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A New Approach to an Analysis of Henry Type Integral Inequalities and Their Bihari Type Versions

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In this paper we propose a new method to solve integral inequalities of Henry–Gronwall type and their Bihari nonlinear version. Nonlinear integral inequalities with weakly singular kernels and with multiple integrals as well as a modification of the Ou–Iang–Pachpatte inequality are also treated. © 1997 Academic Press

1. INTRODUCTION

In the contemporary geometric theory of semilinear parabolic differential equations, PDEs are studied as evolution equations in appropriate infinite dimensional Banach spaces. Linear operators defining linear parts of such equations are unbounded linear operators and it is impossible to apply all standard methods generally used in the theory of ODEs with finite dimensional state spaces. Special semilinear PDEs lead to so-called sectorial evolution equations whose linearizations are defined by sectorial operators (see D. Henry [4] and J. K. Hale [3]). These equations can be written as Volterra integral equations with weakly singular kernels. Therefore there are problems with many modifications of estimates standardly used in the theory of ODEs on finite dimensional spaces. One of the basic tools of finite dimensional theory is the well-known Gronwall linear inequality and also the well-known Bihari nonlinear inequality (see [1, 2, 6–10]). The infinite dimensional theory requires us to solve integral inequalities with singular kernels. D. Henry proposed in his book [4] a method to find solutions of such inequalities and proved some results concerning linear integral inequalities of this type. A modification of

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Henry's theorem concerning more general linear integral inequalities has been recently proved by H. Sano and N. Kunimatsu [12]. All these results are proved by an iteration argument and the resulting estimation formulas are expressed as integrals with singular kernels from functions defined by power series of very complicated form which are sometimes not very convenient for applications.

In this paper we present a new method to solve nonlinear integral inequalities of the Henry type and also their Bihari nonlinear version. Our estimates are quite simple and the resulting formulas are similar to these for the classical Gronwall–Bihari inequalities. We also present results on integral inequalities containing multiple integrals which are some modifications of the author's results published recently in [9] (see also [8]). Some modification of a recent result by B. G. Pachpatte [10] concerning the classical type of integral inequalities is also proved there.

2. HENRY–GRONWALL–BIHARI TYPE INEQUALITIES

A new approach to an analysis of nonlinear integral inequalities with weakly singular kernels is used in the proof of our first theorem concerning a nonlinear integral inequality. A linear integral inequality investigated by D. Henry in his book [4] (see also [12]) is a special case of this nonlinear one.

First let us define a special class of nonlinear functions.

DEFINITION 1. Let $q > 0$ be a real number and $0 < T \leqq \infty$. We say that a function ω: $R^+ \to R$ ($R^+ = \langle 0, \infty)$) satisfies a condition (q), if

$$e^{-qt}[\omega(u)]^q \leqq R(t)\,\omega(e^{-qt}u^q) \qquad \text{for all } u \in R^+,\ t \in \langle 0, T), \quad \text{(q)}$$

where $R(t)$ is a continuous, nonnegative function.

Remark. If $\omega(u) = u^m, m > 0$ then

$$e^{-qt}[\omega(u)]^q = e^{(m-1)qt}\omega(e^{-qt}u^q) \tag{1}$$

for any $q > 1$, i.e., the condition (q) is satisfied with $R(t) = e^{(m-1)qt}$.

Let $\omega(u) = u + au^m$, where $0 \leqq a \leqq 1, m \geqq 1$. We shall show that ω satisfies the condition (q).

We need now and also in the sequel the well-known inequality

$$(A_1 + A_2 + \cdots + A_n)^r \leqq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r) \tag{2}$$

for any nonnegative real numbers $A_1, A_2, \dots, A_n$, where $r > 1$ is a real number and n is a natural number. This inequality is a consequence of Jensen's inequality (see, e.g., [5, 11]. Using (2) with $r = q$ and $n = 2$ we have

$$e^{-qt}[\omega(u)]^q = e^{-qt}(u + au^m)^q \leqq 2^{q-1}e^{-qt}(u^q + a^q u^{qm}), \qquad (3)$$

$$\begin{aligned} 2^{q-1}e^{qmt}\omega(e^{-qt}u^q) &= 2^{q-1}e^{qmt}[e^{-qt}u^q + ae^{-qmt}u^{qm}] \\ &= 2^{q-1}e^{-qt}[e^{qmt}u^q + au^{qm}] \geqq 2^{q-1}e^{-qt}[u^q + a^q u^{qm}] \end{aligned}$$

and thus the inequality (3) yields the condition (q) with $R(t) = 2^{q-1}e^{qmt}$.

THEOREM 1. *Let $a(t)$ be a nondecreasing, nonnegative C^1-function on $\langle 0, T)$, $F(t)$ be a continuous, nonnegative function on $\langle 0, T)$, $\omega: R^+ \to R$ be a continuous, nondecreasing function, $\omega(0) = 0$, $\omega(u) > 0$ on $(0, T)$, and $u(t)$ be a continuous, nonnegative function on $\langle 0, T)$ with*

$$u(t) \leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s)\,\omega(u(s))\,ds, \qquad t \in \langle 0, T), \qquad (4)$$

where $\beta > 0$. Then the following assertions hold:

(i) *Suppose $\beta > 1/2$ and ω satisfies the condition* (q) *with $q = 2$. Then*

$$u(t) \leqq e^t\left\{\Omega^{-1}\left[\Omega\left(2a(t)^2\right) + g_1(t)\right]\right\}^{1/2}, \qquad t \in \langle 0, T_1\rangle, \qquad (5)$$

where

$$g_1(t) = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}} \int_0^t R(s) F(s)^2\,ds,$$

where Γ is the gamma function, $\Omega(v) = \int_{v_0}^{v} (dy/\omega(y))$, $v_0 > 0$, Ω^{-1} is the inverse of Ω, and $T_1 \in R^+$ is such that $\Omega(2a(t)^2) + g_1(t) \in \mathrm{Dom}(\Omega^{-1})$ for all $t \in \langle 0, T_1\rangle$.

(ii) *Let $\beta \in (0, 1/2\rangle$ and ω satisfies the condition* (q) *with $q = z + 2$, where $z = (1 - \beta)/\beta$ (i.e., $\beta = 1/(z+1)$). Then*

$$u(t) \leqq e^t\left\{\Omega^{-1}\left[\Omega\left(2^{q-1}a(t)^q\right) + g_2(t)\right]\right\}^{1/q}, \qquad t \in \langle 0, T_1\rangle, \qquad (6)$$

where

$$g_2(t) = 2^{q-1} K_z^q \int_0^t F(s)^q R(s)\, ds,$$

$$K_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}} \right]^{1/p}, \qquad \alpha = \frac{z}{z+1},\ p = \frac{z+2}{z+1}, \tag{7}$$

$T_1 \in R^+$ *is such that* $\Omega(2^{q-1}a(t)^q) + g_2(t) \in \mathrm{Dom}(\Omega^{-1})$ *for all* $t \in \langle 0, T_1 \rangle$.

Proof. First we shall prove the assertion (i). Using the Cauchy−Schwarz inequality we obtain from (4)

$$\begin{aligned} u(t) &\leqq a(t) + \int_0^t (t-s)^{\beta-1} e^s F(s) e^{-s} \omega(u(s))\, ds \\ &\leqq a(t) + \left[\int_0^t (t-s)^{2\beta-2} e^{2s}\, ds \right]^{1/2} \left[\int_0^t F(s)^2 e^{-2s} \omega(u(s))^2\, ds \right]^{1/2}. \end{aligned} \tag{8}$$

For the first integral in (8) we have the estimate

$$\begin{aligned} \int_0^t (t-s)^{2\beta-2} e^{2s}\, ds &= \int_0^t \tau^{2\beta-2} e^{2(t-\tau)}\, d\tau \\ &= e^{2t} \int_0^t \tau^{2\beta-2} e^{-2\tau}\, d\tau = \frac{2e^{2t}}{4^\beta} \int_0^t \sigma^{2\beta-2} e^{-\sigma}\, d\sigma \\ &< \frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1). \end{aligned}$$

Therefore we obtain from (8)

$$u(t) \leqq a(t) + \left[\frac{2e^{2t}}{4^\beta} \Gamma(2\beta-1) \right]^{1/2} \left[\int_0^t F(s)^2 e^{-2s} \omega(u(s))^2\, ds \right]^{1/2}.$$

Using the inequality (2) with $n = 2$, $r = 2$ we obtain

$$u(t)^2 \leqq 2a(t)^2 + \frac{e^{2t}\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t F(s)^2 e^{-2s} \omega\big(u(s)^2\big)\, ds$$

and applying the condition (q) with $q = 2$ we have

$$v(t) \leqq \alpha(t) + K\int_0^t F(s)^2 R(s)\,\omega(v(s))\,ds, \tag{9}$$

where

$$v(t) = \left(e^{-t}u(t)\right)^2, \qquad \alpha(t) = 2a(t)^2, \qquad K = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}. \tag{10}$$

Now we shall proceed in a standard way. Let $V(t)$ be the right-hand side of (9). Then $\omega(v(t))[\omega(V(t))]^{-1} \leqq 1$ and this yields

$$\left[\alpha'(t) + KF(t)^2 R(t)\,\omega(v(t))\right][\omega(V(t))]^{-1}$$
$$\leqq \alpha'(t)[\omega(\alpha(t))]^{-1} + KF(t)^2 R(t),$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \leqq \frac{\alpha'(t)}{\omega(\alpha(t))} + KF(t)^2 R(t)$$

or

$$\frac{d}{dt}\,\Omega(V(t)) \leqq \frac{d}{dt}\,\Omega(a(t)) + KF(t)^2 R(t).$$

Integrating this inequality from 0 to t we obtain

$$\Omega(V(t)) \leqq \Omega(\alpha(t)) + g_1(t),$$

where

$$g_1(t) = K\int_0^t F(s)^2 R(s)\,ds$$

and thus

$$v(t) \leqq V(t) \leqq \Omega^{-1}\left[\Omega(\alpha(t)) + g_1(t)\right].$$

Using (10) we obtain (5).

Now let us prove the assertion (ii). Obviously, $\beta - 1 = -\alpha = -z/(z+1)$. Let p, q be as in the theorem. Then $1/p + 1/q = 1$ and using the

Hölder inequality we obtain

$$\begin{aligned} u(t) &\leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s)\,\omega(u(s))\,ds \\ &= a(t) + \int_0^t (t-s)^{-\alpha} e^s F(s) e^{-s} \omega(u(s))\,ds \\ &\leqq a(t) + \left[\int_0^t (t-s)^{-\alpha p} e^{ps}\,ds\right]^{1/p} \left[\int_0^t F(s)^q e^{-qs} \omega(u(s))^q\,ds\right]^{1/q}. \end{aligned} \tag{11}$$

For the first integral in (11) we have the estimate

$$\begin{aligned} \int_0^t (t-s)^{-\alpha p} e^{ps}\,ds = e^{pt} \int_0^t \tau^{-\alpha p} e^{-p\tau}\,d\tau = \frac{e^{pt}}{p^{1-\alpha p}} \int_0^{pt} \sigma^{-\alpha p} e^{-\sigma}\,d\sigma \\ < \frac{e^{pt}}{p^{1-\alpha p}} \Gamma(1-\alpha p). \end{aligned}$$

Obviously, $1 - \alpha p = 1/(z+1)^2 > 0$ and so $\Gamma(1-\alpha p) \in R$. Thus (8) and the condition (q) yield

$$u(t) \leqq a(t) + e^t K_z \left[\int_0^t F(s)^q R(s)\,\omega\left(e^{-qs} u(s)^q\right) ds\right]^{1/q}, \tag{12}$$

where K_z is defined by (7). Now using the inequality (2) with $n = 2$, $r = q$ we obtain

$$u(t)^q \leqq 2^{q-1} a(t)^q + 2^{q-1} e^{qt} K_z^q \int_0^t F(s)^q R(s)\,\omega\left(e^{-qs} u(s)^q\right) ds \tag{13}$$

and this yields

$$v(t) \leqq \phi(t) + 2^{q-1} K_z^q \int_0^t F(s)^q R(s)\,\omega(v(s))\,ds, \tag{14}$$

where

$$v(t) = \left(e^{-t} u(t)\right)^q, \qquad \phi(t) = 2^{q-1} a(t)^q. \tag{15}$$

Now we shall proceed in the standard way. Let $V(t)$ be the right-hand side of (14). Then $\omega(V(t))[\omega(V(t))]^{-1} \leqq 1$ and this yields

$$\begin{aligned} &\left[\phi'(t) + 2^{q-1} K_z^q F(t)^q R(t)\,\omega(v(t))\right]\left[\omega(V(t))\right]^{-1} \\ &\qquad \leqq \phi'(t)\left[\omega(\phi(t))\right]^{-1} + 2^{q-1} K_z^q F(t)^q R(t), \end{aligned}$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \leqq \frac{\phi'(t)}{\omega(\phi(t))} + 2^{q-1} K_z^q F(t)^q R(t),$$

or

$$\frac{d}{dt}\Omega(V(t)) \leqq \frac{d}{dt}\Omega(\phi(t)) + 2^{q-1} K_z^q F(t)^q R(t). \tag{16}$$

Integrating (16) from 0 to t we obtain

$$\Omega(V(t)) \leqq \Omega(\phi(t)) + g_2(t),$$

where

$$g_2(t) = 2^{q-1} K_z^q \int_0^t F(s)^q R(s)\, ds$$

and this yields

$$v(t) \leqq V(t) \leqq \Omega^{-1}\left[\Omega(\phi(t)) + g_2(t)\right].$$

Using (15) we obtain (6). ∎

As a consequence of Theorem 1 we have

THEOREM 2. *Let* $0 < T \leqq \infty$, $a(t)$, $F(t)$ *be as in Theorem* 1, *and* $u(t)$ *be a continuous, nonnegative function on* $\langle 0, T)$ *with*

$$u(t) \leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s) u(s)\, ds, \tag{17}$$

where $\beta > 0$. *Then the following assertions hold*:

(i) *If* $\beta > 1/2$ *then*

$$u(t) \leqq \sqrt{2}\, a(t) \exp\left[\frac{2\Gamma(2\beta-1)}{4^{\beta}} \int_0^t F(s)^2\, ds + t\right], \qquad t \in \langle 0, T). \tag{18}$$

(ii) *If* $\beta = 1/(z+1)$ *for some* $z \geqq 1$, *then*

$$u(t) \leqq (2^{q-1})^{1/q} a(t) \exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t F(s)^q\, ds + t\right], \qquad t \in \langle 0, T), \tag{19}$$

where K_z *is defined by* (7), $q = z + 2$.

The method used in the proof of Theorem 1 enables us to prove the following theorem concerning the inequality (17), where $a(t)$, $F(t)$, and $u(t)$ are integrable on $\langle 0, T)$.

THEOREM 3. *Suppose $a(t)$, $b(t)$ are nonnegative, integrable functions on $\langle 0, T)$ $(0 < T \leqq \infty)$ and $F(t)$, $u(t)$ are integrable, nonnegative functions on $\langle 0, T)$ with*

$$u(t) \leqq a(t) + b(t) \int_0^t (t-s)^{\beta-1} F(s) u(s)\, ds, \qquad a.e.\ on\ \langle 0, T). \quad (20)$$

Then the following assertions hold:

(i) *If $\beta > 1/2$ then*

$$u(t) \leqq e^t \Phi(t)^{1/2} \qquad a.e.\ on\ \langle 0, T), \tag{21}$$

where

$$\Phi(t) = 2a(t)^2 + 2Kb(t)^2 \int_0^t a(s)^2 F(s)^2 \exp\left[K \int_s^t b(r)^2 F(r)^2\, dr\right] ds,$$

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}.$$

(ii) *If $\beta = 1/(z+1)$ for some $z \geqq 1$ then*

$$u(t) \leqq e^t \Psi(t)^{1/q}, \qquad a.e.\ on\ \langle 0, T), \tag{22}$$

where

$$\Psi(t) = 2^{q-1} a(t)^q + 2^{q-1} K_z^q b(t)^q \int_0^t a(s)^q F(s)^q \exp\left[2^{q-1} K_z^q \int_s^t b(r)^q F(r)^q\, dr\right] ds,$$

$q = z + 2$, and K_z is defined by (7).

Proof. First we shall prove the assertion (i). Using the same procedure as in the proof of the assertion (i) of Theorem 1 one can show that

$$v(t) \leqq 2a(t)^2 + \frac{\Gamma(2\beta-1)}{4^{\beta-1}} b(t)^2 \int_0^t F(s)^2 v(s)\, ds,$$

where $v(t) = (e^{-t}u(t))^2$. From [6, Theorem 1.4] we obtain the inequality (21). Using the procedure from the proof of the assertion (ii) of Theorem 1 one can show that

$$v(t) \leqq 2^{q-1}a(t)^q + 2^{q-1}K_z^q B(t)^q \int_0^t F(s)^q v(s)\, ds,$$

where $v(t) = (e^{-t}u(t))^q$ and the inequality (22) is a direct consequence of [6, Theorem 1.4]. ▮

Now we shall prove a result which is a modification of [4, Lemma 7.1.2].

THEOREM 4. *Suppose $a(t)$ is a nonnegative, nondecreasing C^1-function on $\langle 0, T)$ $(0 < T \leqq \infty)$ and $F(t)$ is a continuous, nonnegative function on $\langle 0, T)$. Let $u(t)$ be a nonnegative, continuous function $\langle 0, T)$ with*

$$u(t) \leqq a(t) + \int_0^t (t-s)^{\beta-1} s^{\gamma-1} F(s) u(s)\, ds, \qquad t \in \langle 0, T), \quad (23)$$

where $\beta > 0$, $\gamma > 0$. Then the following assertions hold:

(i) *If $\beta > 1/2$ and $\gamma > 1 - 1/2p$, where $p > 1$ is a real number, then*

$$u(t) \leqq 2^{1-1/2q} a(t) \exp\left[\frac{4^q}{2q} K^q L^q \int_0^t F(s)^{2q} e^{qs}\, ds + t\right], \qquad t \in \langle 0, T), \quad (24)$$

where

$$K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}, \qquad L = \left[\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p}}\right]^{1/p}, \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) *Let $\beta = 1/(m+1)$ for some real number $m \geqq 1, \gamma > 1 - 1/\kappa q$, where $\kappa > 1$ is a real number, $p = (m+2)/(m+1)$, $q = m+2$. Then*

$$u(t) \leqq 2^{(\kappa r-1)/qr} a(t) \exp\left[\frac{P}{\kappa r} \int_0^t e^{rs} F(s)^{rq}\, ds + t\right], \qquad t \in \langle 0, T), \quad (25)$$

where $r > 1$ is such that $1/\kappa + 1/r = 1$,

$$P = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{rq/p} \left[\frac{\Gamma(\kappa q(\gamma-1)+1)}{\kappa^{\kappa q(\gamma-1)-1}}\right]^{r/\kappa}$$

and $-\alpha = \beta - 1 = -m/(m+1)$.

Proof. Let us prove the assertion (i). From the inequality (23) we have

$$\begin{aligned} u(t) &\leqq a(t) + \left[\int_0^t (t-s)^{2\beta-2} e^{2s}\, ds\right]^{1/2} \left[\int_0^t s^{2\gamma-2} F(s)^2 e^{-2s} u(s)^2\, ds\right]^{1/2} \\ &\leqq a(t) + e^t K^{1/2} \left[\int_0^t s^{2\gamma-2} F(s)^2 (e^{-s} u(s))^2\, ds\right]^{1/2}, \end{aligned}$$

where $K = \Gamma(2\beta-1)/4^{\beta-1}$. This yields

$$u(t)^2 \leqq 2a(t)^2 + 2e^{2t} K \int_0^t s^{2\gamma-2} F(s)^2 (e^{-s} u(s))^2\, ds$$

and so

$$v(t) \leqq c(t) + 2K \int_0^t s^{2\gamma-2} F(s)^2 v(s)\, ds, \tag{26}$$

where

$$c(t) = 2a(t)^2, \qquad v(t) = \left(e^{-t} u(t)\right)^2. \tag{27}$$

From (26) we have

$$\begin{aligned} v(t) &\leqq c(t) + 2K \int_0^t s^{2\gamma-2} e^{-s} F(s)^2 e^s v(s)\, ds \\ &\leqq c(t) + 2K \left[\int_0^t s^{(2\gamma-2)p} e^{-ps}\, ds\right]^{1/p} \left[\int_0^t F(s)^{2q} e^q (v(s))^q\, ds\right]^{1/q}, \end{aligned} \tag{28}$$

where $q > 1$, $1/p + 1/q = 1$. For the first integral in (28) we have

$$\int_0^t s^{(2\gamma-2)p} e^{-ps}\, ds < \frac{e^{pt}}{p^{(2\gamma-2)p}}\, \Gamma((2\gamma-2)p+1).$$

Obviously the assumption yields

$$(2\gamma-2)p + 1 > \left[2\left(1 - \frac{1}{2p}\right) - 2\right] p + 1 = 0 \text{ and so}$$

$$\Gamma((2\gamma-2)p+1) \in R.$$

Let L be as in Theorem 4. From (38) we have

$$v(t)^q \leqq 2^{q-1} c(t)^q + \frac{4^q}{2} K^q L^q \int_0^t F(s)^{2q} e^{qs} v(s)^q \, ds$$

and this yields

$$v(t)^q \leqq 2^{q-1} c(t)^q \exp\left[\frac{4^q}{2} K^q L^q \int_0^t F(s)^{2q} e^{qs} \, ds \right].$$

From this inequality and (27) we obtain (24). Now let us prove the assertion (ii). From the inequality (23) we obtain

$$\begin{aligned} u(t) &\leqq a(t) + \left[\int_0^t (t-s)^{-p\alpha} e^{ps} \, ds \right]^{1/p} \left[\int_0^t s^{q(\gamma-1)} e^{-qs} F(s)^q u(s)^q \, ds \right]^{1/q} \\ &\leqq a(t) + e^t \left[\frac{\Gamma(1-\alpha p)}{p^{(1-\alpha p)}} \right]^{1/p} \left[\int_0^t s^{\kappa q(\gamma-1)} e^{-\kappa s} \, ds \right]^{1/\kappa q} \\ &\quad \times \left[\int_0^t e^{rs} F(s)^{rq} (e^{-s} u(s))^{rq} \, ds \right]^{1/rq} \\ &\leqq a(t) + e^t \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}} \right]^{1/p} \frac{\Gamma(\kappa q(\gamma-1)+1)^{1/\kappa q}}{\kappa^{\kappa q(\gamma-1)-1}} \\ &\quad \times \left[\int_0^t e^{rs} F(s)^{rq} (e^{-s} u(s))^{rq} \, ds \right]^{1/rq}, \end{aligned}$$

where r is as in the theorem. We assume that $\gamma > 1 - 1/\kappa q$ and thus we have $\kappa q(\gamma - 1) + 1 > \kappa q(-1/\kappa q) + 1 = 0$, i.e., $\Gamma(\kappa q(\gamma - 1) + 1) \in R$. The above inequality yields

$$v(t) \leqq 2^{qr-1} \left[a(t)^{qr} + P \int_0^t e^{rs} F(s)^{rq} v(s) \, ds \right],$$

where $v(t) = (e^{-t} u(t))^{rq}$ and P is defined as in the theorem. Therefore we obtain

$$v(t) \leqq 2^{rq-1} a(t)^{rq} \exp\left[P \int_0^t e^{rs} F(s)^{rq} \, ds \right]$$

and this yields the inequality (25). ∎

3. INEQUALITIES WITH MULTIPLE INTEGRALS

The following theorem is a modification of [9, Theorem 1] and Theorem 1.

THEOREM 5. *Let* $a(t), a'(t), \dots, a^{(m-1)}(t)$ $(a^{(i)} = d^i a/dt^i)$ *be nonnegative, continuous functions on* $\langle 0, T)$ $(0 < T \leqq \infty)$, $F_i(t)$ $(i = 1, 2, \dots, m)$ *be nonnegative, continuous functions on* $\langle 0, T)$, $\omega\colon R^+ \to R$, $d\omega(u)/du$ *be continuous, nondecreasing functions,* $\omega(0) = 0$, $\omega(u) > 0$ *on* $(0, T)$, *and* $u(t)$ *be a continuous, nonnegative function on* $\langle 0, T)$ *with*

$$\begin{aligned} u(t) \leqq a(t) &+ \int_0^t (t-s)^{\beta_1 - 1} F_1(s)\, \omega(u(s))\, ds \\ &+ \int_0^t \int_0^{t_1} (t_1 - s)^{\beta_2 - 1} F_2(s)\, \omega(u(s))\, ds\, dt_1 + \cdots \\ &+ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} (t_{m-1} - s)^{\beta_m - 1} F_m(s)\, \omega(u(s))\, ds \dots dt_1, \end{aligned} \tag{29}$$

where $\beta_i > 1/2$ $(i = 1, 2, \dots, m)$ *and* ω *satisfies the condition* (q) *with* $q = 2$. *Then*

$$u(t) \leqq e^t \chi(t)^{1/2}, \qquad t \in \langle 0, T_1\rangle, \tag{30}$$

where $\chi(t) = \Omega^{-1}[\Omega\{(m+1)a(t)^2\} + G(t)]$,

$$G(t) = h_1(t) + \int_0^t h_2(s)\, ds + \cdots + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s)\, ds \dots dt_1,$$

$$h_i(t) = \eta_i (m+1) F_i(t)^2 R(t), \qquad \eta_i = \frac{\Gamma(2\beta_i - 1)}{2^{2\beta_i + m - 1}}, \; i = 1, 2, \dots, m \tag{31}$$

and $T_1 \in R^+$ *is such that* $\Omega\{(m+1)a(t)^2\} + G(t) \in \mathrm{Dom}(\Omega^{-1})$ *for all* $t \in \langle 0, T_1\rangle$.

Proof. The inequality (29) yields

$$\begin{aligned} u(t) \leqq a(t) &+ \left[\int_0^t (t-s)^{2\beta_1 - 2} e^{2s}\, ds\right]^{1/2} \left[\int_0^t F_1(s)^2 e^{-2s} \omega(u(s))^2\, ds\right]^{1/2} \\ &+ \cdots + \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} (t_{m-1} - s)^{2\beta_m - 2} e^{2s}\, ds \dots dt_1\right]^{1/2} \end{aligned}$$

$$\times\left[\int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}}F_1(s)^2e^{-2s}\omega(u(s))^2\,ds\dots dt_1\right]^{1/2}$$

$$\leqq a(t)+e^t\eta_1^{1/2}\left[\int_0^t F_1(s)^2e^{-2s}\omega(u(s))^2\,ds\right]^{1/2}+\cdots$$

$$+e^t\eta_m^{1/2}\left[\int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}}F_m(s)^2e^{-2s}\omega(u(s))^2\,ds\dots dt_1\right]^{1/2},\quad (32)$$

where η_i, $i=1,2,\dots,m$, are defined by (31). We have used there the estimate

$$\int_0^t\int_0^{t_1}\cdots\int_0^{t_{i-1}}(t_{i-1}-s)^{2\beta_i-1}e^{2s}\,ds\dots dt_1$$

$$=\int_0^t\int_0^t\cdots\int_0^{t_{i-2}}e^{2t_{i-1}}\int_0^{t_{i-1}}\sigma^{2\beta_i-1}e^{-2\sigma}\,d\sigma\dots dt_1$$

$$\leqq\frac{e^{2t}}{2^{2\beta_i}}\Gamma(2\beta_i-1)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{i-2}}e^{2t_{i-1}}\,dt_{i-1}\dots dt_1$$

$$\leqq\frac{e^{2t}\Gamma(2\beta_i-1)}{2^{2\beta_i+i-1}},\qquad i=1,2,\dots,m.$$

The inequalities (32) and (2) yield

$$u(t)^2\leqq(m+1)\bigg[a(t)^2+e^{2t}\eta_1\int_0^t F_1(s)^2e^{-2s}\omega(u(s))^2\,ds+\cdots$$

$$+e^{2t}\eta_m\int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}}F_m(s)^2e^{-2s}\omega(u(s))^2\,ds\dots dt_1\bigg]$$

and using the property (q) with $q=2$ we obtain

$$v(t)\leqq(m+1)\bigg[a(t)^2+\eta_1\int_0^t F_1(s)^2R(s)\,\omega(v(s))\,ds+\cdots$$

$$+\eta_m\int_0^t\int_0^{t_1}\cdots\int_0^{t_{m-1}}F_m(s)\,R(s)\,\omega(v(s))\,ds\dots dt_1\bigg],\quad (33)$$

where

$$v(t)=\left(e^{-t}u(t)\right)^2.\qquad (34)$$

Let $V(t)$ be the right-hand side of (33) and

$$\alpha(t) = (m+1)a(t)^2, \qquad h_i(t) = c_i^2\eta_i(m+1)F_i(t)^2 R(t). \quad (35)$$

Then

$$V'(t) - \alpha'(t) - h_1(t)\,\omega(v(t)) = V_1(t), \quad (36.1)$$

$$V_1'(t) - h_2(t)\,\omega(v(t)) = V_2(t), \quad (36.2)$$

$$\cdots \quad (36.m-1)$$

$$V_{m-2}'(t) - h_{m-1}(t)\,\omega(v(t)) = V_{m-1}(t),$$

$$V_{m-1}'(t) = h_m(t)\,\omega(v(t)) \leqq h_m(t)\,\omega(V(t)) \quad (36.m)$$

for $t \in \langle 0, T)$. ▮

We need the following lemma.

LEMMA 1. *If $H(t)$ is a C^1-function on $\langle 0, T)$, $H(t) \geqq 0$ for $t \in \langle 0, T)$, and $H(0) = 0$ then*

$$\int_0^t \frac{H'(s)}{\omega(V(s))}\,ds \geqq \frac{H(t)}{\omega(V(t))}, \qquad t \in \langle 0, T). \quad (37)$$

Proof. Integrating by parts of the left hand-side of (37) we obtain

$$\int_0^t \frac{H'(s)}{\omega(V(s))}\,ds = \frac{H(t)}{\omega(V(t))} + \int_0^t H(s)\,\frac{\omega'(V(s))}{[\omega(V(s))]^2}\,V'(s)\,ds$$

$$\geqq \frac{H(t)}{\omega(V(t))}.$$

Now let us continue the proof of the theorem. Using (37) and (36.m) we have

$$\frac{V_{m-1}(t)}{\omega(V(t))} \leqq \int_0^t \frac{V_{m-1}'(s)}{\omega(V(s))}\,ds \leqq \int_0^t h_m(s)\,ds. \quad (38)$$

The equality (36.$m-1$) and the inequalities (37), (38) yield

$$\frac{V_{m-2}(t)}{\omega(V(t))} \leqq \int_0^t \frac{V_{m-2}'(s)}{\omega(V(s))}\,ds \leqq \int_0^t h_{m-1}(s)\,ds + \int_0^t \frac{V_{m-1}(s)}{\omega(V(s))}\,ds$$

$$\leqq \int_0^t h_{m-1}(s)\,ds + \int_0^t \int_0^{t_1} h_m(s)\,ds\,dt_1. \quad (39)$$

Proceeding in this way one can prove that

$$\frac{V_1(t)}{\omega(V(t))} \leqq \int_0^t h_2(s)\,ds + \int_0^t \int_0^{t_1} h_3(s)\,ds\,dt_1 + \cdots$$
$$+ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s)\,ds\,dt_{m-1} \dots dt_1.$$

Using this inequality we obtain

$$\frac{V'(t)}{\omega(V(t))} - \frac{\alpha'(t)}{\omega(\alpha(t))} \leqq \frac{V'(t) - \alpha'(t)}{\omega(V(t))} \leqq h_1(t) + \frac{V_1(t)}{\omega(V(t))}$$
$$\leqq h_1(t) + \int_0^t h_2(s)\,ds + \int_0^t \int_0^{t_1} h_3(s)\,ds\,dt_1 + \cdots$$
$$+ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s)\,ds\,dt_{m-1} \dots dt_1 := G(t)$$

and thus we have

$$v(t) \leqq \Omega^{-1}[\Omega(\alpha(t)) + G(t)]$$

for all $t \in \langle 0, T_1)$, where $T_1 \in R^+$ is as in Theorem 5.

Using (34) we obtain (30). ∎

Remark. The assertion for the case $\beta_j = 1/(z+1)$, $z \geqq 1$ for all j and its proof is similar to the assertion (ii) of Theorem 1. We do not formulate it here. The case $\beta_i > 1/(z+1)$ for a real number $z \geqq 1$ is more complicated and we also do not formulate any result concerning this case.

4. HENRY'S VERSION OF THE OU–IANG–PACHPATTE INEQUALITY

We shall study the inequality

$$u(t)^2 \leqq a(t) + \int_0^t (t-s)^{\beta-1} F(s)\,\omega(u(s))\,ds, \qquad \beta > 0. \tag{40}$$

Inequalities of such type with $\beta = 1$ and F continuous have been recently studied by B. G. Pachpatte in [10].

We shall prove the following theorem.

THEOREM 6. *Let $a(t)$ be a nondecreasing, nonnegative C^1-function on $\langle 0, T)$ $(0 < T \leqq \infty)$, $F(t)$ be a continuous, nonnegative function, ω be as in Theorem* 5, *and $u(t)$ be a continuous, nonnegative function satisfying the inequality* (40). *Then the following assertions hold*:

(ii) *Suppose $\beta > 1/2$ and ω satisfies the condition* (q) *with $q = 2$. Then*

$$u(t) \leqq e^t \left\{ \Lambda^{-1} \left[\Lambda\left(2a(t)^2\right) + K \int_0^t F(s)^2 R(s)\, ds \right] \right\}^{1/4}, \tag{41}$$

$t \in \langle 0, T_1)$, *where*

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta - 1}}, \qquad \Lambda(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sqrt{\sigma})}, v_0 > 0 \tag{42}$$

and $T_1 \in R^+$ is such that $\Lambda(2a(t)^2) + K\int_0^t F(s)^2 R(s)\, ds \in \mathrm{Dom}(\Lambda^{-1})$ for all $t \in \langle 0, T_1\rangle$.

(ii) *Let $\beta \in (0, 1/2\rangle$ and ω satisfies the condition* (q) *with $q = z + 2$, where $z = (1 - \beta)/\beta$, i.e., $\beta = 1/(z + 1)$. Then*

$$u(t) \leqq e^t \left\{ \Lambda^{-1} \left[\Lambda\left(2^{q-1} a(t)^q\right) + 2^{q-1} K_z^q \int_0^t F(s)^q R(s)\, ds \right] \right\}^{1/2q}, \qquad t \in \langle 0, T_1\rangle, \tag{43}$$

where

$$K_z = \left[\frac{\Gamma(1 - \beta p)}{p^{1-\beta p}} \right]^{1/p}, \qquad \beta = \frac{1}{z+1}, \qquad p = \frac{z+2}{z+1}, \tag{44}$$

$T_1 \in R^+$ is such that $\Lambda(2^{q-1} a(t)^q) + 2^{q-1} K_z^q \int_0^t F(s)^q R(s)\, ds \in \mathrm{Dom}(\Lambda^{-1})$ for all $t \in \langle 0, T_1\rangle$.

Proof. First let us prove the assertion (i). Following the proof of Theorem 1 one can show that

$$v(t)^2 \leqq \alpha(t) + K \int_0^t F(s)^2 R(s)\, \omega(v(s))\, ds, \tag{45}$$

$$v(t) = \left(e^{-t} u(t)\right)^2, \qquad \alpha(t) = 2a(t)^2, \qquad K = \frac{\Gamma(2\beta - 1)}{4^{\beta - 1}}. \tag{46}$$

Let $V(t)$ be the right-hand side of (45). Then $v(t) \leqq \sqrt{V(t)}$. This yields $\omega(v(t)) \leqq \omega(\sqrt{V(t)})$ and thus

$$\frac{V'(t)}{\omega\left(\sqrt{V(t)}\right)} = \frac{\alpha'(t) + KF(t)^2 R(t)\,\omega(v(t))}{\omega\left(\sqrt{V(t)}\right)}$$

$$\leqq \frac{\alpha'(t)}{\omega\left(\sqrt{\alpha(t)}\right)} + KF(t)^2 R(t).$$

This yields

$$\frac{d}{dt}\int_0^{V(t)} \frac{d\sigma}{\omega\left(\sqrt{V(\sigma)}\right)} \leqq \frac{d}{dt}\int_0^{\alpha(t)} \frac{d\sigma}{\omega\left(\sqrt{\alpha(\sigma)}\right)} + KF(t)^2 R(t). \quad (47)$$

Thus we have

$$\frac{d}{dt}\Lambda(V(t)) \leqq \frac{d}{dt}\Lambda(\alpha(t)) + KF(t)^2 R(t),$$

where Λ is defined by (42). This yields

$$V(t) \leqq \Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F(s)^2 R(s)\,ds\right]$$

and thus we have

$$v(t) \leqq \sqrt{V(t)} \leqq \left\{\Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F(s)^2 R(s)\,ds\right]\right\}^{1/2}.$$

Using (46) we obtain (45).

Now we shall prove the assertion (ii). Following the proof of the assertion (ii) of Theorem 1 one can show that

$$v(t)^2 \leqq \phi(t) + 2^{q-1}K_z^q \int_0^t F(s)^q R(s)\,\omega(v(s))\,ds, \quad (48)$$

where

$$v(t) = \left(e^{-t}u(t)\right)^q, \qquad \phi(t) = 2^{q-1}a(t)^q. \quad (49)$$

Following the procedure from the proof of the assertion (i) we obtain

$$v(t) \leqq \left\{\Lambda^{-1}(\Lambda(\phi(t))) + 2^{q-1}K_z^q \int_0^t F(s)^q R(s)\,ds\right\}^{1/2}$$

and using (46) we obtain (43). ∎

Remark. Of course it is possible to prove a result of that type as in Theorem 5 for an inequality which is an analogue of the inequality (40) with multiple integrals. We do not formulate such type of results because their formulation would be technically very complicated.

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