10. Applications. Let us first summarize the results obtained so far, given that one of the following two conditions holds true; here, $F$ and $G$ denote the cumulative distribution functions defined by (8.1).

(i) $F(y)$ is of the form (9.7) when $y > 0$.

(ii) $F$ and $G$ correspond to integer valued random variables such that, for large positive integers $j$, $p_j = F(j + 0) - F(j - 0)$ is of the form (9.16); (for instance, $p_j = 0$ for $j$ large). Moreover, $x > 0$ is an integer.

Our main aim was to find explicit formulae for the generating functions $C_k$ defined by (3.4). From

$$C_{2k} = e^{T^-} I_{2k}, \quad C_{2k+1} = e^{T^+} I_{2k+1}^-,$$

(cf. (6.5)), it suffices to determine $e^{T^-}$, $e^{T^+}$ and $I_{2k}^-$, $I_{2k+1}^+$. The latter can be obtained from the recursion scheme (8.4)–(8.9), provided that (8.3) holds. This is true under either of the conditions (i) and (ii), (where $R_0 = R$ and $R_0 = I$, respectively).

In fact, $\lambda^+_s$ is then given by (9.8) or (9.18), (9.19). Further, $a_s(\chi^-)$ is given by

$$a_s(\chi^-) = q t b_s(e^{t^+} + L^+ \lambda^+_s),$$

(cf. (9.2)), and (9.9) or (9.19). Moreover, it is an easy matter to find an explicit formula for $e^{t^+}$, (hence, cf. (6.8), for $e^t$), and for $I_0^-$, cf. (9.10), (9.12), (9.20) and (9.21). Also in view of

$$T^- \lambda^+_s = t[\Phi e^{t^+} + L^+ \lambda^+_s]^-, $$

(cf. (6.3)), there only remains the problem of finding a useful explicit formula for $e^{t^+}$, (or for $e^{L^+}$, cf. (6.9)).

There is no real difficulty if $q = 1$ and $\varphi = \Phi$, for, then $L^+ = l^+$. Explicit formulae for $e^{t^+}$ are also easily obtained when $G(y)$ has either for $y < 0$ or for $y > 0$ a simple behavior of the same type as $F(y)$ for $y > 0$.

In working out the details, we shall restrict ourselves to the important special case where

$$q = 1 \text{ and } \varphi(s) \equiv \Phi(s).$$
Then the increments $X_n = z_n - z_{n-1}$ are independent,
\[ E(e^{sX_n}) = \varphi(s), \quad \Pr(X_n < y) = F(y), \]
while
\[ L^- = l^-, \quad L^+ = l^+. \]
Moreover, the definition (3.4) of $C_k$ reduces to
\[ C_k = C_k(s) = \sum_{n=0}^{\infty} t^n E(\{N_n = k\} e^{s_n} | z_0 = - x). \]
For convenience, we shall also assume that (8.3) holds for $r = 1$, in other words, that
\[ T_+ \chi^- = a\{\chi^-\} \lambda^+ \quad \text{if} \quad \chi^- \in B_0^-. \]
Here, $\lambda^+ = \lambda^+ (s)$ denotes a fixed element in $B_0^+$, while $a\{\chi^-\}$ is independent of $s$. From (8.5)–(8.9), this implies
\[ \Gamma_{2k+1}^+ = P Q^k \lambda^+, \quad \Gamma_{2k+2}^+ = P Q^k T_- \lambda^+, \]
\[ (k = 0, 1, 2, \ldots), \quad \text{where} \]
\[ P = a\{\Gamma_0^-\}, \quad Q = a\{T_+ \lambda^+\} \]
and
\[ T_- \lambda^+ = t[\varphi \epsilon^{it^+ \lambda^+}]^- \]
By the way, letting $\sigma = [Q]_{k=1}$, it follows from (10.6) and Theorem 7.1 that
\[ \Pr(N_\infty = 2k + 1 | z_0 = - x) = (1 - \sigma) \sigma^k, \quad (k \geq 0), \]
if $A < \infty$, (that is, $0 < E(X) < \infty$), while
\[ \Pr(N_\infty = 2k + 2 | z_0 = - x, N_\infty > 0) = (1 - \sigma) \sigma^k, \quad (k \geq 0), \]
if $B < \infty$, (that is, $-\infty < E(X) < 0$). Note that both right hand sides are independent of $x$.
From the results of section 9, the basic assumption (10.5) holds in each of the following three cases and essentially only in these cases, (except for the uninteresting possibility that all the jumps $X_n$ are $< 0$).

I. There exist constants $0 < d < \alpha$ such that
\[ F'(y) = d \epsilon^{-\alpha y} \quad \text{if} \quad y > 0. \]

II. $x \geq 0$ is an integer, while $F = G$ corresponds to an integer valued random variable. Moreover, there exist positive constants $d$ and $\alpha, d + 1 < e^\alpha$, such that
\[ p_j = \Pr(X_n = j) = d \epsilon^{-\alpha j} \quad \text{if} \quad j = 1, 2, \ldots. \]

III. $x \geq 0$ is an integer, while $F = G$ corresponds to an integer valued random variable. Moreover, $p_j = \Pr(X_n = j)$ satisfies
\[ p_1 > 0, \quad p_j = 0 \quad \text{if} \quad j \geq 2. \]
The remaining sections are concerned with the determination of $C_k$ and $\Pr(N_n = k | z_0 = -x)$ for each of these cases.

11. The exponential case. Here, we assume $\eta = 1$, $F = G$ and, further, that, for $y > 0$, $F(y)$ has a derivative of the form

$$F'(y) = d \, e^{-ay}, \quad (y > 0),$$

where $0 < d < \alpha$. From (9.10),

$$e^{-L^+} = e^{-t^+} = (\xi - s)(\alpha - s)^{-1},$$

where $\xi = \xi(t)$ denotes the unique number for which

$$\varphi(\xi) = t^{-1}, \quad \text{and} \quad \Re(\xi) > 0,$$

$(\xi \to \alpha$ as $t \to 0)$. From (6.8),

$$e^{L^-} = e^t = (\xi - s)[(\alpha - s)(1 - t\varphi(s))]^{-1},$$

hence, using (11.1),

$$e^{L^-(s)} = e^{t(s)} = (\alpha - \xi)/(td).$$

It follows from (9.8), (9.9) and (9.2) that (10.5) holds with

$$\lambda^+(s) = (\alpha - s)^{-1},$$

$$\lambda^-(s) = (td)^{-1} (\alpha - \xi)^2 \chi^- (s).$$

Further, from (9.12),

$$\Gamma_0^-(s) = e^{-sx}(\alpha - s)(\xi - s)^{-1} - (\alpha - \xi)(\xi - s)^{-1} e^{-\xi x},$$

hence, from (10.1) and (11.4),

$$C_0 = (1 - t\varphi(s))^{-1} (e^{-sx} - e^{-\xi x}(\alpha - \xi)(\alpha - s)^{-1}),$$

while, from (10.7) and (11.5),

$$P = (td)^{-1} (\alpha - \xi)^2 e^{-\xi x}.$$

Using (10.8), (11.2) and (11.5), it is not difficult to show that

$$T \lambda^+ = t(\xi - s)^{-1} [(\alpha - \xi) \bar{\varphi}'(\xi) - (\alpha - s)(\xi - s)^{-1}(\bar{\varphi}(\xi) - \bar{\varphi}(s))].$$

Here,

$$\bar{\varphi}(s) = \int_{-\infty}^{0} e^{sy} dF(y) = \varphi(s) - d(\alpha - s)^{-1}.$$

Consequently, from (10.7) and (11.5)

$$\bar{Q} = - d^{-1} (\alpha - \xi)^2 \bar{\varphi}'(\xi).$$

Let us first determine the distribution of $N_\infty$. If $t$ tends to 1 then $\xi$ tends to $\xi_0$ (say), where $\xi_0 < \alpha$ is equal to the largest real root of $\varphi(s) = 1$. 


In fact, \( \xi_0 = 0 \) if \( 0 < E(X_n) < \infty \) and \( 0 < \xi_0 < \alpha \) if \( -\infty < E(X_n) < 0 \). Hence, by (11.10), if \( E(X_n) > 0 \) then (10.9) holds with
\[
\sigma = -\varphi'(0) \frac{\alpha^2}{d} = 1 - E(X_n) \frac{\alpha^2}{d}.
\]
Similarly, if \( -\infty < E(X_n) < 0 \) then (10.10) holds with
\[
\sigma = \varphi'(\xi_0) (\alpha - \xi_0)^2 / d.
\]
Finally, from (11.6) and (7.1),
\[
Pr(N_n \geq 0 | z_0 = -x) = \begin{cases} 
(1 - \xi_0 / \alpha) e^{-\xi x} & \text{if } E(X_n) < 0, \\
1 & \text{if } E(X_n) > 0.
\end{cases}
\]
The remaining cases are determined by Theorem 7.1.

Next, let us determine the distribution of \( N_n \). Substituting the above formulae in (10.6), we have from (10.1) that, for \( k = 0, 1, 2, \ldots \),
\[
t_{C_{2k+1}} = ((\alpha - \xi)^2 / d)^{k+1} \left( -\varphi'(\xi) \right)^k e^{-\xi x} (\xi - s)^{-1}
\]
and
\[
t_{C_{2k+2}} = ((\alpha - \xi)^2 / d)^{k+1} \left( -\varphi'(\xi) \right)^k e^{-\xi x} \frac{(\xi - s)}{(\alpha - s)} \frac{T_{-\lambda^+}}{(1 - t\varphi(s))},
\]
where \( T_{-\lambda^+} \) is given by (11.8). Thus, by \( \varphi(0) = 1 \) and \( t = \varphi'(\xi)^{-1} \), the generating function
\[
t_{C_k(0)} = \sum_{n=0}^{\infty} Pr(N_n = k | z_0 = -x) t^{n+1}
\]
is an explicitly known function of \( \xi \); (also if \( k = 0 \), by (11.6)).

Transforming the usual contour integral formula for the coefficients in (11.13) to the new integration variable \( \xi \), it follows from (11.3) that
\[
Pr(N_n = k | z_0 = -x)
\]
is equal to the coefficient of \((\alpha - \xi)^{n-k}\)
in the expansion of the function
\[
(\alpha - \xi)^{-k-1} (tC_k(0)) \varphi_1(\xi)^n \varphi_2(\xi)
\]
in powers of \( \alpha - \xi \), \(|\alpha - \xi| \) small. Here,
\[
\varphi_1(\xi) = (\alpha - \xi) \varphi(\xi), \quad \varphi_2(\xi) = (\alpha - \xi)^2 \varphi'(\xi).
\]
As an illustration, let
\[
F'(y) = \frac{1}{2} e^{-|y|}
\]
for all \( y \).

Then
\[
\varphi(s) = (1 - s^2)^{-1}, \quad t = 1 - \xi^2, \quad \varphi(s) = \frac{1}{2} (1 + s)^{-1},
\]
and
\[
T_{-\lambda^+} = (1 - \xi) (1 + \xi)^{-1} (1 + s)^{-1}.
\]
Hence, from (11.11) and (11.12), (11.14) reduces to
\[ 4 \left( 1 + \xi \right)^{-n-k-1} e^{-\xi}, \quad (k \geq 1). \]
Expanding about \( \xi = 1 \), one obtains
\[ \Pr(N_n = k | z_0 = -x) = 2^{-2n+1} \sum_{j=0}^{n-k} \binom{2n-j}{n+k} (2x)^j e^{-x/j!}, \quad (k \geq 1). \]
The case \( x = 0 \) of (11.16) is due to Baxter [2] p. 219. Similarly, letting \( 1 - \xi = z \), we have from (11.6) that \( \Pr(N_n = 0 | z_0 = -x) \) is equal to the coefficient of \( z^n \) in the expansion about \( z = 0 \) of the function
\[ 2^{-n} \left( 1 + z/2 \right)^{-n-1} \left( 1 - e^{-x} ze^{xz} \right) (1-z)^{-1}. \]

12. The geometric case. Here, \( \varrho = 1, x \geq 0 \) is an integer, the \( X_n = z_n - z_{n-1} \) are independent and integral valued, while \( p_j = \Pr(X_n = j) \) satisfies
\[ p_j = d \ e^{-\alpha j} = d \ \Theta^j \quad (j = 1, 2, \ldots); \]
here, \( d \) and \( \alpha \) are positive constants, \( d + 1 < e^\alpha \). Further,
\[ \Theta = e^{-\alpha}. \]
From (9.20),
\[ e^{-L^+ (s)} = e^{-t^+ (s)} = (1 - w/\eta) (1 - \Theta w)^{-1}, \]
where \( w = e^s \) and \( \eta \) denotes the unique number for which
\[ \psi(\eta) = t^{-1}, \text{ and } |\eta| > 1. \]
Here,
\[ \psi(w) = \sum_j p_j w^j = \sum_{j = -\infty}^0 p_j w^j + d \Theta w (1 - \Theta w)^{-1}. \]
Thus, \( \varphi(s) = \varphi(e^s) \) and \( \eta \rightarrow e^\alpha = \Theta^{-1} \) as \( t \rightarrow 1 \). From (6.8),
\[ e^{L^- (s)} = e^{t^- (s)} = (1 - w/\eta) [(1 - \Theta w) (1 - t \varphi(w))]^{-1}, \]
\[ e^{L^- (s)} = e^{t^- (s)} = (td)^{-1} (e^\alpha/\eta - 1). \]
It follows from (9.18) and (9.2) that (10.5) holds with
\[ \lambda^+(s) = (e^{s\alpha} - 1)^{-1} = \Theta w (1 - \Theta w)^{-1}, \]
\[ \alpha \{ \chi^- \} = (td)^{-1} (\Theta \eta)^{-2} (1 - \Theta \eta)^2 \chi^- (x). \]
Further, from (9.21),
\[ \Gamma_0^-(s) = w^{-x} (1 - \Theta w) (1 - w/\eta)^{-1} - (1 - \Theta \eta) w(1 - w/\eta)^{-1} \eta^{-x-1}, \]
hence, from (10.1) and (12.5),
\[ C_0 = (1 - t \varphi(w))^{-1} [w^{-x} \eta^{-x-1} (1 - \Theta \eta) w(1 - \Theta w)^{-1}], \]
while, from (10.7) and (12.6),

\[(12.8)\quad P = (t \theta d^2)^{-1} (1 - \theta \eta)^2 \eta^{-x-2}.\]

Using (10.8), (12.2) and (12.6), it is not difficult to show that

\[(12.9)\quad T' = t \theta \eta^2 \omega(\eta - w)^{-1} [(1 - \theta \eta) \bar{\psi}'(\eta) - (1 - \theta w) (\eta - w)^{-1} (\bar{\psi}(\eta) - \bar{\psi}(w))].\]

Here,

\[(12.10)\quad \bar{\psi}(w) = \sum_{j=-\infty}^{0} \eta^j_\omega \psi(w) - d \omega(w - 1 - \theta w)^{-1}.

It follows from (10.7), (12.6) and (12.9) that

\[(12.11)\quad Q = - (\theta d)^{-1} (1 - \theta \eta)^2 \bar{\psi}'(\eta).\]

Let us first determine the distribution of \(N_\infty\). If \(t\) tends to 1 then \(\eta\) tends to a number \(\eta_0\) (say). Here, \(\eta_0 = 1\) if and only if \(\psi(1) = E(X_\eta) > 0\). Otherwise, \(1 < \eta_0 < e^\xi\) and \(\psi(\eta_0) = 1\). Hence, by (12.11), if \(E(X_\eta) < 0\) then (10.9) holds with

\[\sigma = - (\theta d)^{-1} (1 - \theta \eta)^2 \bar{\psi}'(1) = 1 - (\theta d)^{-1} (1 - \theta)^2 E(X_n) \eta_0^{-x-1} \]

Similarly, if \(-\infty < E(X_\eta) < 0\) then (11.10) holds with

\[\sigma = - \bar{\psi}'(\eta_0) (\theta d)^{-1} (1 - \theta \eta_0)^2 \eta_0^{-x-1} \]

Finally, from (12.7) and (7.1),

\[\Pr(N_\infty > 0 | z_0 = -x) = (1 - \theta)^{-1} (1 - \theta \eta_0) \eta_0^{-x-1} \]

\[= 1 \quad \text{if} \ E(X_n) < 0, \]

\[= 1 \quad \text{if} \ E(X_n) > 0. \]

The remaining cases are determined by Theorem 7.1.

Next, let us determine the distributions of \(N_n\). Substituting the above formulae in (10.6), we have from (10.1) that, for \(k = 0, 1, 2, \ldots\),

\[(12.12)\quad t C_{2k+1} = (1 - \theta \eta)^2 / (\theta d)^k \bar{\psi}'(\eta)^{k-1} \eta^{-x-1} \omega(\eta - w)^{-1} \]

and

\[(12.13)\quad t C_{2k+2} = (1 - \theta \eta)^2 / (\theta d)^{k+1} \bar{\psi}'(\eta)^k \eta^{-x-3} \frac{(\eta - w) T_{\lambda^+}}{(1 - \theta \eta)(1 - \theta \omega)} \]

where \(T_{\lambda^+}\) is given by (12.9). Thus, by \(w = e^\xi\), \(\psi(1) = 1\) and \(t = \psi(\eta)^{-1}\), \(tC_k(0)\) is an explicitly known function of \(\eta\).

Transforming the usual contour integral formula for the coefficients in (11.13) to the new integration variable \(\eta\), it follows from (12.3) that \(\Pr(N_n = k | z_0 = -x)\) is equal to the coefficient of

\[\Theta^{-1} (1 - \Theta \eta)^{n-k} \]

in the expansion of the function

\[(12.14)\quad \Theta^{-1} (1 - \Theta \eta)^{n-k} \psi_1(\eta)^n \psi_2(\eta) \]

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in powers of $1 - \Theta \eta$, ($|1 - \Theta \eta|$ small). Here,

$$\psi_1(\eta) = (1 - \Theta \eta) \psi(\eta), \quad \psi_2(\eta) = (1 - \Theta \eta)^2 \psi'(\eta).$$

As an illustration, let

$$(12.15) \quad p_j = d \Theta^j; \quad \text{for all } j,$$

where $d = (1 - \Theta)/(1 + \Theta)$. Then

$$\psi(w) = (1 - \Theta^2) w(1 - \Theta w)^{-1} (w - \Theta)^{-1},$$

$$\tilde{\psi}(w) = (1 - \Theta) (1 - \Theta)^{-1} w(w - \Theta)^{-1},$$

hence, from (12.9),

$$T^{-\lambda^+} = t(n \Theta)^2 (1 - \Theta)^2 (\eta - \Theta)^{-2} w(w - \Theta)^{-1}.$$

One obtains from (12.12) and (12.13) that

$$(12.16) \quad C_k(0) = \Theta^{-1}(1 - \Theta^2) [(1 - \Theta \eta) (\eta - \Theta)^{-1}]^k (\eta - 1)^{-1} \eta^{-x}$$

for each $k \geq 1$. Further, (12.14) reduces to

$$\Theta^{-n-k-2}(1 + \Theta) (\eta - \Theta)^{-n-k-1} \eta^{n-x-1}(\eta + 1)$$

if $k \geq 1$. Expanding in powers of $1 - \Theta \eta = z$ (say) one obtains that, for $k \geq 1$,

$$(12.17) \quad \Pr(N_n = k | z_0 = -x) = \Theta^{x+k} (1 + \Theta)^{-2n+1} [d_{n,k}(n-k) - (1 - \Theta) d_{n,k}(n-k-1)],$$

where the $d_{n,k}(r)$ are defined by

$$(1-u)^{-n-k-1} (1-(1-\Theta^2)u)^{n-x-1} = \sum_{r=0}^{\infty} d_{n,k}(r) u^r, \quad (|u| < 1).$$

Without going into further details, we mention that in the special case

$$(12.18) \quad p_j = 2^{-j-2} \quad \text{if } j \geq -1,$$

($p_j = 0$ for $j < -2$), a similar computation easily yields

$$(12.19) \quad \Pr(N_n = k | z_0 = -x) = 2^{-2n-x} \binom{x+2n+1}{n-k},$$

whenever $k \geq 1$.

13. The Bernoulli case. Here, $\Theta = 1$, $x \geq 0$ is an integer, the $X_n = z_n - z_{n-1}$ are independent and integral valued, while $p_j = \Pr(X_n = j)$ satisfies

$$(13.1) \quad p_1 > 0, \quad p_j = 0 \quad \text{if } j > 2.$$  

Thus, from (9.20),

$$(13.2) \quad e^{-L^+(\Theta)} = e^{-L(\Theta)} = 1 - w/\eta,$$
where \( w = e^s \) while \( \eta \) is uniquely determined by

\[
\psi(\eta) = t^{-1}, \quad |\eta| > 1,
\]

\((\eta \to \infty \text{ as } t \to 0)\); here,

\[
\psi(w) = \sum_{j=-\infty}^{+1} p_j w^j,
\]

thus, \( \varphi(s) = \psi(e^s) \). Hence, from (6.8),

\[
e^{L^\tau(s)} = e^{L^\tau(e^s)} = (1-w/\eta) (1-t\varphi(w))^{-1}.
\]

If \( x = \sum \chi_j w^j \) then by \([x]^0\) we shall denote the constant term \( \chi_0 \). In particular,

\[
[x^L]^0 = [x^0]^0 = \lim_{w \to \infty} e^{L^\tau(s)} = (tp_1 \eta^0)^{-1}.
\]

Hence, from (9.19) and (9.2), (10.5) holds with

\[
\begin{cases}
\lambda^+(s) = e^s = w, \\
\lambda(-s) = (tp_1 \eta^0)^{-1} [x^-]^0.
\end{cases}
\]

Further, from (9.21),

\[
\Gamma_0^-(s) = \sum_{j=-\infty}^0 \eta^{-x-j} w^j,
\]

hence, from (10.1) and (13.5),

\[
C_0 = (1 - tp(w)^{-1} w^{-x} (1 - (w/\eta)^{x+1}),
\]

while, from (10.7) and (13.6),

\[
P = (tp_1)^{-1} \eta^{-x-2}.
\]

Using (10.8), (13.2) and (13.6), it is not difficult to show that

\[
\begin{cases}
T^{-\lambda^+} = t \omega \eta^2 (\eta - w)^{-1} [\bar{\psi}(\eta) - (\eta - w)^{-1} (\psi(\eta) - \bar{\psi}(w))]
\end{cases}
\]

\[
\sum_{j=1}^{\infty} p_{-j} t \sum_{h=1}^{j} h w^{h-j} \eta^{1-h},
\]

where

\[
\bar{\psi}(w) = \sum_{j=-\infty}^0 p_j w^j = \psi(w) - p_1 w.
\]

Consequently, from (10.7) and (13.6),

\[
Q = -p_1^{-1} \bar{\psi}(\eta) = 1 - p_1^{-1} \psi(\eta).
\]

Let us first determine the distribution of \( N_\infty \). Observe that as \( t \to 1 \) then \( \eta \to 1 \) if \( E(X_n) > 0 \) and \( \eta \to \eta_0 \) if \( -\infty < E(X_n) < 0 \), where \( \eta_0 \) is the unique real number with \( \psi(\eta_0) = 1 \) and \( \eta_0 > 1 \). Hence, from (13.11), if \( E(X_n) > 0 \) then (10.9) holds with \( \sigma = 1 - p_1^{-1} E(X_n) \); if \( -\infty < E(X_n) < 0 \)
then (10.10) holds with $\sigma = 1 - p_1^{-1} \varphi'(\eta_0)$. Finally, from (13.7) and (7.1),

$$\Pr(N_n > 0 | z_0 = -x) = \eta_0^{-x-1} \quad \text{if } E(X_n) < 0,$$

$$= 1 \quad \text{if } E(X_n) > 0.$$ 

The remaining cases are determined by Theorem 7.1.

Next, let us determine the distribution of $N_n$. Substituting the above formulae in (10.6), we have from (10.1) that, for $k = 0, 1, 2, \ldots,$

$$t C_{2k+1} = p_1^{-k-1} (-\varphi'(\eta))^k \eta^{-x-1} \psi(w)^{-1}\eta - w)^{-1}$$

and

$$t C_{2k+2} = p_1^{-k-1} (-\varphi'(\eta))^k \eta^{-x-3} (1 - t \psi(w))^{-1} (\eta - w)^T_\lambda^+,$$

where $T_\lambda^+$ is given by (13.9). Thus, by $w = e^\lambda$, $\varphi(1) = 1$ and $t = \psi(\eta)^{-1}$, $t C_k(0)$ is an explicitly known function of $\eta$.

Transforming the usual contour integral formula for the coefficients in (11.3) to the integration variable $\eta$, it follows from (13.3) that

$$\Pr(N_n = k | z_0 = -x)$$

is equal to the coefficient of $\eta^{-1}$ in the expansion of the function

$$t C_k(0) \varphi(\eta)^n \varphi'(\eta)$$

in descending powers of $\eta$, (|$\eta$| large).

As an illustration, let

$$\varphi(w) = pw + qw - r,$$

where $0 < p = 1 - q$ and $r > 1$. Letting $\eta^{-1} = z$, one finds that

$$\Pr(N_n = 2k + 1 | z_0 = -x)$$

is equal to the coefficient of

$$z^n - x - 1 - k(r+1)$$

in the expansion of the function

$$(rq/p)^k (1 - z^{r+1} rq/p) (1 - z)^{-1} (p + q z^{r+1})^n$$

in powers of $z$, (|$z$| small). In the still more special case where $rq = p$, (thus, $E(X_n) = 0$), this yields

$$\Pr(N_n = 2k + 1 | z_0 = -x) = r^k \left( \frac{n}{\lambda - k} \right) p^{n - \lambda} q^k,$$

where $\lambda$ denotes the largest integer $< (n - x - 1)/(r + 1)$. The case $r = 1, x = 0$ of (13.16) is due to BAXTER [2] p. 220, (where $[n - k]/2$ is misprinted as $[n - k/2]$).

Similarly, (13.15) implies that $\Pr(N_n = 2k + 2 | z_0 = -x)$, ($k > 0$), is equal to the coefficient of

$$z^n - x - 2 - k(r+1)$$
in the expansion of the function
\[ r^k (q/p)k+1 \sum_{j=0}^{r-1} (j+1)z^j (p+qz^{r+1})^n h(z) \]
in powers of \( z \), \(|z| \) small. Here,
\[ h(z) = (1-z^{r+1}rq/p)(1-z)(1-z/p+z^{r+1}q/p)^{-1}. \]
In the special case \( r=1, p=q=1/2 \), this yields
\[ \Pr(N_n = 2k+2|z_0 = -x) = 2^{-n} \left( \frac{n}{[(n-x)/2] - k - 1} \right), \]
\((k > 0)\). The case \( x=0 \) of (13.17) is due to BAXTER [2] p. 220.

Without giving further details, we mention that in the special case
\[ p_j = 2^{j-2} \quad \text{if } j < 1, \]
\((p_j=0 \text{ for } j \geq 2)\), a similar computation easily yields
\[ tC_k(0) = 2\eta^{-x-1}(2\eta-1)^{-k+1}(\eta-1)^{-1}, \quad (k \geq 1), \]
and
\[ \Pr(N_n = k|z_0 = -x) = 2^{-2n+x+1} \left( \frac{2n-x}{n+k} \right), \quad (k \geq 1). \]

REFERENCES