inverse relation expressed in Eq. (29) leads to

$$\varepsilon(t) = \frac{1}{E_\beta}\int_0^t (t-\bar{t})^{\beta-1}\mathcal{E}_\beta\left(-\frac{E_\infty}{E_\beta}(t-\bar{t})\right)\dot{\sigma}(\bar{t})d\bar{t} \tag{31}$$

where $\mathcal{E}_\beta(\cdot)$ is the Mittag–Leffler function defined as (Podlubny (1999))

$$\mathcal{E}_\beta(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)} \tag{32}$$

By inserting Eq. (31) into Eq. (30) we get

$$\begin{aligned}\varepsilon(t) &= \frac{1}{E_\beta}\sum_{k=0}^{\infty}\frac{(-E_\infty/E_\beta)^k}{\Gamma(\beta k+1)}\int_0^t (t-\bar{t})^{\beta-1+k}\dot{\sigma}(\bar{t})d\bar{t} \\ &= \frac{1}{E_\beta}\sum_{k=0}^{\infty}\frac{(-E_\infty/E_\beta)^k}{\Gamma(\beta k+1)}\Gamma(\beta+k)\left(D_{0^+}^{-(\beta+k-1)}\sigma\right)(t)\end{aligned} \tag{33}$$

This is the case of fractional Kelvin–Voigt model depicted in Fig. 3.

Moreover, since Eqs. (13) and (14) remain valid, inserting them into Eq. (32) the quasi static case returns

$$E_\beta\frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] = \sum_{k=0}^{\infty}\frac{(-E_\infty/E_\beta)^k}{\Gamma(\beta k+1)}\Gamma(\beta+k)(D_{0^+}^{-(\beta+k-1)}q_y) \times (z,t) \tag{34}$$

Being the system linear, Eq. (33) may be easily derived as previously seen by supposing that $q_y(z,t) = \bar{q}_y(z)\psi(t)$.

As in fact in this case Eq. (33) may be written as

$$\begin{aligned}E_\beta\frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2 v(z,t)}{\partial z^2}\right] &= \bar{q}_y(z)\sum_{k=0}^{\infty}\frac{(-E_\infty/E_\beta)^k}{\Gamma(\beta k+1)}\Gamma(\beta+k)(D_{0^+}^{-(\beta+k-1)}\psi)(t) \\ &= \bar{q}_y(z)Q(t)\end{aligned} \tag{35}$$

Then, in virtue of the linearity of the system, we may solve Eq. (34) by assuming that $v(z,t) = \bar{v}(z)Q(t)$ and performing the static problem

$$E_\beta\frac{\partial^2}{\partial z^2}\left[I_x(z)\frac{\partial^2\bar{v}(z)}{\partial z^2}\right] = \bar{q}_y(z) \tag{36}$$

with the relevant boundary conditions (kinematics and mechanics). Once Eq. (36) is integrated the response $v(z,t)$ will be obtained by amplifying the static response $\bar{v}(z)$ of $Q(t)$. Similar arguments may be considered for more sophisticated stress–strain fractional constitutive law in which the stress is related to the strain involving a summation of two fractional derivatives of different order. In this case inverse relationships involve two parameters Mittag–Leffler functions (Mainardi and Spada, 2011).

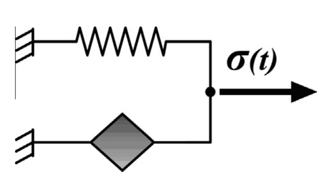

Fig. 3. Fractional Kelvin–Voigt model.

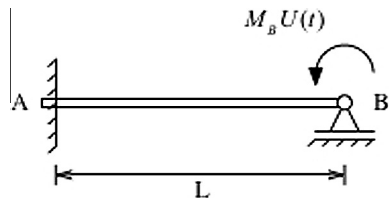

Fig. 4. Clamped-simply supported beam under a bending moment at the hinge.

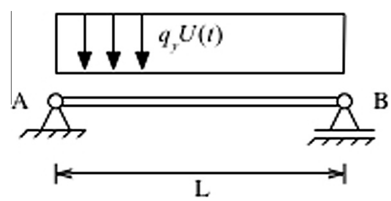

Fig. 5. Simply supported beam.

6. Example

In order to elucidate the previous concepts a simple example is presented. Let be a clamped–simply supported beam depicted in Fig. 4 loaded by an external moment $M(t) = M_B U(t)$.

For simplicity's sake we suppose that $I_x(z) = I_x = const.$ (uniform beam).

The elastic solution for such a system and a purely fractional visco-elastic beam is

$$\bar{v}(z) = \frac{M_B}{4LE_\beta I_x} z^2(L - z) \tag{37}$$

On the other hand since $v(z,t) = \bar{v}(z)(D_{0^+}^{-\beta} U)(t)$ then

$$v(z,t) = \frac{M_B z^2 (L - z) t^\beta}{4LE_\beta I_x \Gamma(1+\beta)} = \bar{v}(z) E_\beta D(t) \tag{38}$$

While, from Eq. (24) we get that bending moment and shear coalesce with the elastic ones being

$$M_x(z,t) = -\frac{M_B}{2}\left(1 - \frac{3z}{L}\right) U(t); \quad T_y(z,t) = \frac{3}{2L} M_B U(t) \tag{39}$$

The above results confirm the first correspondence principle.

7. Virtual work principle

In this section the application to virtual work principle for an Euler–Bernoulli flexural beam with fractional local constitutive law is presented.

The virtual work principle in the dual form for a three dimensional body is written as

$$\int_V \delta\boldsymbol{\sigma}^T \boldsymbol{\varepsilon}(t) dV = \int_S \delta\mathbf{p}_n^T \mathbf{u}(t) dS + \int_V \delta\mathbf{b}^T \mathbf{u}(t) dV \tag{40}$$

where $\delta\boldsymbol{\sigma}$ are virtual stress corresponding to equilibrated tractions $\delta\mathbf{p}_n$ on the surface and body forces $\delta\mathbf{b}$. While $\boldsymbol{\varepsilon}(t)$ and $\mathbf{u}(t)$ are kinematically compatible strain and corresponding displacements. Eq. (39) has to be verified at each time instant t.

The virtual work principle exploited for an Euler–Bernoulli beam in which shear and axial deformation are neglected, taking into account Eq. (11) is written as

$$\int_L \delta M_x(z,t) \frac{1}{E_\beta I_x(z)} D_{0^+}^{-\beta} M_x(z,t) dz = \int_L \delta q_y(z) v(z,t) dz \tag{41}$$

The virtual work principle may be used to evaluate displacement of a statically determined beams: since in virtue of the correspondence principle the moment distribution along the beam is already known. Such an example for the simply supported beam in Fig. 5 the displacement at the midspan of the beam may be obtained simply by loading the beam with an unitary transverse load in $L/2$, and in this case $\delta M_x(z) = \frac{1}{2} z, (0 \leqslant z < L/2)$ and for $I_x(z) = I_x$, $M_x(z,t)$ is the same as in the elastic case $M_x(z,t) = (q_y \frac{L}{2} z - \frac{q_y}{2} z^2) U(t)$ and then

$$\frac{2}{E_\beta I_x \Gamma(1+\beta)} \int_0^{L/2} \frac{1}{2} z (q_y \frac{L}{2} z - \frac{q_y}{2} z^2) t^\beta dz = v\left(\frac{L}{2}, t\right) \tag{42}$$

leading to the displacement at the midspan.

Once this result is archived also redundant beam with fractional visco-elastic constitutive law may be easily derived by using superposition principle and force method. Such an example for the clamped–simply supported beam in Fig. 4, an equivalent system is that composed of a cantilever loaded by $M_B U(t)$ and the redundant unknown reaction $X(t)$. $X(t)$ has to be selected in such a way that $v(L,t) = 0\ \forall t$.

The compatibility condition in L is given

$$v(L,t) = v^{(0)}(L,t) + v^{(1)}(L,t) X(t) = 0 \tag{43}$$

where $v^{(0)}(L,t)$ is the displacement of the principal system (Fig. 6a) and $v^{(1)}(L,t)$ is the displacement of the auxiliary system (Fig. 6b) $X(t) = 1$. By using Eq. (41) it follows

$$\begin{aligned} v^{(0)}(L,t) &= \frac{M_B t^\beta}{E_\beta I_x \Gamma(1+\beta)} \int_0^L (z - L) dz \\ &= -\frac{M_B t^\beta}{E_\beta I_x \Gamma(1+\beta)} \frac{L^2}{2}; \quad \forall t \geqslant 0 \end{aligned} \tag{44}$$

$$\begin{aligned} v^{(1)}(L,t) &= \frac{t^\beta}{E_\beta I_x \Gamma(1+\beta)} \int_0^L (z - L)^2 dz \\ &= \frac{t^\beta}{E_\beta I_x \Gamma(1+\beta)} \frac{L^3}{3}; \quad \forall t \geqslant 0 \end{aligned} \tag{45}$$

leading to

$$X(t) = -\frac{v^{(0)}(L,t)}{v^{(1)}(L,t)} = \frac{3}{2L} M_B U(t) \tag{46}$$

that is the correct result.

With these results in mind extension to arches, frames and complex structures may be derived in a very simple way.

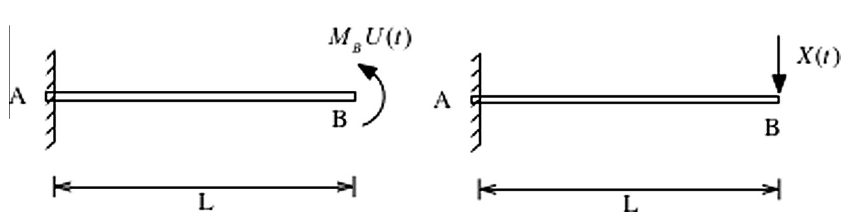

Fig. 6. (a) Principal system; (b) auxiliary system.

8. Conclusions

Euler–Bernoulli beam with fractional constitutive law has been treated. It has been shown that for the case in which the external load is splitted into products of spatial and temporal functions $q_y(z,t) = \sum_{j=1}^{n} \bar{q}_{y_j}(z)\psi_j(t)$ the quasi static case solution in terms of displacements may be readily found as summation of displacements history evaluated for the static load $\bar{q}_{y_j}(z)$ performed in the purely elastic case. Each displacement $\bar{v}_j(z)$ corresponding to elastic case, has to be amplified by the Riemann–Liouville fractional integral of the load history $\psi_j(t)$.

Moreover it has been demonstrated that both correspondence principles also hold for the Euler–Bernoulli beam with fractional constitutive law. In virtue of these principles bending moment and shear distribution, displacements may properly be derived from the elastic case. Extension to fractional Kelvin–Voigt constitutive law as well as virtual work principle has been formulated showing the simplicity of solving redundant beams or for displacement evaluation of Euler–Bernoulli beams under quasi static loads.

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