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C^∞-ALGEBRAS FROM THE FUNCTIONAL ANALYTIC VIEW POINT

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The notion of C^∞-algebra goes back to Lawvere [8], although the main examples appear much earlier in differential topology, singularity theory, and counterexamples can be found in harmonic analysis. Recently the notion of C^∞-algebra became important for the foundations of synthetic differential geometry: Kock [5], Dubuc [1], and the forthcoming book of Moerdijk and Reyes [10].

In this paper we put a 'natural' locally-m-convex topology on each C^∞-algebra, using the framework of convenient vector spaces of Kriegl [6, 7].

For important examples this topology is non-Hausdorff (germs of flat functions are always cluster points of 0), but if it is Hausdorff, we are able to derive nice results:

The C^∞-homomorphisms are exactly the continuous algebra homomorphisms, countably generated Hausdorff C^∞-algebras are nuclear, the action of smooth functions on Hausdorff C^∞-algebras is smooth and continuous, and the coproduct equals the bornological tensor product in the most important cases. We also consider C^∞-modules.

Working on this paper we came to believe that the notion of C^∞-algebra is adequate as long as it is finitely generated. Infinitely generated algebras should at least be product preserving functors on convenient vector spaces, say. Then the free ones would be the 'usual' spaces of smooth functions on products of factors $\mathbb{R}$.

The most important open question seems to be whether the C^∞-algebra structure is already determined by the algebra structure. See 6.7 for a partial result.

There is an obvious generalisation of the notion of C^∞-algebra: C^r-algebra ($0\le r\le\infty$ or $r=\omega$ for real analytic). Note that each commutative C^*-algebra is a C^0-algebra. The spectral theorems of C^*-algebras now indicate how to define a non-commutative C^0-algebra: one can apply $f\in C^0(\mathbb{R}^n,\mathbb{R})$ only to n commuting elements of the algebra. This in turn shows that non-commutative C^∞-algebras could be algebras, where each (finitely generated?) commutative subalgebra is a C^∞-algebra.

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1. C^∞-algebras and C^∞-modules

1.1. A commutative real algebra A can be looked at as follows. Let Pol be the category of all finite-dimensional real vector spaces and polynomial mappings between them. A real commutative algebra is then a product preserving functor $A:\mathsf{Pol}\to\mathsf{Set}$. We will identify $A(\mathbb{R})$ with the algebra itself. The multiplication is given by $A(m):A(\mathbb{R}^2)=A\mathbb{R}\times A\mathbb{R}\to A\mathbb{R}$, where $m:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the usual multiplication. Note that $A(\mathbb{R}^0)=A(\text{point})=\text{point}$ and the unit 1_A of $A\mathbb{R}$ is given by $A(1:\text{point}\to\mathbb{R}):\text{point}\to A\mathbb{R}$. If $1_A=0_A$, then $A=0$.

1.2. Now a C^∞-algebra A is a product preserving functor A from the category C^∞ of all finite-dimensional real vector spaces and C^∞-mappings, into Set. So a C^∞-algebra is roughly speaking, a commutative real algebra, in which one may not only evaluate polynomials but also C^∞-functions. See [10] for a thorough introduction to C^∞-algebras.

1.3. A module M over a commutative real algebra A is now defined in the setting of 1.1 as follows: For a real vector space M let $\mathrm{Pol}(\cdot,M):\mathsf{Pol}^{\mathrm{op}}\to\mathsf{Set}$ be the functor, where $\mathrm{Pol}(\mathbb{R}^n,M)$ is the space of all polynomial mappings $\mathbb{R}^n\to M$, $\mathrm{Pol}(\mathbb{R}^n,M)=\mathrm{Pol}(\mathbb{R}^n,\mathbb{R})\otimes_{\mathbb{R}}M$.

An A-module structure η on M is then a dinatural transformation (see MacLane [8a]) from the bifunctor $\mathrm{Pol}(\cdot,M)\times A(\cdot\,\cdot):\mathsf{Pol}^{\mathrm{op}}\times\mathsf{Pol}\to\mathsf{Set}$ into the constant bifunctor M, which is linear in the first variable. Thus for any $p\in\mathrm{Pol}(\mathbb{R}^n,\mathbb{R}^m)$ we have the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Pol}(\mathbb{R}^m,M)\times A(\mathbb{R}^m) & & \\
 & {\scriptstyle \mathrm{Id}\times(p)}\nearrow & & \searrow{\scriptstyle \eta_m} & \\
\mathrm{Pol}(\mathbb{R}^m,M)\times A(\mathbb{R}^n) & & & & M \\
 & {\scriptstyle p^*\times\mathrm{Id}}\searrow & & \nearrow{\scriptstyle \eta_n} & \\
 & & \mathrm{Pol}(\mathbb{R}^n,M)\times A(\mathbb{R}^n) & &
\end{array}$$

and $\eta_n(-,a):\mathrm{Pol}(\mathbb{R}^n,M)\to M$ is linear for all $a\in A(\mathbb{R}^n)=A^n$, where $\mathrm{Pol}(\mathbb{R}^n,M)$ has the pointwise linear structure.

1.4. So what should be a C^∞-module over a C^∞-algebra A? It should be an A-module M, such that not only polynomials with values in M may be evaluated at elements of A, but also C^∞-mappings $f:\mathbb{R}^n\to M$. To specify these we need a locally convex topology on M.

Definition. Let A be a C^∞-algebra. A *C^∞-module* M over A is a (Hausdorff) locally convex vector space M and a dinatural transformation η from the bifunctor $C^\infty(\cdot,M)\times A(\cdot\,\cdot):(\mathsf{C}^\infty)^{\mathrm{op}}\times\mathsf{C}^\infty\to\mathsf{Set}$ into the constant bifunctor M such that

$\eta_{\mathbb{R}^n}(-,a): C^\infty(\mathbb{R}^n, M) \to M$ is linear. M is called a *continuous C^∞-module* if $\eta_{\mathbb{R}^n}(-,a)$ is also continuous.

This definition is tentative and does not catch some examples which should be C^∞-modules: germs of smooth section of vector bundles should form a C^∞-module over the C^∞-algebra of germs of smooth functions; but what is $C^\infty(\mathbb{R}^n, M)$ then?
Also the space $\mathfrak{D}'$ of distributions on $\mathbb{R}$ is *not* a continuous C^∞-module over $C^\infty(\mathbb{R}, \mathbb{R})$, and we believe that it is not a C^∞-module at all (see 4.6).
If M is complete, we will give a simpler characterisation of continuous C^∞-modules below (4.5).

Remark. If we are contend with a dinatural transformation

$$\tilde{\eta}_{\mathbb{R}^n}: (C^\infty(\mathbb{R}^n) \otimes M) \times A \to M$$

(on the algebraic tensorproduct only), then any algebraic module over a C^∞-algebra A becomes a C^∞-module in this sense.

2. The natural topology on a C^∞-algebra

2.1. Let A be a C^∞-algebra. For $a = (a_1, \ldots, a_n) \in A^n$ consider the mapping $\varepsilon_a: C^\infty(\mathbb{R}^n, \mathbb{R}) = C^\infty(\mathbb{R}^n) \to A$, $\varepsilon_a(f) = A(f)(a_1, \ldots, a_n)$. Then ε_a is a homomorphism of C^∞-algebras for each $a \in A^n$.

2.2. We equip each $C^\infty(\mathbb{R}^n)$ with the *compact C^∞-topology*. In detail we use the following family of seminorms:

$$\|f\|_K^k := 2^k \max_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha f(x)|,$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex, $|\alpha| = |(\alpha_1, \ldots, \alpha_n)| = \alpha_1 + \cdots + \alpha_n$, K is compact in $\mathbb{R}^n$ and

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Then it turns out that $\|f \cdot g\|_K^k \leq \|f\|_K^k \cdot \|g\|_K^k$ for all $k \in \mathbb{N}_0$, K compact in $\mathbb{R}^n$. So $C^\infty(\mathbb{R}^n)$ is a locally-m-convex algebra in the sense of Michael [9]; see also Husain [2]. It is well known that $C^\infty(\mathbb{R}^n)$ is a nuclear Fréchet space ((NF)-space), and hence bornological.

2.3. Definition. The *natural topology* on a C^∞-algebra A is the finest locally convex topology on A such that all mappings $\varepsilon_a: C^\infty(\mathbb{R}^n) \to A$, $a \in A^n$, $n \in \mathbb{N}$, are continuous, where $C^\infty(\mathbb{R}^n)$ bears the compact C^∞-topology of 2.2. The natural topology will be denoted n_A or n. It is non-Hausdorff in important cases.

2.4. Theorem. (1) *Let $\varphi : A \to B$ be a C^∞-homomorphism between C^∞-algebras. Then $\varphi : (A, n_A) \to (B, n_B)$ is continuous for the natural topologies.*

(2) *Let $\varphi : A \to B$ be a surjective algebra homomorphism (φ is then automatically a C^∞-homomorphism). Then $\varphi : (A, n_A) \to (B, n_B)$ is a quotient mapping for the natural topologies.*

(3) *Let $\varphi : A \to B$ be an algebra homomorphism, where A, B are C^∞-algebras. If φ is continuous for the natural topologies and n_B is Hausdorff, then φ is a C^∞-homomorphism.*

Proof. (1) and (2): Let $a_1, \ldots, a_n \in A$. Then

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \xrightarrow{\varepsilon_{a_1,\ldots,a_n}} & A \\ & \searrow^{\varepsilon_{\varphi(a_1),\ldots,\varphi(a_n)}} & \downarrow \varphi \\ & & B \end{array}$$

commutes, since φ is a C^∞-homomorphism. Consequently φ is continuous and (1) is proved.

If φ is surjective, then by Moerdijk and Reyes it is a C^∞-homomorphism. Any $b_i \in B$ is of the form $\varphi(a_i)$ for some $a_i \in A$, so

$$\varepsilon_{b_1,\ldots,b_n} = \varepsilon_{\varphi(a_1),\ldots,\varphi(a_n)} = \varphi \circ \varepsilon_{a_1,\ldots,a_n}$$

and thus φ is a quotient mapping.

(3) For polynomials $p \in \mathrm{Pol}(\mathbb{R}^n, \mathbb{R}) \subset C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} \varphi \circ \varepsilon_{a_1,\ldots,a_n}(p) &= \varphi(A(p)(a_1,\ldots,a_n)) = B(p)(\varphi(a_1),\ldots,\varphi(a_n)) \\ &= \varepsilon_{\varphi(a_1),\ldots,\varphi(a_n)}(p). \end{aligned}$$

Thus the continuous mappings $\varphi \circ \varepsilon_{a_1,\ldots,a_n}, \varepsilon_{\varphi(a_1),\ldots,\varphi(a_n)} : C^\infty(\mathbb{R}^n) \to B$ coincide on the dense subspace $\mathrm{Pol}(\mathbb{R}^n, \mathbb{R})$, and (B, n_B) is Hausdorff, so they coincide on the whole of $C^\infty(\mathbb{R}^n)$, thus φ is a C^∞-homomorphism. □

2.5. Lemma. *Let $\{A_\alpha\}$ be a directed family of C^∞-subalgebras of a C^∞-algebra A with $A = \bigcup_\alpha A_\alpha$. Then $(A, n_A) = \varinjlim_\alpha (A_\alpha, n_{A_\alpha})$ in the category of locally convex spaces.*

Proof. Let $T_\alpha : (A_\alpha, n_{A_\alpha}) \to F$ be continuous and linear and $T_\alpha | A_\beta = T_\beta$ for $A_\beta \subsetneq A_\alpha$. Then $T : A \to F$ is well defined and linear. For continuity we have to show $T \circ \varepsilon_a^A : C^\infty(\mathbb{R}^n) \to A \to F$ is continuous for all $a = (a_1, \ldots, a_n) \in A^n$. Since $\{A_\alpha\}$ is directed there is some β such that all $a_i \in A_\beta$. But then $\varepsilon_a^A = i \circ \varepsilon_a^{A_\beta} : C^\infty(\mathbb{R}^n) \to A_\beta \to A$ and $T \circ \varepsilon_a^A = T_\beta \circ \varepsilon_a^{A_\beta}$ is continuous. □

3. The topological structure of free C^∞-algebras

3.1. *Free C^∞-algebras.* Let us denote by $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda, \mathbb{R}) = C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ the free C^∞-algebra on Λ generators. It equals the inductive limit (in **Set**) of the system

$$\{C^\infty(\mathbb{R}^{F_1}) \to C^\infty(\mathbb{R}^{F_2}) : F_1 \subset F_2 \text{ finite subsets of } \Lambda\},$$

where the mapping is induced by the projection $\mathbb{R}^{F_2} \to \mathbb{R}^{F_1}$.

3.2. Lemma. *If $\Lambda = k$ is finite, then $(C^\infty_{\text{fin}}(\mathbb{R}^k), n)$ equals $C^\infty(\mathbb{R}^k)$ with the compact C^∞-topology.*

Proof. $\varepsilon_{(f_1,\dots,f_m)} : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^k)$, $f_i \in C^\infty(\mathbb{R}^k)$, is given by $g \to g \circ (f_1, \dots, f_m)$, and this mapping is continuous for the compact C^∞-topologies, so n is finer. On the other hand

$$\varepsilon_{(\text{pr}_1,\dots,\text{pr}_k)} : C^\infty(\mathbb{R}^k) \to (C^\infty(\mathbb{R}^k), n)$$

is continuous by construction of n and equals the identity, so n is coarser than the compact C^∞-topology. □

3.3. Theorem. *$(C^\infty_{\text{fin}}(\mathbb{R}^\Lambda), n)$ is the regular strict inductive limit of the direct summands ($C^\infty(\mathbb{R}^F)$, compact C^∞-topology) for all finite subsets F of Λ. (Regular means, that each bounded set is contained and bounded in some $C^\infty(\mathbb{R}^F)$.)*

Hence $(C^\infty_{\text{fin}}(\mathbb{R}^\Lambda), n)$ is Hausdorff and a conuclear bornological Montel space. $(C^\infty_{\text{fin}}(\mathbb{R}^{\mathbb{N}}), n)$ is a complete webbed nuclear strict (LF)*-space.*

Proof. The diagram

$$\begin{array}{ccc} \mathbb{R}^F \cong \mathbb{R}^F \times 0 & \xrightarrow{\ i\ } & \mathbb{R}^\Lambda = \mathbb{R}^F \times \mathbb{R}^{\Lambda \setminus F} \\ & {\scriptstyle \text{Id}} \searrow \quad \swarrow {\scriptstyle \text{pr}} & \\ & \mathbb{R}^F & \end{array}$$

commutes, and induces a commuting diagram

$$\begin{array}{ccc} C^\infty(\mathbb{R}^F) & \xleftarrow{\ \text{restriction} = i^*\ } & C^\infty_{\text{fin}}(\mathbb{R}^\Lambda) \\ & {\scriptstyle \text{Id}} \nwarrow \quad \nearrow {\scriptstyle \text{pr}^*} & \\ & C^\infty(\mathbb{R}^F) & \end{array}$$

of C^∞-homomorphisms, which are continuous by 2.5. Therefore each $C^\infty(\mathbb{R}^F)$ is a direct summand in $(C^\infty_{\text{fin}}(\mathbb{R}^\Lambda), n)$. By 2.4, we have $(C^\infty_{\text{fin}}(\mathbb{R}^\Lambda), n) = \varinjlim_F C^\infty(\mathbb{R}^F)$, where F runs over all finite subsets of Λ and the limit is in the category of locally convex spaces. So this limit is strict; and if it were not regular, then there were a bounded subset B of $(C^\infty_{\text{fin}}(\mathbb{R}^\Lambda), n)$, which is not contained in any $C^\infty(\mathbb{R}^F)$ for finite

F. So we can find an increasing sequence F_n of finite subsets of Λ and $b_n \in B$ with $b_n \in C^\infty(\mathbb{R}^{F_{n+1}}) \setminus C^\infty(\mathbb{R}^{F_n})$. Put $\Lambda_0 := \bigcup_n F_n$, a countable subset of Λ. Then $\{b_n\} \subset B$, so $\{b_n\}$ is bounded, and $\{b_n\}$ is contained in the direct summand $(C^\infty_{\text{fin}}(\mathbb{R}^{\Lambda_0}), n)$, but $b_n \notin C^\infty(\mathbb{R}^{F_n}, \mathbb{R})$. Since $(C^\infty_{\text{fin}}(\mathbb{R}^{\Lambda_0}), n)$ is the strict inductive limit of the Fréchet spaces $C^\infty(\mathbb{R}^{F_n})$, it is a regular inductive limit (see Jarchow [3, p. 84]), and $\{b_n\} \subseteq C^\infty(\mathbb{R}^{F_m})$ for some m, a contradiction.

So $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is Montel (since every bounded B is bounded in some Montel space $C^\infty(\mathbb{R}^F)$ and thus pre-compact), is bornological (as inductive limit of bornological spaces), and is conuclear (the dual space $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)'_b$ with the strong topology is the projective limit of the nuclear spaces $C^\infty(\mathbb{R}^F)'_b = \mathfrak{D}'(\mathbb{R}^F)$ by the regularity of the inductive limit, so the dual is nuclear).

If Λ is countable, $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is a strict (LF)-space, hence webbed, complete and nuclear (see Jarchow [3, p. 92, p. 86, p. 481]). □

3.4. Remark. If Λ is uncountable, then $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is not webbed and not a Schwartz space, hence not nuclear.

For $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ contains the direct sum $\mathbb{R}^{(\Lambda)}$ of Λ copies of $\mathbb{R}$ as a direct summand and we may invoke Jarchow [3, p. 202, p. 98].

3.5. Corollary. (1) *Every smooth curve $c: \mathbb{R} \to C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is locally a smooth curve into some $C^\infty(\mathbb{R}^F)$ for finite F.*

(2) *For any $f \in C^\infty(\mathbb{R}^k)$ the induced mapping $f_*: C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)^k \to C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is smooth in the sense of Frölicher–Kriegl, and is also continuous.*

(3) *The multiplication on $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is continuous, and $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is even a locally-m-convex algebra.*

Proof. (1) For a compact interval I the set $c(I)$ is bounded and hence lies in some $C^\infty(\mathbb{R}^F)$, so $c|I$ is a smooth curve there.

(2) Smoothness follows immediately from (1), since $f_*: C^\infty(\mathbb{R}^F)^k \to C^\infty(\mathbb{R}^F)$ is smooth. Continuity (which does not follow automatically from smoothness in the Frölicher–Kriegl calculus) is much more difficult. Let first $\Lambda = N$ be finite.

Then $C^\infty(\mathbb{R}^N)^k = C^\infty(\mathbb{R}^N, \mathbb{R}^k)$. Let $f \in C^\infty(\mathbb{R}^k)$, $g \in C^\infty(\mathbb{R}^N, \mathbb{R}^k)$. We claim that for every closed disked 0-neighbourhood V in $C^\infty(\mathbb{R}^N)$ there is another one U in $C^\infty(\mathbb{R}^N, \mathbb{R}^k)$ such that

$$f_*(g + \lambda U) \subseteq f_*(g) + \lambda V \quad \text{for all } |\lambda| \le 1.$$

The map $(f_*)': C^\infty(\mathbb{R}^N, \mathbb{R}^k)^2 \to C^\infty(\mathbb{R}^N)$ is smooth, so continuous since all spaces are Fréchet (Kriegl [7]); thus we may choose U with $(f_*)'((g+U) \times U) \subseteq V$. Then

$$f_*(g + \lambda h) - f_*(g) = \lambda \int_0^1 (f_*)'(g_0 + t\lambda h)(h)\, dt \in \lambda V \quad \text{for } h \in U.$$

Now let Λ be arbitrary. Then $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)^k = C^\infty_{\text{fin}}(\mathbb{R}^\Lambda, \mathbb{R}^k)$ is a (locally convex hence topological) quotient of the direct sum $\bigoplus_{F \text{ finite}} C^\infty(\mathbb{R}^F, \mathbb{R}^k)$. It remains to show that $(f_*)_F: \bigoplus_F C^\infty(\mathbb{R}^F, \mathbb{R}^k) \to \bigoplus_F C^\infty(\mathbb{R}^F)$ is continuous. Let $(g_F) \in \bigoplus_F C^\infty(\mathbb{R}^F, \mathbb{R}^k)$ and let V be a closed disked 0-neighbourhood in $\bigoplus_F C^\infty(\mathbb{R}^F)$. Then $V_F = V \cap C^\infty(\mathbb{R}^F)$ is a closed disked 0-neighbourhood in $C^\infty(\mathbb{R}^F)$ and by the special case there are 0-neighbourhoods U_F in $C^\infty(\mathbb{R}^F, \mathbb{R}^k)$ with $f_*(g_F + \lambda U_F) \subseteq f_*(g_F) + \lambda V_F$ for $|\lambda| \leq 1$.

Now the disked hull U of $\bigcup_F U_F$ is a 0-neighbourhood in $\bigoplus_F C^\infty(\mathbb{R}^F, \mathbb{R}^k)$ (see Jarchow [3, p. 111]) and $f_*((g_F + \lambda_F h_F)_F) \in (f_*(g_F) + \lambda_F V_F)_F \subseteq f_*(g_F) + V$ for $h_F \in U_F$ and $\sum |\lambda_F| \leq 1$.

(3) Multiplication is given by $m_*: C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)^2 \to C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$, where $m(x, y) = x \cdot y$ is in $C^\infty(\mathbb{R}^2)$, so multiplication is continuous by (2). We will show directly now that $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda, \mathbb{R})$ is locally-m-convex. Since $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is a Hausdorff quotient of $\bigoplus_{F \text{ finite}} C^\infty(\mathbb{R}^F)$ it suffices to use the following two lemmas. □

3.6. Lemma. *The locally convex direct sum $\bigoplus_\alpha A_\alpha$ of locally-m-convex algebras is locally-m-convex.*

Proof. Let $(\|\cdot\|^i_\alpha)_{i \in I(\alpha)}$ be a generating system of seminorms on A_α consisting of submultiplicative seminorms: $\|ab\|^i_\alpha \leq \|a\|^i_\alpha \|b\|^i_\alpha$. For $(a_\alpha) \in \bigoplus_\alpha A_\alpha$ put $\|(a_\alpha)\|_{(M,j)} = \sum_\alpha M_\alpha \|a_\alpha\|^{j(\alpha)}_\alpha$, where $M = (M_\alpha > 1)$ and $j = (j(\alpha) \in I(\alpha))$. Then clearly each $\|\cdot\|_{(M,j)}$ is a submultiplicative seminorm, and these seminorms generate the topology on $\bigoplus_\alpha A_\alpha$. □

3.7. Lemma. *A Hausdorff quotient of a locally m convex algebra is locally-m-convex.*

Proof. Let $\pi: A \to B$ be the quotient mapping. Let $(\|\cdot\|^A_\alpha)$ be a generating system of submultiplicative seminorms on A. Then $\|b\|^B_\alpha := \inf\{\|a\|^A_\alpha : \pi(a) = b\}$ is a system of submultiplicative seminorms on B generating the topology. □

4. The topological structure of C^∞-algebras

4.1. Let us consider an arbitrary C^∞-algebra A. Then A is a quotient of a free C^∞-algebra $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ for some Λ, and by 2.5 the mapping $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda) \to A$ is a quotient mapping for the natural topologies. Hence n_A is Hausdorff if and only if the corresponding ideal in $C^\infty_{\text{fin}}(\mathbb{R}^\Lambda)$ is closed in the natural topology.

4.2. Theorem. *Let A be a Hausdorff C^∞-algebra. Then we have:*

(1) *A is an ultrabornological locally convex space.*

(2) *If A is countably generated, then A is a webbed nuclear* (LF)*-space.*

(3) *If A is finitely generated, then A is a nuclear Fréchet space.*

(4) *A is a locally-m-convex algebra.*

(5) *If $f \in C^\infty(\mathbb{R}^n)$, then $A(f): A^n \to A$ is smooth and continuous as well as all its*

derivatives (*of class* C_c^∞ *of Keller* [4]).

(6) *If* M *is any locally convex space* (*or subset of such*), *then* $C^\infty(M,A)$ *is again a Hausdorff* C^∞*-algebra.*

(7) $C^\infty(\mathbb{R}^n,\mathbb{R})\xrightarrow{A}C^\infty(A^n,A)$ *is a* (*continuous*) *homomorphism of* C^∞*-algebras.*

Proof. (1)–(3): A Hausdorff quotient of a nuclear Fréchet space is a nuclear Fréchet space, that of a webbed nuclear (LF)-space is a webbed nuclear (LF)-space (Jarchov [3, p. 481, p. 90]).

$C^\infty_{\text{fin}}(\mathbb{R}^A)$ is Montel bornological by 3.3, hence ultrabornological and its quotient A is again ultrabornological.

(4) follows from 3.7 and 3.5(3).

$$(5)\qquad \begin{aligned} A(f)'(a)(b) &= \lim_{t\to 0}\frac{1}{t}(A(f)(a+tb)-A(f)(a)), \quad a,b\in A^n \\ &= \lim_{t\to 0}\frac{1}{t}(A(f\circ(x+ty))-A(f\circ x))(a,b) \\ &= \varepsilon_{a,b}\left(\lim_{t\to 0}\frac{f\circ(x+ty)-f\circ x}{t}\right) \\ &= \varepsilon_{a,b}((f'\circ x)\cdot y) = A((f'\circ x)\cdot y)(a,b) \\ &= A(f')(a)\cdot b \end{aligned}$$

Now let A be the quotient of a free C^∞-algebra. Then

$$\begin{array}{ccc} C^\infty_{\text{fin}}(\mathbb{R}^A,\mathbb{R}^n) & \xrightarrow{f_*} & C^\infty_{\text{fin}}(\mathbb{R}^A,\mathbb{R}^n) \\ \downarrow & & \downarrow \\ A^n & \xrightarrow{A(f)} & A^n \end{array}$$

commutes, f_* is continuous by (3.5)(2), so $A(f)$ is continuous, and $A(f^{(k)})$ is continuous for all k. Thus all derivatives $A(f)^{(k)}: A^n\times(A^n)^k\to A^n$ are continuous, so $A(f)$ is C_c^∞.

(6) Let M be a locally convex vector space (the argument for subsets is the same). Let $f\in C^\infty(\mathbb{R}^n)$. Then $A(f): A^n\to A$ is smooth, so $A(f)_*: C^\infty(M,A)^n = C^\infty(M,A^n)\to C^\infty(M,A)$ is well defined and gives $C^\infty(M,A)$ the structure of a C^∞-algebra. Now for $g\in C^\infty(M,A)^n$ the mapping $\varepsilon_g: C^\infty(\mathbb{R}^n)\to C^\infty(M,A)$, $\varepsilon_g(f) = A(f)\circ g$, is smooth and linear in the sense of Kriegl, where $C^\infty(M,A)$ bears the Hausdorff bornological topology of Kriegl, so ε_g is continuous. So the natural topology on $C^\infty(M,A)$ is finer than the bornological structure, and thus Hausdorff.

(7) follows from 2.5(1). □

4.3. Remark. A Hausdorff C^∞-algebra A might not be complete, not even C^∞-

complete in the sense of Kriegl [6]: see the counterexample below. We do not know whether some completion of A is then again a C^∞-algebra.

4.4. Corollary. *The natural topology n_A on a C^∞-algebra A is the finest locally convex topology on A which makes A into a semitopological algebra and for which all the mappings $C^\infty(\mathbb{R}) \xrightarrow{\varepsilon_a} A$, $a \in A$, are continuous. (The same holds for locally-m-convex topologies.)*

Proof. *Step 1:* The compact C^∞-topology CO^∞ on $C^\infty(\mathbb{R})$ is the finest among all locally convex topologies τ such that multiplication is separately continuous and all $\varepsilon_f : (C^\infty(\mathbb{R}), CO^\infty) \to (C^\infty(\mathbb{R}), \tau)$ are continuous, for then

$$\varepsilon_{\mathrm{Id}} = \mathrm{Id} : (C^\infty(\mathbb{R}), CO^\infty) \to (C^\infty(\mathbb{R}), \tau)$$

is continuous.

Step 2: The compact C^∞-topology on $C^\infty(\mathbb{R}^n)$ is the finest among all locally convex topologies, which make multiplication separately continuous and all mappings $\varepsilon_f : C^\infty(\mathbb{R}) \to (C^\infty(\mathbb{R}^n), \tau)$, $f \in C^\infty(\mathbb{R}^n)$, continuous.

Induction on n. For $n=1$ see Step 1. Suppose this is true for n. Consider such a topology τ on $C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$, and the following diagram

$$\begin{array}{ccccc}
(C^\infty(\mathbb{R}^{n+1}), CO^\infty) \cong C^\infty(\mathbb{R}^n) \tilde{\otimes}_\pi C^\infty(\mathbb{R}) & & \xleftarrow{\;\; i \;\;} & & C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}) \\
 & \searrow^{\mathrm{Id}} & \downarrow j & \swarrow_{k} & \\
 & & (C^\infty(\mathbb{R}^{n+1}), \tau) & &
\end{array}$$

where $i(f,g) = f \otimes g$ is the canonical bilinear mapping, j is induced by the identity, and $k(f,g)(x,t) = f(x) \cdot g(t)$. We also used the fact that $(C^\infty(\mathbb{R}^{n+1}), CO^\infty) = C^\infty(\mathbb{R}^n) \tilde{\otimes}_\pi C^\infty(\mathbb{R})$ for the (completed) projective tensor product, since both spaces are nuclear.

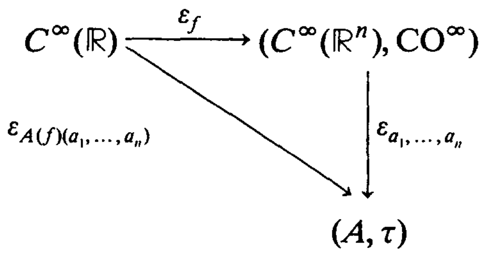
In order to show that j is continuous we have to check that k is jointly continuous, and for that it suffices that k is separately continuous, since both spaces are Fréchet spaces (Jarchov [3, p. 89]). But $g \mapsto k(f,g)$ equals $g \mapsto \varepsilon_{\mathrm{pr}_2}(g) \mapsto (f \circ \mathrm{pr}_1) \cdot \varepsilon_{\mathrm{pr}_2}(g)$, where $\mathrm{pr}_1 : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\mathrm{pr}_2 : \mathbb{R}^{n+1} \to \mathbb{R}^1$. The first mapping $\varepsilon_{\mathrm{pr}_2} : (C^\infty(\mathbb{R}), CO^\infty) \to (C^\infty(\mathbb{R}^{n+1}), \tau)$ is continuous by requirement, the second mapping $h \mapsto (f \circ \mathrm{pr}_1) \cdot h$ is continuous since multiplication is separately continuous in $(C^\infty(\mathbb{R}^{n+1}), \tau)$.

Now we consider the diagram

$$\begin{array}{ccc}
C^\infty(\mathbb{R}) & \xrightarrow{\;\;\varepsilon_f\;\;} & (C^\infty(\mathbb{R}^n), CO^\infty) \\
 & \searrow^{\varepsilon_{(f \circ \mathrm{pr}_1)}} & \downarrow (\mathrm{pr}_1)^* \\
 & & (C^\infty(\mathbb{R}^{n+1}), \tau).
\end{array}$$

pr_1^* is an algebra homomorphism. It pulls back τ to a topology τ' on $C^\infty(\mathbb{R}^n)$ for which multiplication is separately continuous and for which $\varepsilon_f : C^\infty(\mathbb{R}) \to (C^\infty(\mathbb{R}^n), \tau')$ is continuous for each $f \in C^\infty(\mathbb{R}^n)$, since $(\mathrm{pr}_1)^* \circ \varepsilon_f = \varepsilon_{(f \circ \mathrm{pr}_1)} : C^\infty(\mathbb{R}) \to (C^\infty(\mathbb{R}^{n+1}), \tau)$ is continuous. By induction CO^∞ is the finest such topology on $C^\infty(\mathbb{R}^n)$, so $(\mathrm{pr}_1)^* : (C^\infty(\mathbb{R}^n), \mathrm{CO}^\infty) \to (C^\infty(\mathbb{R}^{n+1}), \tau)$ is continuous. Now $f \to k(f, g)$ equals the composition $f \to (\mathrm{pr}_1)^*(f) \to (\mathrm{pr}_1)^*(f) \cdot (g \circ \mathrm{pr}_2)$ of continuous mappings.

Step 3: Proof of the general result. Consider a semitopological algebra locally convex topology τ on A such that all ε_a are continuous.

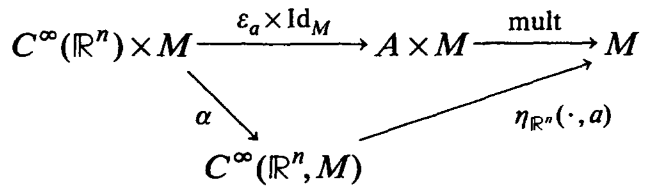


$\varepsilon_{a_1,\ldots,a_n}$ is an algebra homomorphism, so it pulls back τ to a locally convex τ' on $C^\infty(\mathbb{R}^n)$ making all ε_f continuous, since the diagram commutes. By step 2, $\varepsilon_{a_1,\ldots,a_n}$ is continuous for all $a_i \in A$, $n \in \mathbb{N}$. So n_A is finer than τ, and by 2.12(4), n_A is semitopological. □

4.5. Now comes the simple characterisation of some C^∞-modules promised in 1.4.

Theorem. *Let M be a Hausdorff complete locally convex vector space. Let A be a C^∞-algebra and suppose that M is a module over the underlying algebra $\bar{A}$ of A. Then this module structure extends (uniquely) to a continuous C^∞-module structure of M over A if and only if the $\bar{A}$-module structure* $\mathrm{mult} : (A, n_A) \times M \to M$ *is continuous.*

Proof. Let us suppose first that mult is a C^∞-algebra structure. For each $a \in A^n$ consider the following commutative diagram



where ε_a is from 2.1, $\eta_{\mathbb{R}^n}(\cdot, a)$ is from 1.4, is continuous and linear, and where $\alpha(f, m)(x) = f(x) \cdot m$, so α is bilinear and continuous. Thus $\mathrm{mult} \circ (\varepsilon_a \times \mathrm{Id}_M)$ is bilinear and continuous, thus mult is continuous.

Let us suppose conversely that $\mathrm{mult} : (A, n_A) \times M \to M$ is continuous. Then

$$\mathrm{mult} \circ (\varepsilon_a \otimes \mathrm{Id}_M) : C^\infty(\mathbb{R}^n) \otimes_\varepsilon M = C^\infty(\mathbb{R}^n) \otimes_\pi M \to A \otimes_\pi M \to M$$

is also continuous, where $\otimes_\varepsilon$ is the (not completed) inductive tensor product, $\otimes_\pi$ is the projective tensor product, and these tensor products agree since $C^\infty(\mathbb{R}^n)$

is nuclear. Consequently the continuous linear extension to the completion $C^\infty(\mathbb{R}^n)\tilde{\otimes}_\varepsilon M$ (which coincides with $C^\infty(\mathbb{R}^n, M)$ by Treves [17, p. 449]) is a map $\eta_{\mathbb{R}^n}(\cdot, a)$. This is the looked for dinatural transformation of 1.4. □

Remark. The first part of the theorem is true without completeness assumptions. The second part is true if M is assumed to be C^∞-complete only, one then has to consider the Mackey-completion of $C^\infty(\mathbb{R}^n)\otimes_\varepsilon M$ in the proof.

4.6. Important counterexample. The space $\mathfrak{D}'$ of distributions (on the real line for simplicity's sake) is a module over $C^\infty(\mathbb{R})$ but not a continuous C^∞-module, because the multiplication is not jointly continuous. If $\delta^{(n)}$ is the n-th derivative of the δ-distribution at 0 and $f\in C^\infty(\mathbb{R}, \mathbb{R})$, we have

$$f\cdot \delta^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \langle \delta^{(n-k)}, f\rangle \cdot \delta^{(k)}.$$

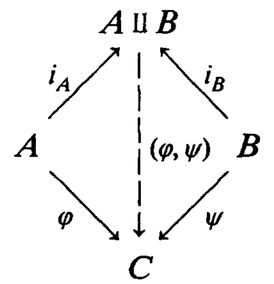
Consider the embedding $\bigoplus_{\mathbb{N}} \mathbb{R} = \mathbb{R}^{(\mathbb{N})} \to \mathfrak{D}'$, $(c_i) \to \sum c_i \delta^{(i)}$, which is an isomorphism onto a closed submodule, and the multiplication $C^\infty(\mathbb{R}) \times \bigoplus_{\mathbb{N}} \mathbb{R} \to \bigoplus_{\mathbb{N}} \mathbb{R}$ factors to the quotient algebra $\prod_{\mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$ of Taylor series at 0 of smooth functions on $\mathbb{R}$, and the coefficient of δ is then the duality pairing (up to signs) of $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{(\mathbb{N})}$, which is not jointly continuous.

The same argument works for any space of distribution sections of a vector bundle, with or without compact support.

We have not been able to find a non-continuous (but linear) η in the sense of 1.4. This seems to indicate that the operators $u \mapsto f\cdot u$, $\Delta' \to \Delta'$, do not admit spectral theorems for C^∞-functions.

5. The coproduct of C^∞-algebras

5.1. If A, B are C^∞-algebras, then the coproduct $A \amalg B$ is a C^∞-algebra with the following universal property:



There are C^∞-algebra homomorphisms i_A, i_B such that for all C^∞-algebra homomorphisms $\varphi: A \to C$, $\psi: B \to C$ there is a unique C^∞-algebra homomorphism $(\varphi, \psi): A \amalg B \to C$ with $(\varphi, \psi)\circ i_A = \varphi$, $(\varphi, \psi)\circ i_B = \psi$. For free algebras we have

$$C^\infty_{\text{fin}}(\mathbb{R}^{\Lambda_1}) \amalg C^\infty_{\text{fin}}(\mathbb{R}^{\Lambda_2}) = C^\infty_{\text{fin}}(\mathbb{R}^{\Lambda_1 \sqcup \Lambda_2}),$$

where $A_1 \sqcup A_2$ is the disjoint union. See [10] for that. Using the universal property above one may easily check that $A \amalg \cdot$ commutes with inductive limits (in Set) of C^∞-algebras.

5.2. In Kriegl [6] it is shown that the category of C^∞-complete (i.e., bornologically complete) bornological vector spaces (also called convenient vector spaces) and bounded linear mappings is a closed monoidal category, with the C^∞-completed bornological tensor product $\tilde{\otimes}_b$ as product.

So we have the general exponential law $L(E \tilde{\otimes}_b F, G) = L(E, L(F, G))$, hence $E \tilde{\otimes}_b \cdot$ is a left adjoint internal functor and therefore commutes with inductive limits (and quotients) in the category of convenient vector spaces.

Let us denote in this section (A, n_A) by A_{nat}. Then $A \to A_{\text{nat}}$ gives a functor from the category of C^∞-algebras to that of locally convex spaces with continuous linear mappings. We are going to prove now, that in the most important cases these two functors coincide. First two lemmas on the formation of limits.

5.3. Lemma. *Let $\{A^\alpha\}$ be a directed family of C^∞-subalgebras of a C^∞-algebra A with $A = \bigcup_\alpha A^\alpha$, and such that all A^α, A have Hausdorff and c^∞-complete natural topologies. Then*

$$A_{\text{nat}} = (\text{cvs})\text{-}\varinjlim_\alpha A^\alpha_{\text{nat}}.$$

Proof. By 2.5, $A_{\text{nat}} = (\text{lcs})\text{-}\varinjlim_\alpha A^\alpha_{\text{nat}}$ and since by assumption A_{nat} is Hausdorff and c^∞-complete this is the cvs-limit. □

Next the corresponding statement for quotients. However, since an ideal I is in general not a C^∞-algebra ($1 \notin I$!) we have to formulate it in a somewhat more complicated way.

5.4. Lemma. *Let $B \xrightarrow{e} A$ be a surjective algebra homomorphism, where A and B have Hausdorff and c^∞-complete natural topologies, and let*

$$\pi : C^\infty_{\text{fin}}(\mathbb{R}^\Lambda) \to B \times_A B = \{(x_1, x_2) \in B \times B : e(x_1) = e(x_2)\}$$

be a surjection. Then $B_{\text{nat}} \xrightarrow{e} A_{\text{nat}}$ is the (cvs)-coequalizer of $(\text{pr}_1 \circ \pi, \text{pr}_2 \circ \pi)$.

Proof. It is obvious, that $B \xrightarrow{e} A$ is the coequalizer of $\text{pr}_i : B \times_A B \to B$ for $i = 1, 2$ in the category of C^∞-algebras and vector spaces, and since π is onto, it is also the coequalizer of $\text{pr}_i = \pi : C^\infty_{\text{fin}}(\mathbb{R}^\Lambda) \to B$ in both categories. By 2.4, $B_{\text{nat}} \xrightarrow{e} A_{\text{nat}}$ is a quotient mapping, hence it is the coequalizer in the category of not necessarily Hausdorff lcs. But since A_{nat} is Hausdorff and c^∞-complete, it is the coequalizer in cvs as well. □

5.5. Theorem. *Let A and B be C^∞-algebras, which carry as well as their coproduct $A \amalg B$ Hausdorff c^∞-complete natural topologies. Then*

$$(A \amalg B)_{\text{nat}} = A_{\text{nat}} \tilde{\otimes}_{\text{b}} B_{\text{nat}}.$$

Proof. We will show this in several steps.

First for free, finitely generated C^∞-algebras, which have automatically Hausdorff and complete natural topologies as has their coproduct.

$$C^\infty(\mathbb{R}^N) \amalg C^\infty(\mathbb{R}^M) = C^\infty(\mathbb{R}^{N \cup M}).$$

On the other hand

$$\begin{aligned}
C^\infty(\mathbb{R}^{N\cup M}, \mathbb{R}) &= C^\infty(\mathbb{R}^N, C^\infty(\mathbb{R}^M, \mathbb{R})) && \text{(by cartesian closedness)}\\
&= C^\infty(\mathbb{R}^N, \mathbb{R}) \tilde{\otimes}_\varepsilon C^\infty(\mathbb{R}^M, \mathbb{R}) && \text{(by Treves [17] since } C^\infty(\mathbb{R}^M) \text{ is Frechet)}\\
&= C^\infty(\mathbb{R}^N) \tilde{\otimes}_\pi C^\infty(\mathbb{R}^M) && \text{(since } C^\infty(\mathbb{R}^N) \text{ is nuclear)}\\
&= C^\infty(\mathbb{R}^N) \tilde{\otimes}_{\text{b}} C^\infty(\mathbb{R}^M) && \text{(since both factors are Frechet-spaces, hence the bounded bilinear maps are continuous).}
\end{aligned}$$

And the c^∞-completion $\tilde{\otimes}_{\text{b}}$ of the algebraic tensor product is the completion, since $C^\infty(\mathbb{R}^{N \cup M})$ is Frechet.

Next for free, arbitrarily generated C^∞-algebras. Since $C^\infty_{\text{fin}}(\mathbb{R}^A)$ is the union of the finitely generated subalgebras $C^\infty(\mathbb{R}^N)$ with $N \subseteq A$, and $C^\infty_{\text{fin}}(\mathbb{R}^A) \amalg C^\infty_{\text{fin}}(\mathbb{R}^B)$ of $C^\infty(\mathbb{R}^N) \amalg C^\infty(\mathbb{R}^M)$ with $N \subseteq A$, $M \subseteq B$ finite and all occuring spaces have Hausdorff and c^∞-complete natural topologies, we can apply the above lemmas and the fact that $\amalg$ preserves colimits of C^∞-algebras and $\tilde{\otimes}_{\text{b}}$ colimits of cvs.

Now for finitely generated C^∞-algebras A and B. Then A and B are quotients of $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^m)$. And by 5.4,

$$A_{\text{nat}} = \text{cvs-coequ}(f_1, f_2) \quad \text{and} \quad B_{\text{nat}} = \text{cvs-coequ}(g_1, g_2)$$

with

$$f_i : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^n) \quad \text{and} \quad g_i : C^\infty(\mathbb{R}^M) \to C^\infty(\mathbb{R}^m).$$

Since $\amalg$ preserves colimits of C^∞-algebras and $\tilde{\otimes}_{\text{b}}$ colimits of cvs and since $(C^\infty(\mathbb{R}^N) \amalg C^\infty(\mathbb{R}^M))_{\text{nat}} = C^\infty(\mathbb{R}^N)_{\text{nat}} \tilde{\otimes}_{\text{b}} C^\infty(\mathbb{R}^M)_{\text{nat}}$ (similar for n and m), we obtain $(A \amalg B)_{\text{nat}} = A_{\text{nat}} \tilde{\otimes}_{\text{b}} B_{\text{nat}}$, provided that $(A \amalg B)_{\text{nat}}$ is Hausdorff (it is c^∞-complete then).

Finally the general case. Since every C^∞-algebra A is union of its finitely generated subalgebras A', which are Hausdorff (and hence c^∞-complete) if A is so, $A_{\text{nat}} = \text{cvs-}\varinjlim (A')_{\text{nat}}$ by (5.3). Hence $A \amalg B = C^\infty\text{-alg-}\varinjlim_{A',B'} A' \amalg B'$. Consequently, $(A \amalg B)_{\text{nat}} = \text{cvs-}\varinjlim_{A',B'} (A' \amalg B')_{\text{nat}}$ provided $(A \amalg B)_{\text{nat}}$ is Hausdorff and c^∞-complete, and furthermore

$$\text{cvs-lim}_{A',B'} (A' \amalg B')_{\text{nat}} = \text{cvs-lim}_{A',B'} A'_{\text{nat}} \tilde{\otimes}_{\text{b}} B'_{\text{nat}}$$

$$= (\text{cvs-lim}\, A'_{\text{nat}}) \tilde{\otimes}_{\text{b}} (\text{cvs-lim}\, B'_{\text{nat}}) = A_{\text{nat}} \tilde{\otimes}_{\text{b}} B_{\text{nat}}. \qquad \square$$

Remark. We do not know wether $(A \amalg B)_{\text{nat}}$ is Hausdorff (resp. c^∞-complete) provided A_{nat} and B_{nat} are (even in case of finitely generated C^∞-algebras).

For countably generated C^∞-algebras it is enough to assume Hausdorffness (then they are c^∞-complete automatically (4.2.2)).

On the other hand there are Hausdorff C^∞-algebras which are not c^∞-complete, see Example 6.10.

In general $A \amalg B$ might turn out to be a suitably completed tensor product of A and B. For this the exact degree of completeness of Hausdorff C^∞-algebras has to be determined. They are ultrabornological in general!

6. Examples and counterexamples

6.1. Let $C_0^\infty(\mathbb{R})$ be the C^∞-algebra of germs at 0 of smooth functions $\mathbb{R} \to \mathbb{R}$. Let $\pi: C^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})$ be the quotient mapping. By 2.5(2) it is a quotient mapping for the natural topologies. Let $f \in C^\infty(\mathbb{R})$ be a function which is infinitely flat at 0. Define $g: \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} x & \text{if } y \geq 0, \\ x+y & \text{if } y \leq 0 \text{ and } x \geq -y \\ 0 & \text{if } y \leq 0 \text{ and } y \leq x \leq -y, \\ x-y & \text{if } y \leq 0 \text{ and } x \leq y. \end{cases}$$

Since f is flat at 0, the mapping $x \to f(g(x,y)) =: f_y(x)$ is smooth for each y, and $y \to f_y$ is a continuous curve in $C^\infty(\mathbb{R})$. But the germ at 0 of f_y equals $\pi(f)$ for $y \geq 0$ and equals 0 for $y < 0$. So the continuous curve $y \to \pi(f_y)$ in $C_0^\infty(\mathbb{R})$ has only two values, $\pi(f)$ and 0. So $\pi(f)$ is a cluster point of 0.

Clearly this proof may be generalized to show the following:

Proposition. *Let M be a (finite-dimensional second-countable) manifold, let $C_x^\infty(M)$ be the C^∞-algebra of germs at $x \in M$ of smooth functions on M. Then any germ which is flat at x is in the closure of 0 in the natural topology. The Hausdorff C^∞-algebra associated to $C_x^\infty(M)$ is the C^∞-algebra of formal power series in $\dim M$ variables.*

Remark. This proposition is also an immediate consequence of Whitney's spectral theorem (see Whitney [14] or Tougeron [16]) which reads as follows:

Theorem. *Let M be a (finite-dimensional second-countable) manifold and let I be an ideal in $C^\infty(M) = C^\infty(M, \mathbb{R})$. Then the closure $\bar{I}$ of I in the compact C^∞-topology (which equals the natural topology) consists of all functions f such that the ∞-jet $j^\infty f(x)$ at x lies in the ideal $j^\infty(I)(x)$ for all $x \in M$.*

6.2. Now let B be a Hausdorff locally-m-convex algebra, which we assume to be C^∞-complete (this is weaker than sequentially complete). We want to determine if there is a C^∞-algebra-structure on B such that the natural topology is finer than the given one. Suppose that this is the case. Then for any $n \in N$ and $b \in B^n$ the mapping $C^\infty(\mathbb{R}^n) \supset \mathrm{Pol}(\mathbb{R}^n, \mathbb{R}) \xrightarrow{\varepsilon_b} B$ is continuous in the topology induced on $\mathrm{Pol}(\mathbb{R}^n, \mathbb{R})$, and this in turn suffices to extend ε_b by continuity to the whole of $C^\infty(\mathbb{R}^n)$. In the following we want to characterize this continuity.

6.3. Theorem (of Markov, Duffin, Shaffer, see Rivlin [14, p. 119]). *If p is a polynomial in one variable of degree n and* $\sup\{|p(x)|: |x| \le 1\} \le 1$, *then for* $-1 \le x \le 1$, $1 \le k \le n$ *we have for the k-th derivative of p*:

$$|p^{(k)}(x)| \le \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdots(2k-1)} = T_n^{(k)}(1),$$

with equality holding only if $p = \pm T_n$, the n-th Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, and $x = \pm 1$.

6.4. Corollary. *For any polynomial p in one variable we have*

$$\sup_{|y|\le M} \max_{i \le k} |p^{(i)}(y)| \le T^{(k)}_{\deg p}(1) \cdot \sup_{|y| \le M} |p(y)|$$

for $M \ge 0$, $k \in \mathbb{N}$, where $\deg p$ is the degree of p.

Note that $T_n^{(k)}(1)$ is increasing with respect to k and n.

6.5. Definition. A C^∞-complete commutative locally-m-convex algebra B with unit is called a *Chebyschev algebra* if the following holds:

For each $b \in B$ and submultiplicative seminorm q on B there are $k \in \mathbb{N}$, $M > 0$, $\delta > 0$ such that

$$p \in \mathrm{Pol}(\mathbb{R}, \mathbb{R}),\ \sup_{|y| \le M} |p(y)| \le \frac{\delta}{T^{(k)}_{\deg p}(1)} \quad \text{implies} \quad q(p(b)) < 1.$$

6.6. Theorem. *Any Chebyshev algebra is a C^∞-algebra such that the natural topology is finer than the given one.*

Proof. $\varepsilon_b: \mathrm{Pol}(\mathbb{R}, \mathbb{R}) \to B$ is linear and continuous for the topology induced by the embedding $\mathrm{Pol}(\mathbb{R}, \mathbb{R}) \subset C^\infty(\mathbb{R})$ onto a dense subspace. Since B is C^∞-complete, ε_b may be extended to a continuous algebra homomorphism $\varepsilon_b: C^\infty(\mathbb{R}) \to B$. Then for $b_1, b_2 \in B$ the bilinear continuous mapping $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to B$, $(f, g) \mapsto \varepsilon_{b_1}(f) \cdot \varepsilon_{b_2}(g)$ extends to a continuous linear mapping

$$C^\infty(\mathbb{R}^2) = C^\infty(\mathbb{R}) \tilde{\otimes}_b C^\infty(\mathbb{R}) \xrightarrow{\varepsilon_{b_1, b_2}} B$$

and in turn to $\varepsilon_b: C^\infty(\mathbb{R}^n, \mathbb{R}) \to B$ for any $b \in B^n$. On the dense subset $\mathrm{Pol}(\mathbb{R}^n, \mathbb{R})$

these mappings have the required properties, so they make B into a C^∞-algebra, whose natural topology is finer than the given one. $\square$

6.7. It is not yet clear in general wether a commutative algebra may carry several different C^∞-algebra structures. Some of them may not:

Lemma. *Let A be a C^∞-algebra consisting of functions on some set S with pointwise multiplication. Then the action of $f\in C^\infty(\mathbb{R})$ on $a\in A$ is by composition, $(Af)(a)=f\circ a$ (i.e., A is a sub-C^∞-algebra of some $\mathbb{R}^S$ provided A is a subalgebra of $\mathbb{R}^S$).*

Proof. If $p\in \mathrm{Pol}(\mathbb{R},\mathbb{R})$, then clearly $p(a)=p\circ a$. Let $s\in S$. Then $\mathrm{ev}_s : A\to\mathbb{R}$ is a multiplicative linear functional. So for $a\in A$ the mapping

$$C^\infty(\mathbb{R},\mathbb{R}) \xrightarrow{\varepsilon_a} A \xrightarrow{\mathrm{ev}_s} \mathbb{R}$$

is a multiplicative linear functional on $C^\infty(\mathbb{R},\mathbb{R})$, so it is of the form ev_x for some $x\in\mathbb{R}$ ('Milnor's exercise').

Testing with the identity in $\mathrm{Pol}(\mathbb{R},\mathbb{R})$ we see that $x=a(s)$, so $(\mathrm{ev}_s\circ\varepsilon_a)(f)=\mathrm{ev}_{a(s)}(f)$, i.e., $(Af)(a)(s)=f(a(s))$ or $(Af)(a)=f\circ a$. $\square$

6.8. Example. Let T be a compactly generated topological space, let $C(T)$ be the algebra of continuous functions on T, with the topology of uniform convergence on compact subsets. Then $C(T)$ is a Chebyschev algebra: For compact $K\subset T$ and $f\in C(T)$ let $f(K)\subseteq[-M,M]$. If $p\in\mathrm{Pol}(\mathbb{R},\mathbb{R})$ and

$$\sup_{|y|\le M} |p(y)| \le \frac{0.9}{T_{\deg p}(1)} < 1,$$

then $|p(f(x))|<1$ for all $x\in K$.

6.9. Example. Let $S(\mathbb{R}^n)$ be the nuclear Fréchet space of rapidly decreasing (in all derivatives separately) functions on $\mathbb{R}^n$. We equip $S(\mathbb{R}^n)$ with a unit, so we consider $\mathbb{R}\oplus S(\mathbb{R}^n)$, the algebra of all functions which decrease rapidly to a constant function. Then $\mathbb{R}\oplus S(\mathbb{R}^n)$ is a C^∞-algebra and even a Chebyshev algebra.

Proof. By 6.6, it suffices to show that $f\circ g\in\mathbb{R}\oplus S(\mathbb{R}^n)$ for $f\in C^\infty(\mathbb{R})$ and $g\in\mathbb{R}\oplus S(\mathbb{R}^n)$. Then $g(x)=\lambda+\bar g(x)$ for $\bar g\in S(\mathbb{R}^n)$, $\lambda\in\mathbb{R}$, and $f(t)=f(\lambda)+h(t-\lambda)$, where $h(0)=0$. Then $f(g(x))=f(\lambda)+h(\bar g(x))$ and it remains to show that $h\circ\bar g\in S(\mathbb{R}^n)$, that is,

$$\lim_{|x|\to\infty} \partial^\alpha(h\circ\bar g)(x)\cdot(1+|x|^2)^k=0 \quad \text{for all } k\in\mathbb{N}.$$

Using the following result we see that this is really the case and that $A\to h\circ\bar g$ is continuous, $\{h\in C^\infty(\mathbb{R}): h(0)=0\}\to S(\mathbb{R}^n)$, so $\mathbb{R}\oplus S(\mathbb{R}^n)$ is a Chebyshev algebra.

Theorem. *Let $f\in C^\infty(\mathbb{R},\mathbb{R})$, $g\in C^\infty(\mathbb{R}^n,\mathbb{R})$. Then for $x\in\mathbb{R}^n$ and each multiindex $\gamma\in\mathbb{N}_0^n$ we have*

$$\partial^\gamma(f\circ g)(x)=\sum_{k\in\mathbb{N}_0} f^{(k)}(g(x))\cdot\sum_{\lambda\in V}\frac{\gamma!}{\lambda!}\cdot\prod_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha>0}}\left(\frac{1}{\alpha!}\right)^{\lambda_\alpha}\cdot\partial^\alpha g(x)$$

$$=\sum_{\lambda\in V'} f^{(\Sigma\lambda_\alpha)}(g(x))\cdot\frac{\gamma!}{\lambda!}\prod_{\alpha>0}\left(\frac{1}{\alpha!}\right)^{\lambda_\alpha}\partial^\alpha g(x)$$

where

$$V=\left\{\lambda:\lambda=(\lambda_\alpha)\in\mathbb{N}_0^{\mathbb{N}_0^n\setminus 0},\ \sum_\alpha\lambda_\alpha=k,\ \sum_\alpha\lambda_\alpha\cdot\alpha=\gamma\right\}$$

$$V'=\left\{\lambda:\lambda=(\lambda_\alpha)\in\mathbb{N}_0^{\mathbb{N}_0^n\setminus 0},\ \sum_\alpha\lambda_\alpha\cdot\alpha=\gamma\right\}$$

The one-dimensional analogue of this formula is folklore under the name Faa di Bruno formula.

6.10. Example (of a Hausdorff C^∞-algebra which is not C^∞-complete). Let E_0 be the space of all sequences $x=(x_n)_{n=1}^\infty$ in $\mathbb{R}$ such that the support $\operatorname{supp}(x)=\{n\in\mathbb{N}: x_n\neq 0\}$ has density 0 in $\mathbb{N}_0$, where

$$\operatorname{density}(A)=\overline{\lim_{n\to\infty}}\frac{\operatorname{cardinality}(A\cap\lceil 0,n\rceil)}{n}.$$

We equip E_0 with a unit, so we consider $E_1=\mathbb{R}\oplus E_0$, the space of all sequences $x\in\mathbb{R}^{\mathbb{N}_0}$ which differ from a constant on a set of density 0. This is an algebra of functions, a subalgebra of $\mathbb{R}^{\mathbb{N}_0}$. For $f\in C^\infty(\mathbb{R}^n)$ and $x^1,\dots,x^n\in E_1$ we easily see that

$$f\circ(x^1,\dots,x^n)=(f(x_0^1,\dots,x_0^n),f(x_1^1,\dots,x_1^n),f(x_2^1,\dots,x_2^n),\dots)$$

differs from the constant $f(c(x^1),\dots,c(x^n))$ at most on the set $\bigcup_{i=1}^n\{k: x_k^i\neq c(x^i)\}$, where $c(x^i)$ is the constant of the sequence x^i. This set is again of density 0, so E_1 is a C^∞-algebra.

We equip E_1 with the trace topology from $\mathbb{R}^{\mathbb{N}_0}$. Then for any set A of density 0 in $\mathbb{N}_0$ the algebra $\mathbb{R}^A\oplus\mathbb{R}$ is a C^∞-subalgebra of E_1 (even an ideal), so by 2.5, we see that $(E_1,n)=\varinjlim_A\mathbb{R}^A\oplus\mathbb{R}$ in the category lcs, where A runs through all subsets of density 0 on $\mathbb{N}_0$ and $\mathbb{R}^A\oplus\mathbb{R}$ has the product topology which coincides with the natural topology. Valdivia [18] showed that $(E_0$, trace topology$)=\varinjlim_A\mathbb{R}^A$. So the natural topology on E_1 equals the trace topology from $\mathbb{R}^{\mathbb{N}_0}$ and we have the following consequences of Valdivia [18].

(1) (E_1,n) is a Hausdorff locally convex space but not C^∞-complete, since the Mackey closure of E_1 is just $\mathbb{R}^{\mathbb{N}_0}$.

(2) (E_1,n) is the locally convex inductive limit of all its finitely generated sub-

algebras, but this limit is not regular, i.e., bounded sets are not bounded in some step, because these subalgebras are nuclear Fréchet spaces (4.2(3)) and E_1 would be complete if the limit were regular.

(3) The mapping $\varphi : E_1 \to \mathbb{R}$, defined by $\varphi(x) = \psi(x - c(x))$, $\psi(y) = \sum_{n=1}^{\infty} n^n y_1 \cdots y_n$, is not smooth, although $\varphi \mid \mathbb{R}^A \oplus \mathbb{R}$ is smooth for any set A of density 0. Furthermore $\varphi \circ \pi : C^\infty_{\text{fin}}(\mathbb{R}^A) \to E_1 \to \mathbb{R}$ is smooth for any quotient mapping π.

Proof. If density $A = 0$, then there is some $k \in \mathbb{N}_0 \backslash A$, such that for $y \in \mathbb{R}^A$, $\psi(y) = \sum_{n=1}^{k-1} n^n y_1 \cdots y_n$, which is obviously smooth. But

$$x^n = \left(\frac{1}{n}, \dots, \frac{1}{n}, 0, 0, \dots\right) \in E_0,$$

$x^n \to 0$ in the sense of Mackey, but $\varphi(x^n) = 1 \not\to 0 = \varphi(0)$. So $\varphi : E_1 \to \mathbb{R}$ is not smooth.

Now $\bar\varphi = \varphi \circ \pi : C^\infty_{\text{fin}}(\mathbb{R}^A) \to E_1 \to \mathbb{R}$ is smooth iff $\bar\varphi \mid C^\infty(\mathbb{R}^F)$ is smooth for finite $F \subset A$. But then $C^\infty(\mathbb{R}^F)/C^\infty(\mathbb{R}^F) \cap \ker \pi =: A_F$ is a finitely generated subalgebra of E_1, so is contained in some $\mathbb{R}^B$ with density $B = 0$, so $\bar\varphi \mid C^\infty(\mathbb{R}^F, \mathbb{R})$ is smooth. □

6.11. Example. Let $R[a,b]$ be the space of real functions on the interval $[a,b]$ which are Riemann integrable. These are exactly the functions g which are bounded, and continuous almost everywhere in the Lebesgue sense. Thus

$$f \circ (g_1, \dots, g_n) \in R[a,b] \quad \text{for } f \in C^\infty(\mathbb{R}^n, \mathbb{R}),\ g_i \in R[a,b],$$

and $R[a,b]$ is a C^∞-algebra with pointwise multiplication.

6.12. Example. Let $C^\omega[a,b]$ be the algebra of real analytic functions on $[a,b]$. Then by 6.7, $C^\omega[a,b]$ is *not* a C^∞-algebra.

6.13. Example. Let $T = S^1$ be the torus. Consider the algebra $A(T) = \{\hat f, f \in l^1(\mathbb{Z})\}$ of functions on T whose Fourier coefficients are in l^1, with pointwise multiplication.

Theorem. (Rudin [15, Chapter 6]). *If $F : [-1,1] \to \mathbb{R}$ and $F \circ f \in A(T)$ for all $f \in A(T)$ with $-1 \le f \le 1$, then F is real analytic in* $[-1,1]$.

So $A(T)$ is not a C^∞-algebra with pointwise multiplication by 6.7, so $l^1(\mathbb{Z})$ is not a C^∞-algebra with convolution.

Likewise the multiplier algebra $C_0M_p(G)$ for the group $G = \mathbb{R}^n, \mathbb{Z}^n, T^n$ is not a C^∞-algebra with the pointwise multiplication. This follows in an analogous way from Zafran [20].

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