MATHEMATICS

ON MATRIX TRANSFORMATIONS OF CERTAIN SEQUENCE SPACES

BY

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In this paper we determine the characteristics of certain matrix transformation spaces ([1], 298), with special emphasis on the transformations of the space of bounded sequences. An extension of part of a theorem of H. S. Allen ([2], Theorem 3, (a)) to convergence free spaces is also given.

We have, from Allen's theorem, the following:

If \( Z < \alpha < \sigma_\infty \) ([3], 273), then \( Z \to \beta = \alpha \to \beta = \sigma_\infty \to \beta \), where \( \beta \) is any perfect sequence space. ([3], 275). Thus in considering the transformations of such an \( \alpha \) to various perfect sequence spaces, we may consider merely the transformations of \( \sigma_\infty \); and this we shall do. The results will then be immediately applicable to \( \alpha \to \beta \) with \( \alpha \) as defined. In particular, \( \alpha \) may be \( Z \) or \( \Gamma \).

Theorem I. \( \sigma_\infty \to \sigma_r \) \((r > 1)\) is the set of matrices \( A = (a_{n,k}) \) such that

\[
\sum_{n=1}^{\infty} \left| \sum_{k \in E} a_{n,k} \right|^r \leq M',
\]

where \( E \) is an arbitrary subset of the positive integers, and \( M \) is independent of \( E \).

To prove necessity we shall employ the method of proof adopted by Lorentz ([4], 244) for the particular case \( r = 1 \). For sufficiency, which we consider first, we employ a theorem of Köthe and Toeplitz. (See [1], 288.)

Let \( E \) be an arbitrary subset of the positive integers. Let \( X \) be the set of all \( x = \{x_k\} \) such that \( |x_k| < 1 \), \((k = 1, 2, \ldots)\). Then \( X \) is \( p \)-bd. in \( \sigma_\infty \) ([3], 298, (10.4, III)).

Hence, if \( A = (a_{n,k}) \in \sigma_\infty \to \sigma_r \), then \( AX \) is \( p \)-bd. in \( \sigma_r \) ([1], 288, (6.2, VIII)), i.e., if \( y = Ax, x \in X \), then \( \sum \frac{|y_n|^r}{M'} \), where \( M \) is a constant, for every \( y \) ([3], 299, (10.4, V); 298, (10.4, IV)).

Thus

\[
\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{n,k} x_k \right|^r \leq M'
\]

for every \( x \) in \( X \).
Take $x_k=1$, $k \in E$, and $x_k=0$ otherwise. Then $x \in X$, and hence
\[ \sum_{n=1}^{\infty} \left| \sum_{k \in E} a_{n,k} \right|^r \leq M', \]
and the condition has been proved necessary.

In proving sufficiency, we consider first real $(a_{n,k})$ and real bounded sequences, and then extend the result to complex sequences and matrices.

Let $\{x_k\}$ be a sequence of positive terms, such that, as $k$ varies, $x_k$ assumes only a finite number of different values, and let $\bar{d}_k \ |x_k| < 1$. We may then always write $x = \sum b_p x^{(p)}$, where the $x^{(p)} = \{x_k^{(p)}\}$ are sequences of 0’s and 1’s, and $b_p > 0$, $\sum b_p < 1$, and $p$ assumes only a finite number of values in the summation.

Consider the function
\[ F(x) = \left\{ \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|^r \right\}^{1/r}. \]

In this case,
\[ F(x) = \left\{ \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \sum_{p} b_p x_k^{(p)} \right|^r \right\}^{1/r}, \]
and, the $b_p$ being finite in number, this may be rearranged as
\[ F(x) = \left\{ \sum_{n=1}^{\infty} \left| \sum_{p} b_p \sum_{k=1}^{\infty} a_{nk} x_k^{(p)} \right|^r \right\}^{1/r}. \]

Hence, by Minkowski’s inequality,
\[ F(x) \leq \sum_p b_p F(x^{(p)}) \leq M \]
if the condition on $A = (a_{n,k})$ is satisfied, since $F(x^{(p)}) < M$ and $\sum b_p < 1$.

If negative values of $x_k$ are admitted, we obtain $F(x) < 2M$ (again by a simple application of Minkowski’s inequality).

But these $\{x_k\}$, with $x_k$ assuming a finite number of different values between $-1$ and $+1$ as $k$ varies, are dense in the set of real bounded sequences with unit bound, since for a given $\varepsilon > 0$, the range $-1$ to $+1$ divided equally into $[2/\varepsilon] + 1$ segments will provide a value of $x_k$, for each $k$, within a distance $\varepsilon$ of every point in the range. Hence $F(x) < 2M$ for any sequence in $\sigma_\infty$ with $\|x\| < 1$, and $F(x) < \infty$ for all sequences in $\sigma_\infty$.

This proves the theorem for real sequences and matrices.

Now consider the sufficiency of the condition if the matrix, still regarded as real, is applied to complex bounded sequences $\{z_k\}$, where $z_k = x_k + iy_k$, and the $x_k$ and $y_k$ are both real and bounded.

We have, if the condition holds,
\[ \sum_{n=1}^{\infty} \left| \sum_{k \in E} a_{nk} \right|^r \leq M'. \]
Now
\[ \lim_{n \to \infty} \left| \sum_{n=1}^{\infty} (a_{nk} x_k + i a_{nk} y_k) \right|^r = \lim_{n \to \infty} |A_n + i B_n|^r, \]
where \( A_n = \sum_k a_{nk} x_k, \) \( B_n = \sum_k a_{nk} y_k, \) and \( A_n \) and \( B_n \) are real.

Since the matrix applies to all real bounded sequences and transforms them to sequences in \( \sigma_r, \) we have also
\[ \sum_n |A_n|^r < \infty, \quad \sum_n |B_n|^r < \infty. \]

Now
\[ (\sum_{n=1}^{\infty} |A_n + i B_n|^r)^{1/r} \leq (\sum_{n=1}^{\infty} |A_n|^r)^{1/r} + (\sum_{n=1}^{\infty} |B_n|^r)^{1/r} < \infty, \]
by Minkowski's inequality. Thus
\[ \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{nk} (x_k + i y_k) \right|^r < \infty, \]
and the result follows for real matrices.

Similarly, if \( a_{nk} = b_{nk} + ic_{nk}, \) with \( b_{nk} \) and \( c_{nk} \) real for all \( n \) and \( k, \) then for a real sequence \( \{x_k\} \) we require that
\[ \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} b_{nk} c_k + i \sum_{k=1}^{\infty} c_{nk} x_k \right|^r = \sum_{n=1}^{\infty} |P_n + i Q_n|^r \]
say, \( \infty. \)

The proof now proceeds as before, and we see that the convergence of \( \sum_n |P_n|^r \) and \( \sum_n |Q_n|^r \) is necessary and sufficient for the convergence of the left hand expression, which means that the matrices \( (b_{n,k}) \) and \( (c_{n,k}) \) separately satisfy the given condition.

The argument may now be extended to complex sequences and complex matrices, and we see that it is necessary and sufficient for the real and imaginary parts of the matrix elements separately to satisfy the given condition.

It remains only to show that, if
\[ \sum_{n=1}^{\infty} \left| \sum_{k \in \mathcal{E}} a_{nk} \right|^r \leq M^r, \]
then
\[ \sum_{n=1}^{\infty} \left| \sum_{k \in \mathcal{E}} b_{nk} \right|^r \text{ and } \sum_{n=1}^{\infty} \left| \sum_{k \in \mathcal{E}} c_{nk} \right|^r \]
are each bounded; and this is obvious, since
\[ \left| \sum_{k \in \mathcal{E}} (b_{n,k} + i c_{n,k}) \right|^r \geq \sum_{k \in \mathcal{E}} b_{n,k}^r \text{ or } \sum_{k \in \mathcal{E}} c_{n,k}^r. \]

The theorem is now proved.

\( \mathcal{E}_r \) is the space of all sequences such that \( |x_k| < Nk^r \) \( (r > 0) \) for every \( k \)
([3], 274, xiv).

\( F_r \) is the space of all sequences such that \( \sum_{k=1}^{\infty} k^r |x_k| \) converges \( (r > 0) \)
([3], 274, xv).
These spaces are perfect, and $E_r^* = F_r$ ([3], 277, (10.1, IX)).

The following result will be required in the next matrix transformation theorem.

**Lemma.** A set $Y$ is $E_r$-bd. if, and only if, $|y_n/n^r|$ is bounded for all $y$ in $Y$ and for every $n$; i.e., if $|y_n| < M n^r$ for every $y$ in $Y$.

Suppose first that this condition is satisfied. Then if $u \in F_r$,

$$|\sum y_n u_n| \leq M \sum_n n^r |u_n|,$$

which converges by definition of $F_r$. Thus the condition is sufficient.

To prove necessity, let $v_n = n^r u_n$, then $v$ is in $\sigma_1$, if, and only if, $u$ is in $F_r$. Thus if $y \in Y$,

$$|\sum y_n u_n| = |\sum y_n n^r u_n| = |\sum y_n n^r v_n| \leq N(u),$$

by $p$-boundedness.

But \{y_n\} is in $\sigma_1$, and otherwise arbitrary. Hence \{y_n/n^r\} is $\sigma_\infty$-bd., the condition for which is $|y_n/n^r| < M$ ([3] 10.4, III).

This proves the lemma.

**Theorem II.** $\sigma_\infty \to E_r$ is the set of matrices $A = (a_{n,k})$ such that

$$\sum_k |a_{nk}| < N n^r,$$

for some positive $N$ and all $n$.

The condition is sufficient, since

$$|\sum_k a_{n,k} x_k| \leq \sum_k |a_{n,k}| |x_k| \leq M' \sum_k |a_{n,k}|,$$

where $M' = \overline{bd}_{k^r} |x_k|$.

Thus if $A$ satisfies the condition of the theorem,

$$|y_n| = |\sum_k a_{nk} x_k| \leq M' N n^r.$$

Replacing $M'N$ by $M$ gives $|y_n| < M n^r$, whence $y \in E_r$.

To prove necessity, consider the transformation of the set of sequences in $\sigma_\infty$ given by $x_k = e^{i\theta_k}$. This is collectively bounded as the $\theta_k$ vary, so will transform, by [1], 288, (6.2, VIII(b)), into an $E_r$-bounded set; i.e., from the lemma,

$$|y_n| = |\sum_k a_{nk} e^{i\theta_k}| \leq M n^r$$

for some positive $M$ independent of the choice of the $\theta_k$, and for all $n$. Choose the $\theta_k$ such that, for some fixed $n$, $\arg a_{nk} = -\theta_k$ ($k = 1, 2, \ldots$). Then $|y_n| = \sum_k |a_{n,k}| < M n^r$ for this $n$. But by varying the $\theta_k$ we may obtain this result for all $n$. Hence the condition is necessary.

**Theorem III.** $\sigma_\infty \to F_r$ is the set of all matrices $A = (a_{n,k})$ such that

$$\sum_{n=1}^\infty n^r |\sum_{k \in E} a_{n,k}|$$

converges for any arbitrary set $E$ of the positive integers.

By a method similar to that employed in the lemma preceding Theorem
II, we easily establish that the condition that a set \( Y \in F_r \) is \( p - \text{bd.} \) is that \( \sum n^r|y_n| < M \) for some positive \( M \) and all \( y \in Y \).

Let \( X \) be the set of all \( x = \{x_k\} \) such that \( |x_k| < 1 \) \((k = 1, 2, \ldots)\); then, exactly as in theorem I, we have that, if \( y = Ax, x \in X \), then \( \sum n^r|y_n| < M \).

i.e. \( \sum n^r \sum_{k \in E} a_{n,k} x_k \leq M \).

Let \( x_k = 1, k \in E, x_k = 0 \) otherwise. Then \( x \in X \), and hence

\( \sum_{n=1}^{\infty} n^r \sum_{k \in E} a_{n,k} \leq M. \)

This shows that the condition is necessary.

The proof of sufficiency follows exactly on the lines of the corresponding result for \( \sigma_\infty \rightarrow \sigma_1 \), so will not be given.

We mention also without proof two further transformations of \( \sigma_\infty \), both of which are sufficiently obvious.

(a) \( \sigma_\infty \rightarrow \sigma \) is the set of all matrices with rows in \( \sigma_1 \),
(b) \( \sigma_\infty \rightarrow \phi \) is the set of all column-bounded matrices with rows in \( \sigma_1 \).

As in all these cases \( \beta \), the space to which \( \sigma_\infty \) has been transformed, is perfect, we may apply Allen’s theorem [2], 3(b), that \( (\alpha \rightarrow \beta)' = \beta^* \rightarrow \alpha^* \), and obtain the following corollaries to the above theorems. The conditions on \( A \) in each case are as follows.

\( \sigma_1 \) \( \rightarrow \sigma_1 \), \( \sum_{k=1}^{\infty} |\sum_{n \in E} a_{n,k}| < M' \) for all arbitrary subsets \( E \) of the positive integers.

\( F_r \rightarrow \sigma_1, \sum_{n=1}^{\infty} |a_{n,k}| < Nk' \) for all \( k \) and some positive \( N \).

III(a) In \( E_r \rightarrow \sigma_1, \sum_{k=1}^{\infty} k^r \sum_{n \in E} a_{n,k} \) converges for all arbitrary subsets \( E \) of the positive integers.

(a)' In \( \phi \rightarrow \sigma_1 \), columns of \( A \) are in \( \sigma_1 \),
(b)' In \( \sigma \rightarrow \sigma_1 \), (i) \( A \) is row-bounded, (ii) Columns of \( A \) are in \( \sigma_1 \).

**Theorem IV.** If (i) \( \alpha > \phi \), (ii) \( \beta \) is convergence-free, then \( \alpha \rightarrow \beta \) is the set of all matrices \( A = (a_{n,k}) \) such that

(a) rows of \( A \) are in \( \alpha^* \),
(b) row suffixes of non-zero rows of \( A \) form a \( W \)-set for \( \beta \).

(See [3], 281 for definition of \( W \)-set.)

Condition (a) is necessary in order that \( A \) shall apply absolutely to \( \alpha \).

(a) and (b) are obviously together sufficient. To show that (b) is necessary, assume that the \( p^{th} \) row is not a zero row, and that \( p \) is not in a \( W \)-set for \( \beta \). Let the first non-zero term of the row be \( a_{p,k} \). On applying the matrix to \( e^{(k)} \) we obtain \( y_p = (A e^{(k)})_p = a_{p,k} \neq 0 \). But for all sequences in \( \beta \), \( y_p = 0 \). This contradiction proves the necessity of the condition, and thus completes the proof of the theorem.
Theorem V. If (i) \( \alpha \) is normal and contains \( \phi \), (ii) \( \beta \) is convergence-free, (iii) \( \alpha^{**} > \lambda > \alpha \), then \( \alpha \to \beta = \lambda \to \beta = L(\lambda, \beta) = \alpha^{**} \to \beta \). (cf. [2], 375, theorem 3(a)).

For, since \( \alpha^{**} = \alpha^{*} \), it follows from theorem IV that \( \alpha \to \beta = \alpha^{**} \to \beta \). And since, from (iii), \( \alpha^{**} > \lambda^{*} > \alpha^{*} \), it follows that \( \lambda^{*} = \alpha^{*} \), theorem IV then gives

\[
\alpha \to \beta = \lambda \to \beta = \alpha^{**} \to \beta.
\]

But since \( \alpha \) is normal, \( \alpha \to \beta = L(\alpha, \beta) \); and as

\[
\alpha \to \beta = \lambda \to \beta < L(\lambda, \beta) < L(\alpha, \beta) = \alpha \to \beta,
\]

the theorem follows.

If \( \alpha \) is not normal, but \( \alpha \geq \phi \), we still have

\[
\alpha \to \beta = \lambda \to \beta = \alpha^{**} \to \beta,
\]

where \( \beta \) is convergence-free.

We conclude with a result derived from Allen’s theorem ([2], 375) quoted above.

Theorem VI. If (i) \( \alpha \) is normal and contains \( \phi \), (ii) \( \alpha < \lambda < \alpha^{**} \), then \( \sum(\lambda) \) is a ring ([1], 311, 6.5).

For, since \( \sum(\lambda) < \lambda < \alpha^{**} < \alpha \to \alpha^{**} \), from the definitions; and from (i) and (ii), \( \alpha \to \alpha^{**} = \sum(\alpha^{**}) \), by [2], 375, 3(a) (since \( \alpha^{**} \) is 'perfect'). Thus \( \sum(\lambda) = \sum(\alpha^{**}) = \sum(\lambda^{**}) \), from (ii).

Hence, by [1], 312, (6.5, III), \( \sum(\lambda) \) is a ring.

As an example, \( \phi < Z < \sigma_{\infty} \), and \( Z \) is normal. Thus \( \sum(Z) \) and \( \sum(\Gamma) \) are rings (cf. [1], 313, (6.5, IV) and note).

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Postscript.

Since this paper was submitted for publication, a note by H. S. Allen appeared in the Oxford Quarterly Journal of Mathematics, Volume 8, No. 30, June 1957, giving the result which appears as my Theorem VI above.

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REFERENCES