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# On the Oscillation of Certain Functional Differential Equations

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In the present paper we establish some new criteria for the oscillatory and asymptotic behavior of certain class of functional differential equations of the form

$$x^{(n)}(t) + p(t)x^{(n-1)}(t-h) = H(t, x(g(t))).$$



## 1. INTRODUCTION

We consider the functional differential equation

$$x^{(n)}(t) + p(t)x^{(n-1)}(t-h) = H(t, x(g(t))), \qquad n \geq 2, \tag{E}$$

where $p$: $[t_0, \infty) \to [0, \infty)$, $t_0 \geq 0$, $g$: $[t_0, \infty) \to R = (-\infty, \infty)$, and $H$: $[t_0, \infty) \times R \to R$ are continuous, $h$ is a positive real number, and $g(t) \to \infty$ as $t \to \infty$.

We assume that there exist a continuous function $q$: $[t_0, \infty) \to [0, \infty)$, and not identically zero for all large $t$, and a positive constant $c$ such that

$$H(t, x) \operatorname{sgn} x \geq q(t)\,|x|^c \qquad \text{for } x \neq 0,\ t \geq t_0. \tag{1}$$



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In what follows, we consider only solutions of Eqs. (E) which are defined for all large $t$. The oscillatory character is considered in the usual sense; i.e., a solution of Eq. (E) is called oscillatory if it has no last zero. Otherwise, it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behavior of functional differential equations with deviating arguments has been intensively studied in the last two decades. Most of the literature on the subject has been focused on equations of the form of (E) with $p(t) = 0$. For typical results regarding Eq. (E) with $p(t) = 0$, we refer to [4, 8, 10, 12] and the references cited therein. Although much less is known regarding the oscillatory behavior of solutions of equations with the middle term of order $n - 1$ of the form

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) + H(t, x(g(t))) = 0, \qquad n \text{ is even}, \qquad (E_1)$$

and $p(t)$ satisfies the condition:

$$\int^{\infty} \exp\left(\int_{t_0}^{s} -p(u)\,du\right) ds = \infty;$$

a number of authors have considered this problem. As a recent contribution in this direction we refer the reader to [5–7] and the references cited therein.

Very recently, the present author [1], considered Eq.

$$x^{(n)}(t) = p(t)x^{(n-1)}(t + h) + q(t)f(x(g(t))), \qquad n \text{ is odd}, \qquad (E_2)$$

where $p$, $q$, $g$: $[t_0, \infty) \to R$ and $f$: $R \to R$ are continuous, $p(t) \geq 0$, $q(t) \geq 0$ are not identically zero for all large $t$, $xf(x) > 0$ for $x \neq 0$ and $g(t) \to \infty$ as $t \to \infty$, and we established some new sufficient conditions involving the middle term which ensure that every solution of Eq. $(E_2)$ is oscillatory.

It seems that nothing is known regarding the oscillatory and asymptotic behavior of Eq. (E). Therefore, the purpose of this paper is to study the effect of the middle term on the solutions of Eq. (E) and establish some new sufficient conditions involving the middle term which ensure that every solution $x(t)$ of Eq. (E) is oscillatory if $n$ is even and either $x(t)$ is oscillatory or else $x^{(i)}(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n - 2$, if $n$ is odd. The obtained criteria are independent of our earlier results in [1–3]. Examples are inserted in the text to illustrate the relevance of the theorems.

## 2. Main Results

We introduce the following notation: For $T \geq t_0$, we let

$$A[s, t] = \exp\left(\int_t^s p(u)\, du\right) \qquad \text{for } s \geq t \geq T,$$

$$B[v, u] = \int_u^v A[s, t]\, ds \qquad \text{for } v > u > t \geq T,$$

$$A_g = \{t \in [t_0, \infty): g(t) > t\},$$

and, as is done by Kitamura [8], we define the function $r(t)$ by

$$r(t) = \min\{\max\{s, g(s)\}: s \geq t\}. \tag{*}$$

Note that the following inequality holds:

$$g(s) \geq r(t) \qquad \text{for } t < s < r(t).$$

THEOREM 1. *Suppose that for some $m$, $0 \leq m \leq 1$,*

$$\int^\infty s^m q(s)\, ds = \infty, \tag{2}$$

$$p'(t) \geq 0, \qquad (t^m p(t))'' \leq 0 \qquad \textit{for } t \geq t_0, \tag{3}$$

$$\liminf_{t\to\infty} \int_{t-h}^t p(s)\, ds > 1/e, \tag{4}$$

*and either*

$$\limsup_{t\to\infty} \frac{1}{A[r(t), t]} \int_t^{r(t)} B[r(t), u] q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right) du > 1, \qquad \textit{if } c = 1, \tag{5}$$

*or*

$$\limsup_{t\to\infty} \frac{1}{A[r(t), t]} \int_t^{r(t)} B[r(t), u] q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right)^c du > 0, \qquad \textit{if } c > 1. \tag{6}$$

*Then for $n$ even, every solution of Eq.* (E) *is oscillatory, while for $n$ odd, every solution $x$ of Eq.* (E) *is either oscillatory or else $x^{(i)}(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n-2$.*

*Proof.* Without loss of generality we assume that $x(t)$ is an eventually positive solution of Eq. (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. First, we claim that $x^{(n-1)}(t)$ is eventually of one sign. To prove it, suppose that there exists a $t_1 \geq t_0 + h$ such that $x^{(n-1)}(t_1 - h) = 0$. Then from Eq. (E), we get

$$x^{(n)}(t_1) = H(t_1, x(g(t_1))) > 0.$$

Thus, $x^{(n-1)}(t)$ is increasing at any $t_1$, $(t_1 \geq t_0 + h)$ for which it is zero. Therefore, $x^{(n-1)}(t)$ cannot have any zeros on $(t_1, \infty)$.

There are two possibilities:

(a) $x^{(n-1)}(t) < 0$;

(b) $x^{(n-1)}(t) > 0$ eventually.

(a) Assume $x^{(n-1)}(t) < 0$ eventually. Thus

$$x^{(n)}(t) + p(t)x^{(n-1)}(t - h) = H(t, x(g(t))) \geq 0,$$

eventually. Set $w(t) = x^{(n-1)}(t)$. Then

$$w'(t) + p(t)w(t - h) \geq 0 \qquad \text{eventually.} \tag{7}$$

But, by Theorem 2 in [9], condition (4) implies that inequality (7) has no eventually negative solution. This is a contradiction.

(b) Assume $x^{(n-1)}(t) > 0$ eventually. Now, we distinguish the following three cases:

(i) Suppose $x^{(n-1)}(t) > 0$, $x^{(n-2)}(t) < 0$, and $x'(t) > 0$ eventually. Clearly, $x^{(n-3)}(t)$ cannot be negative eventually, because together with the fact that $x^{(n-2)}(t) < 0$ eventually, we get a contradiction to the positiveness of $x(t)$. Hence $x^{(n-3)}(t) > 0$ eventually. Now, since $x'(t) > 0$ for $t \geq t_1$, there exist a positive constant $k$ and a $t_2 \geq \max\{1, t_1 + h\}$ such that

$$x(g(t)) \geq k \qquad \text{for } t \geq t_2. \tag{8}$$

We multiply Eq. (E) by $t^m$, $0 \leq m \leq 1$, and integrate (by parts) from $t_2$ to $t \geq t_2$ to obtain

$$\begin{aligned} t^m x^{(n-1)}(t) - \int_{t_2}^{t} m s^{m-1} x^{(n-1)}(s)\, ds + t^m p(t) x^{(n-2)}(t - h) \\ - \int_{t_2}^{t} (s^m p(s))' x^{(n-2)}(s - h)\, ds = C + \int_{t_2}^{t} s^m H(s, x(g(s)))\, ds, \end{aligned}$$

where $C$ is a constant. Use (1) and (8) in the above equation and integrate the fourth term by parts; we get

$$t^m x^{(n-1)}(t) - m\int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\,ds - (t^m p(t))' x^{(n-3)}(t-h)$$
$$+ \int_{t_2}^{t} (s^m p(s))'' x^{(n-3)}(s-h)\,ds \geq C^* + k^c \int_{t_2}^{t} s^m q(s)\,ds,$$

where $C^*$ is a constant. Thus,

$$t^m x^{(n-1)}(t) - m\int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\,ds \geq C^* + k^c \int_{t_2}^{t} s^m q(s)\,ds.$$

Therefore, we conclude that

$$\lim_{t\to\infty}\left[t^m x^{(n-1)}(t) - m\int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\,ds\right] = +\infty. \tag{9}$$

Let us define

$$y(t) = \int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\,ds;$$

then

$$y'(t) = t^{m-1}x^{(n-1)}(t).$$

Hence, (9) becomes

$$\lim_{t\to\infty}\left[ty'(t) - my(t)\right] = +\infty.$$

By Lemma 1 of Staikos and Sficas [13], we know that

$$\lim_{t\to\infty} y(t) = \pm\infty.$$

Therefore, since $x^{(n-1)}(t)$ is positive, we have that

$$\lim_{t\to\infty}\int_{t_2}^{t} s^{m-1}x^{(n-1)}(s)\,ds = +\infty.$$

Since $0 \le m \le 1$ and $t \ge 1$, we have that $t^{m-1} \le 1$ and so

$$+\infty = \lim_{t\to\infty} \int_{t_2}^{t} s^{m-1} x^{(n-1)}(s)\, ds \le \lim_{t\to\infty} \int_{t_2}^{t} x^{(n-1)}(s)\, ds,$$

which gives a contradiction to the fact that $x^{(n-2)}(t) < 0$ eventually.

(ii) Suppose $x^{(i)}(t) > 0$, $i = 0, 1, \ldots, n-1$, eventually. Integrating Eq. (E) from $t$ to $s$, $s \ge t \ge T$ (say), we obtain

$$x^{(n-1)}(s) - x^{(n-1)}(t) + p(s)x^{(n-2)}(s-h) - p(t)x^{(n-2)}(t-h) - \int_t^s p'(u)x^{(n-2)}(u-h)\, du \ge \int_t^s q(u)x^c(g(u))\, du$$

or

$$x^{(n-1)}(s) + p(s)x^{(n-2)}(s-h) \ge \int_t^s q(u)x^c(g(u))\, du.$$

In view of the increasing nature of the function $x^{(n-2)}$, we see that

$$x^{(n-1)}(s) + p(s)x^{(n-2)}(s) \ge \int_t^s q(u)x^c(g(u))\, du$$

or

$$\frac{d}{ds}(A[s,t]x^{(n-2)}(s)) \ge \int_t^s A[s,t]q(u)x^c(g(u))\, du. \tag{10}$$

Integrating (10) from $t$ to $r(t) > t \ge T$, where $r(t)$ is defined by (*), we have

$$A[r(t),t]x^{(n-2)}(r(t)) - x^{(n-2)}(t) \ge \int_t^{r(t)} \int_t^s A[s,t]q(u)x^c(g(u))\, du\, ds$$

$$= \int_t^{r(t)} \left( \int_u^{r(t)} A[s,t]\, ds \right) q(u)x^c(g(u))\, du$$

$$= \int_t^{r(t)} B[r(t),u]q(u)x^c(g(u))\, du,$$

or

$$A[r(t),t]x^{(n-2)}(r(t)) \ge \int_t^{r(t)} B[r(t),u]q(u)x^c(g(u))\, du. \tag{11}$$

On the other hand, by Taylor's formula with integral remainder, for $v \geq w \geq T$

$$x(v) = \sum_{j=0}^{n-3} \frac{(v-w)^j}{j!} x^{(j)}(v) + \int_w^v \frac{(v-z)^{n-3}}{(n-3)!} x^{(n-3)}(z)\, dz$$

or

$$x(v) \geq \frac{(v-w)^{n-2}}{(n-2)!} x^{(n-2)}(w). \tag{12}$$

From (12) with $v$ and $w$ be replaced by $g(u)$ and $r(t)$, respectively, and in view of $g(u) \geq r(t)$ for $T < t < u < r(t)$, we see that

$$x(g(u)) \geq \frac{(g(u) - r(t))^{n-2}}{(n-2)!} x^{(n-2)}(r(t)). \tag{13}$$

Using (13) in (11), we have

$$A[r(t), t]x^{(n-2)}(r(t)) \geq \int_t^{r(t)} B[r(t), u]q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right)^c \times (x^{(n-2)}(r(t)))^c\, du$$

or

$$x^{(n-2)}(r(t)))^{1-c} \geq \frac{1}{A[r(t), t]} \int_t^{r(t)} B[r(t), u]q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right)^c du. \tag{14}$$

Now, if $c = 1$, then

$$1 \geq \frac{1}{A[r(t), t]} \int_t^{r(t)} B[r(t), u]q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right) du. \tag{15}$$

Taking the lim sup. of both sides of (15) as $t \to \infty$, we obtain a contradiction to (5).

Next, if $c > 1$, then by taking the lim sup of both sides of (14), we have

$$0 = \limsup_{t\to\infty} (x^{(n-2)}(r(t)))^{1-c}$$
$$\geq \limsup_{t\to\infty} \frac{1}{A[r(t), t]} \int_t^{r(t)} B[r(t), u]q(u) \left(\frac{(g(u) - r(t))^{n-2}}{(n-2)!}\right)^c du,$$

a contradiction to (6).

(iii) Suppose $(-1)^i x^{(i)}(t) > 0$, $i = 0, 1, \ldots, n-2$, eventually. This is the case when $n$ is odd. Clearly $x'(t) < 0$ for $t \geq t^* \geq t_0$. We claim that $x(t) \to 0$ monotonically as $t \to \infty$. Suppose to the contrary that $x(t) \to b > 0$ as $t \to \infty$. Then there exists a $T^* \geq t^*$ such that

$$x(g(t)) \geq b \qquad \text{for } t \geq T^* \tag{16}$$

Integrating Eq. (E) from $T^*$ to $t$, using (16), and proceeding exactly as in case (i), we obtain the desired contradiction. This completes the proof.

We are next concerned with the case when $c > 1$.

THEOREM 2. *Let conditions* (1)–(4) *hold with* $c > 1$. *If for every* $T \geq t_0$

$$\int_{A_g} (A[g(u), T])^{1-c}(g(u) - u)^{(n-2)c+1} q(u)\, du = \infty, \tag{17}$$

*then the conclusion of Theorem 1 holds.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (E), say $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 1, we see that $x^{(n-1)}(t)$ is eventually of fixed sign and by (4), the case when $x^{(n-1)}(t) < 0$ eventually is impossible. Furthermore, if $x^{(n-1)}(t)$ is eventually positive, the proof of the case (i) and (iii) follows exactly as the proof of these cases in Theorem 1 and, hence, is omitted. Thus, we only need to consider case (iii).

(ii) Suppose $x^{(i)}(t) > 0$, $i = 0, 1, \ldots, n-1$, eventually. Integrating eq. (E) from $T \geq t_0$ to $s > T$, we can easily get (10) with $t$ being replaced by $T$, that is,

$$\frac{d}{ds}(A[s, T]x^{(n-2)}(s)) \geq \int_T^s q(u)x^c(g(u))\, du.$$

Dividing this inequality by $(A[S, T]x^{(n-2)}(s))^c$ and integrating it from $T$ to $T^* \geq T$, we obtain

$$\begin{aligned}\int_T^{T^*} \frac{(A[s, T]x^{(n-2)}(s))'}{(A[s, T]x^{(n-2)}(s))^c}\, ds &\geq \int_T^{T^*} \int_T^s (A[s, T])^{1-c} q(u) \left(\frac{x(g(u))}{x^{(n-2)}(s)}\right)^c du\, ds \\ &= \int_T^{T^*} \int_T^{T^*} (A[s, T])^{1-c} q(u) \left(\frac{x(g(u))}{x^{(n-2)}(s)}\right)^c ds\, du \\ &\geq \int_{A_g \cap [T, T^*]} \int_u^{g(u)} (A[s, T])^{1-c} q(u) \left(\frac{x(g(u))}{x^{(n-2)}(s)}\right)^c ds\, du.\end{aligned}$$

From (13) with $r(t)$ replaced by $s$, $g(u) \geq s \geq T$, $t \geq T$, we have

$$\int_{x^{n-2)}(T)}^{A[T^*,T]x^{(n-2)}(T^*)} Y^{-c}\, dY \geq \int_{A_g \cap [T,\infty)} \left( \int_u^{g(u)} (A[s, T])^{1-c} \left( \frac{(g(u)-s)^{n-2}}{(n-2)!} \right)^c ds \right) q(u)\, du$$

Now,

$$\int_u^{g(u)} (A[s, T])^{1-c} \left( \frac{(g(u)-s)^{(n-2)c}}{((n-2)!)^c} \right) ds$$
$$\geq (A[g(u),T])^{1-c} \int_u^{g(u)} \frac{(g(u)-s)^{(n-2)c}}{((n-2)!)^c} ds$$
$$= C^* (A[g(u), T])^{1-c} (g(u)-u)^{(n-2)c+1},$$

where

$$C^* = \frac{1}{((n-2!)^c((n-2)c+1)}.$$

Thus,

$$\infty > \int_{x^{n-1)}(T)}^{A[T^*,T]x^{(n-2)}(T^*)} Y^{-c}\, dY \geq C^* \int_{A_g \cap [T,\infty)} (A[g(u), T])^{1-c} \times (g(u)-u)^{(n-2)c+1} q(u)\, du.$$

This contradicts (17) and, hence, the proof is complete.

*Remark* 1. If $g$ is nondecreasing and $g(t) \geq t$, then $r(t) = g(t)$ and conditions (5) and (6) are equivalent to

$$\limsup_{t\to\infty} \frac{1}{A[g(t), t]} \int_t^{g(t)} B[g(t), u] q(u) \frac{(g(u)-g(t))^{n-2}}{(n-2)!} du > 1, \quad \text{if } c = 1 \qquad (5)'$$

and

$$\limsup_{t\to\infty} \frac{1}{A[g(t), t]} \int_t^{g(t)} B[g(t), u] q(u) \left( \frac{(g(u)-g(t))^{n-2}}{(n-2)!} \right)^c du > 0, \quad \text{if } c > 1, \qquad (6)'$$

respectively.

The following examples are illustrative.

EXAMPLE 1. The functional differential equation

$$x^{(n)}(t) + e^h x^{(n-1)}(t-h) = e^{-t}x(2t), \qquad t > 0,\ he^h > 1, \tag{L$_1$}$$

has a nonoscillatory solution $x(t) = e^t \to \infty$ as $t \to \infty$. Only conditions involved with the function $q(t)$ are not satisfied; i.e., conditions (2) and (5) are violated.

EXAMPLE 2. The functional differential equations

$$x^{(4)}(t) + x^{(3)}\left(t - \frac{3\Pi}{2}\right) = 2x(t + 2\Pi) \tag{L$_2$}$$

$$x^{(5)}(t) + x^{(4)}\left(t - \frac{3\Pi}{2}\right) = 2x\left(t + \frac{5\Pi}{2}\right), \tag{L$_3$}$$

have an oscillatory solution $x(t) = \sin t$, and the equation

$$x^{(3)}(t) + x^{(2)}(t-h) = (e^h - 1)(e^{(2c-1)t})|x(2t)|^c \operatorname{sgn} x(2t), t \geq 0, c \geq 1, he > 1, \tag{L$_4$}$$

has a nonoscillatory solution, $x(t) = e^{-t}$.

It is easy to check that the hypotheses of Theorem 1 are satisfied for Eqs. (L$_2$)–(L$_4$).

EXAMPLE 3. Consider the functional differential equation

$$\begin{aligned} &x^{(n)}(t) + x^{(n-1)}(t-h) = |x(2t)|^c \ln(e + x^2(2t)) \operatorname{sgn} x(2t), \\ &t \geq 0, c \geq 1, eh > 1, n \geq 2. \end{aligned} \tag{L$_5$}$$

Here we take

$$p(t) = 1, \qquad q(t) = 1, \qquad g(t) = 2t = r(t).$$

It follows that

$$\begin{aligned} A[s,t] &= e^{s-t}, \qquad s > t \geq 0, \\ B[2t,u] &= e^t - e^{u-t}, \qquad 2t > u > t \geq 0, \end{aligned}$$

and

$$\left(\frac{2^{n-2}}{(n-2)!}\right)^c e^{-t}\int_t^{2t} (e^t - e^{u-t})(u-t)^{(n-2)c}\,du \to \infty \qquad \text{as } t \to \infty,\ c \geq 1.$$

Thus, all the conditions of Theorem 1 are satisfied and, hence, for $n$ even every solution of Eq. ($L_5$), is oscillatory, while for $n$ odd, every solution $x(t)$ of Eq. ($L_5$) is oscillatory or else $x^{(i)}(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n-2$.

We note that none of the known oscillation criteria cover these examples.

## 3. Some Important Remarks

*Remark* 2. We observe that when $n$ is odd and $m = 0$, condition (4) of Theorems 1 and 2 can be disregarded. To show this, we consider only case (a) when $x^{(n-1)}(t) < 0$ eventually. Since $n$ is odd, we must have $x^{(n-2)}(t) > 0$ and $x'(t) > 0$ eventually. Thus, there exist a $t_2 \geq t_1$ and a positive constant $k$ such that

$$x(g(t)) \geq k \qquad \text{for } t \geq t_2.$$

Integrating Eq. (E) from $t_2$ to $t$, we have

$$x^{(n-1)}(t) - x^{(n-1)}(t_2) + \int_{t_2}^{t} p(s)x^{(n-1)}(s-h)\,ds = \int_{t_2}^{t} H(t, x(g(s))\,ds$$

or

$$x^{(n-1)}(t) \geq x^{(n-1)}(t_2) + k^c \int_{t_2}^{t} q(s)\,ds \to \infty \qquad \text{as } t \to \infty,$$

a contradiction.

Based on this remark, we see that the condition $he > 1$ in Eq. ($L_4$) can be disregarded.

*Remark* 3. If an additional term is added to Eq. (E), i.e., Eq. (E) takes the form

$$x^{(n)}(t) + p(t)x^{(n-1)}(t-h) = Q(t)f(x(G(t))) + H(t, x(g(t))), \qquad n \text{ is even}, \tag{$E_3$}$$

where $Q$, $G$: $[t_0, \infty) \to R$ and $f$: $R \to R$ are continuous, $Q(t) > 0$ for $t \geq t_0$, $G(t) < t$, and $G(t) \to \infty$ as $t \to \infty$, then we can replace condition (4) by "Every bounded solution of the delay equation

$$y^{(n)}(t) = Q(t)f(x(G(t))), \qquad n \text{ is even}, \tag{E$_4$}$$

is oscillatory, provided the constant $m$ in conditions (2) and (3) is zero."

We observe that $x^{(n-1)}(t) < 0$ implies that $x^{(n-2)} > 0$ and either $x'(t) > 0$ or $x'(t) < 0$ eventually. The proof of the first case can be done as in Remark 1. Thus, we consider the case when $x^{(n-1)}(t) < 0$, $x^{(n-2)}(t) > 0$, and $x'(t) < 0$ eventually. In this case, we see that $x(t)$ satisfies

$$\begin{gathered}(-1)^i x^{(i)}(t) > 0, \qquad i = 0, 1, \ldots, n,\ t \geq t_1 \text{ (say)}, \\ x^{(n)}(t) \geq Q(t)f(x(G(t))), \qquad t \geq t_1.\end{gathered} \tag{18}$$

Integrating (18) from $t$ to $u$, repeatedly $n$ − times and letting $u \to \infty$, we find

$$x(t) \geq \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} Q(s)f(x(G(x)))\,ds. \tag{19}$$

But, by a result of Philos [11], if inequality (19) has an eventually positive solution $x(t)$, then the corresponding equation,

$$y(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} Q(s)f(x(G(s)))\,ds,$$

has also an eventually positive solution $y(t)$. It follows then that Eq. (E$_4$) has an eventually positive solution, a contradiction.

For illustration, we consider the following functional differential equation

$$x^{(4)}(t) + kx^{(3)}\left(t - \frac{3}{2}\Pi\right) = kx(t - 2\Pi) + x(t + 2\Pi), \qquad t \geq 0, \tag{L$_6$}$$

where $k$ is a positive constant. It is easy to check that all the bounded solutions of the delay equation,

$$y^{(4)}(t) = kx(t - 2\Pi),$$

are oscillatory if $k > 3/2\Pi^4$ (see [8]).

Now if $k \in (3/2\Pi^4, \infty)$, then by the result in Remark 3, every solution of Eq. (L$_6$) is oscillatory. Equation (L$_6$) has an oscillatory solution $x(t) = \sin t$.

We note that condition (4) fails if we take $k \in (3/2\Pi^4, 2/3e\Pi]$.

*Remark* 4. If $h = 0$, then Eq. (E) takes the form

$$x^{(n)}(t) + p(t)x^{(n-1)}(t) = H(t, x(g(t))),$$

or

$$(P(t)x^{(n-1)}(t))' = P(t)H(t, x(g(t))), \tag{E$_5$}$$

where $P(t) = \exp(\int_{t_0}^{t} p(s)\, ds)$, $t \geq t_0 \geq 0$. In this case, the oscillatory behavior of Eq. ($E_5$) can be investigated by Kitamura's results [8], provided that

$$\int^{\infty} \frac{1}{P(s)}\, ds = \infty.$$

As an application of Theorem 1, we consider the functional differential equation

$$x^{(n)}(t) + px^{(n-1)}(t-h) = qx(t+k), \qquad n \geq 2, \tag{L}$$

where $h$, $k$, $p$, and $q$ are positive real numbers, which is a special case of Eq. (E).

COROLLARY 1. *Let $n$ be even. If*

$$phe > 1, \tag{20}$$

$$\sum_{i=0}^{n-1} (-1)^{i+1} \frac{(pk)^i}{i!} > p^n, \tag{21}$$

*then Eq.* (L) *is oscillatory.*

COROLLARY 2. *Let $n$ be odd and let condition* (21) *hold. Then every solution $x(t)$ of Eq.* (L) *is either oscillatory or else $x^{(i)}(t) \to 0$ monotonically as $t \to \infty$, $i = 0, 1, \ldots, n-2$.*

*Proof.* Here, we apply Theorem 1 when $c = 1$. It is easy to check that

$$\begin{aligned} A[s, t] &= e^{p(s-t)}, \qquad s > t \geq t_0, \\ A[t+k, t] &= e^{pk}, \\ B[t+k, u] &= (1/p)(e^{pk} - e^{p(u-t)}), \qquad t+k \geq u, \end{aligned}$$

and obviously, conditions (4) and (5) are reduced to conditions (20) and (21), respectively.

*Remark* 5. The characteristic equation associated to Eq. (L) is

$$w^n + w^{n-1}e^{-hw} = qe^{kw}. \tag{22}$$

From Corollaries 1 and 2, one may conclude that Eq. (22) has no real roots if $n$ is even and both conditions (20) and (21) are satisfied , while Eq. (22) has no real roots or else has negative real roots if $n$ is odd and condition (21) holds.

*Remark* 6. 1. The results of this paper are presented in a form which is essentially new. From Theorems 1 and 2, there exists a class of equations of type (E) with $n$ even, for which the oscillation situation is completely characterized. The reason for that is due to the presence of the middle term with the retarded argument in Eq. (E).

2. The results of this paper are applicable to equations of type (E) which are linear or superlinear and when the argument $g(t)$ is either advanced or a mixed type.

3. The results of this paper are extendable to more general equations of the form

$$x^{(n)}(t) + p(t)x^{(n-1)}(h(t)) = \sum_{i=1}^{m} H_i(t, x(g_i(t))),$$

where $g_i$, $h$, $p$: $[t_0, \infty) \to R$ and $H_i$: $[t_0, \infty) \times R \to R$, $i = 1, 2, \ldots, m$, are continuous, $h(t) < t$, $h(t) \to \infty$ as $t \to \infty$ and $g_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2, \ldots, m$. The details are left to the reader.

4. It would be interesting to obtain results similar to those presented here for the sublinear case (i.e., the constant $c$ in condition (1) is less than 1) and to present a complete criteria for the oscillation of Eq. (E) when $n$ is odd. Also, to establish results for the oscillation of Eq. (L) via the kinds of roots of its associated characteristic equation (22).

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