Journal of Pure and Applied Algebra 86 (1993) 223–229
North-Holland

On the growth function of direct decompositions associated with homology of free abelianized extensions

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Communicated by K.W. Gruenberg
Received 13 February 1992

Abstract

Kuz'min, Yu.V., On the growth function of direct decompositions associated with homology of free abelianized extensions, Journal of Pure and Applied Algebra 86 (1993) 223–229.

Let G be an arbitrary group given by a free presentation $G = F/N$. We deal with the homology group $H_n(\Phi, \mathbb{Z})$ where $\Phi = F/[N, N]$. It is known that if G has no p-torsion then the p-component of $H_n(\Phi, \mathbb{Z})$ (p odd) has a natural direct decomposition of the form $\bigoplus_k H_{n_k}(G, \mathbb{Z}/p\mathbb{Z})$. The number of direct summands is a function of dimension n. We prove that this function grows faster than n^s for any s but slower than a^n for any $a > 1$. Indeed a more precise asymptotic estimate is given. We also study maximal multiplicity of the group $H_*(G, \mathbb{Z}/p\mathbb{Z})$ in the above decomposition and get information on decomposition of two other periodic groups related to $H_n(\Phi, \mathbb{Z})$.

Let G be an arbitrary group given by a free presentation $G = F/N$. The group $\Phi = F/N'$, where N' is the commutator subgroup, is called a free abelianized extension of G. We briefly recall some results on the homology groups of Φ with trivial coefficients. For a more detailed discussion see [3–5]. The abelian group $M = N/N'$ is a G-module with action coming from conjugation in Φ, and the n-fold exterior power $\wedge^n M$ is a G-module with diagonal action of G. The embedding $M \to \Phi$ induces the corestriction map $H_n(M, \mathbb{Z}) \otimes_G \mathbb{Z} \to H_n(\Phi, \mathbb{Z})$ and, since M is free abelian, it can be written as

$$\bigwedge^n M \otimes_G \mathbb{Z} \to H_n(\Phi, \mathbb{Z}) .$$

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By C_n and K_n denote the cokernel and the kernel of this map and let T_n stand for the torsion subgroup of $H_n(\Phi, \mathbb{Z})$. It was proved in [4] and [5] that C_n, K_n and T_n have finite exponents; moreover, the exponents of $C_n \otimes \mathbb{Z}[\frac{1}{2}]$ and $K_n \otimes \mathbb{Z}[\frac{1}{2}]$ divide $n-1$ while the exponent of $T \otimes \mathbb{Z}[\frac{1}{2}]$ divides n ($\mathbb{Z}[\frac{1}{2}]$ is the ring of 2-rational numbers). To give a description of the odd torsion in C_n, K_n and T_n in terms of the homology groups of G we need some more notation.

For an arbitrary abelian group A and prime number p denote by t_pA the p-component of A. If m is a natural number then by definition $mA = A \oplus \cdots \oplus A$ (with m summands), and $mA = 0$ if $m = 0$. For a polynomial $f(x) = \sum m_k x^k$ with non-negative integral coefficients and any G-module A we set

$$fH_n(G, A) = \bigoplus_k m_k H_{n+k}(G, A) .$$

At last we define polynomials $f_n^{(p)}$ $(n > 1)$ by

$$f_n^{(p)} = \begin{cases} x^2 & \text{if } n = p , \\ 0 & \text{if } n \neq 0,1 \bmod p , \\ x f_{n-1}^{(p)} & \text{if } n = 1 \bmod p , \\ x^2 f_{n-p}^{(p)} + f_{n/p}^{(p)} & \text{if } n = 0 \bmod p \text{ with } n > p . \end{cases} \tag{1}$$

Let p be an odd prime, G a group with no p-torsion, and $n > 1$.

It is proved in [3], that if $n = 1 \bmod p$ then

$$t_pC_n \cong f_{n-1}^{(p)} H_n(G, \mathbb{Z}_p) , \qquad t_pK_n \cong f_{n-1}^{(p)} H_{n+1}(G, \mathbb{Z}_p) \tag{2}$$

and if $n = 0 \bmod p$ then

$$t_pT_n \cong f_n^{(p)} H_n(G, \mathbb{Z}_p) . \tag{3}$$

We remind the reader that for any group G, $t_pC_n = t_pK_n = 0$ if $n \neq 1 \bmod p$ and $t_pT_n = 0$ if $n \neq 0 \bmod p$.

In this note we study the asymptotic behavior of the number of direct summands in the above decompositions (Theorem 1). As a consequence we deduce asymptotic estimates for the maximal multiplicity of the groups $H_*(G, \mathbb{Z}_p)$ in (2) and (3) (Corollary 2). In particular it turns out that these numbers have intermediate growth as functions of the dimension. By definition a function $\gamma(n)$ has intermediate growth if for any natural s, $\lim_{n\to\infty} \gamma(n)/n^s = \infty$, while for any $a > 1$, $\lim_{n\to\infty} \gamma(n)/a^n = 0$. We also calculate the difference between the number of summands of even and odd dimension.

Of course some of the homology groups $H_*(G, \mathbb{Z}_p)$ in the decompositions (2) and (3) may be trivial. This depends on G. However, in general the number of direct summands is evidently

$$f_{n-1}^{(p)}(1) \quad \text{for } t_p C_n, t_p K_n \ (n = 1 \bmod p)\,,$$
$$f_n^{(p)}(1) \quad \text{for } t_p T_n \ (n = 0 \bmod p)\,.$$

Since in both cases p divides the subindex of the polynomial, we consider the following function

$$\gamma(m) = f_{mp}^{(p)}(1) \quad (m = 1, 2, \ldots)\,.$$

From (1) it is clear that $\gamma(1) = 1$ and for $m > 1$

$$\gamma(m) = \begin{cases} \gamma(m-1) & \text{if } m \neq 0{,}1 \bmod p\,, \\ \gamma(m-1) + \gamma((m-1)/p) & \text{if } m = 1 \bmod p\,, \\ \gamma(m-1) + \gamma(m/p) & \text{if } m = 0 \bmod p\,; \end{cases} \tag{4}$$

that is

$$\gamma((k-1)p+1) = \gamma((k-1)p+2) = \cdots = \gamma(kp-1)$$

and

$$\gamma(kp) = \gamma(kp-1) + \gamma(k)\,, \qquad \gamma(kp+1) = \gamma(kp) + \gamma(k)$$

(see Fig. 1).

Functions of this type have been considered before. Let $\omega(n)$ be the number of partitions of the integer n into powers of p. It is easy to prove that

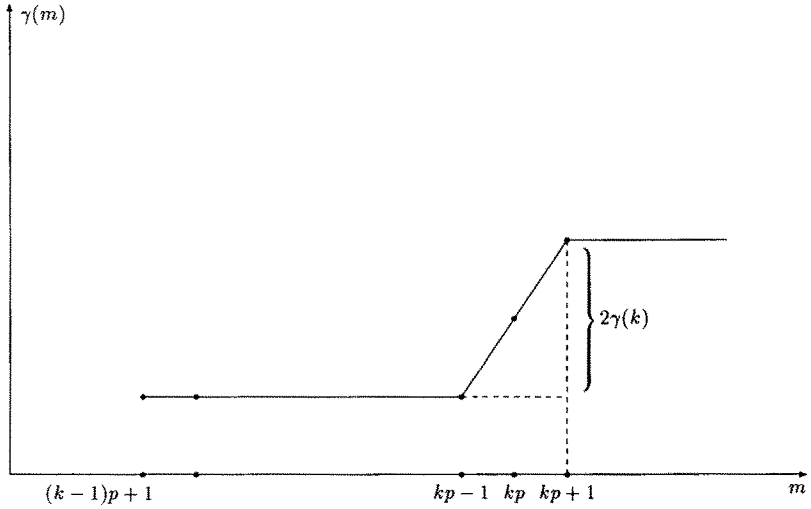

Fig. 1.

$$\omega(n) = \begin{cases} \omega(n-1) & \text{if } p \neq 0 \bmod p\,, \\ \omega(n-1) + \omega(n/p) & \text{if } p = 0 \bmod p\,. \end{cases}$$

For $p = 2$ this function was studied by Euler [1] (see also [2, 6]). Our function $\gamma(n)$ is somewhat different to $\omega(n)$ and although it may be possible to deduce properties of γ from ω, we give direct proofs that are selfcontained and quite short.

The increment of $\gamma(m)$ on each segment $[(i-1)p+1, ip+1]$ is $2\gamma(i)$ ($i = 1, \ldots, k$), hence

$$\gamma(kp+1) = 1 + 2\sum_{i=1}^{k} \gamma(i)\,. \tag{5}$$

It is quite easy to explain why for any natural s, $\lim_{m\to\infty} \gamma(m)/m^s = \infty$. Indeed, by (5) $\gamma(kp+1) \geq 1 + 2k$, therefore $\gamma(m) \geq \varphi_1(m)$ for a linear function $\varphi_1(m)$ (one can take $\varphi_1(m) = 1 + 2(m-1)/p$). Again because of (5)

$$\gamma(kp+1) \geq 1 + 2\sum_{i=1}^{k} \varphi_1(i)\,.$$

The sum $\varphi_1(1) + \cdots + \varphi_1(k)$ is already a polynomial of degree 2. Continuing in this way, we see that there exists a polynomial $\varphi_s(m)$ of degree s such that $\gamma(m) \geq \varphi_s(m)$. Since s is arbitrary, $\lim_{m\to\infty} \gamma(m)/m^s = \infty$.

Now we are going to give a precise estimate. Write $\alpha(m) \prec \beta(m)$ if $\lim_{m\to\infty} \alpha(m)/\beta(m) = 0$.

Theorem 1. *For any* $\varepsilon > 0$,

$$m^{(1/2-\varepsilon)\log_p m} \prec \gamma(m) \prec m^{(1/2)\log_p m}\,.$$

First of all we want to explain where 1/2 comes from. The increment of $\gamma(m)$ on the segment $[(k-1)p+1, kp+1]$ is $2\gamma(k)$. Dividing by p, the length of the segment, one can write

$$(\gamma(kp+1) - \gamma((k-1)p+1))/p = \frac{2}{p}\,\gamma(k)\,.$$

This equality suggests the idea that a continuous model for $\gamma(m)$ should be a function $f(x)$ such that $f'(x) = \lambda f(x/p)$ for some real $\lambda > 0$.

Take for example $f(x) = a^x$ ($a > 1$). Then

$$f'(x)/f(x/p) = \frac{a^x \ln a}{a^{x/p}} = a^{x(1-1/p)} \ln a \to \infty\,,$$

so a^x grows too fast. It is natural to try

$$f(x) = x^{\log_a x} = x^{\alpha \log_p x} .$$

The reader will easily verify that for this choice of $f(x)$,

$$f'(x) = f(x)\,\frac{2\alpha \log_p x}{x}\,, \qquad f(x/p) = f(x)\,\frac{p^{\alpha}}{x^{2\alpha}} .$$

Therefore,

$$\frac{f'(x)}{f(x/p)} = \frac{2\alpha}{p^{\alpha}}\, x^{2\alpha - 1} \log_p x . \tag{6}$$

It follows that

$$\lim_{x\to\infty} \frac{f'(x)}{f(x/p)} = \begin{cases} \infty & \text{if } \alpha \geq 1/2 , \\ 0 & \text{if } \alpha < 1/2 \end{cases}$$

and 1/2 really plays a special role.

Proof of Theorem 1. Consider the upper bound $\gamma(m) < f(m) = m^{(1/2)\log_p m}$. We shall compare the increments of $f(m)$ and $\gamma(m)$ on the segment $[(k-1)p+1, kp+1]$. By the Mean Value Theorem

$$\Delta f = f(kp+1) - f((k-1)p+1) = f'(x)p$$

for some $x \in [(k-1)p+1, kp+1]$. Because of (6), with $\alpha = 1/2$,

$$\Delta f = f(x/p)\,\frac{\log_p x}{\sqrt{p}}\, p > f(k-1)\sqrt{p}\,\log_p (k-1)p .$$

On the other hand,

$$\Delta\gamma = \gamma(kp+1) - \gamma((k-1)p+1) = 2\gamma(k) < 4\gamma(k-1) ,$$

so

$$\frac{\Delta\gamma}{\Delta f} < \frac{4}{\sqrt{p}\,\log_p (k-1)p}\,\frac{\gamma(k-)}{f(k-1)} < \frac{1}{2}\,\frac{\gamma(k-1)}{f(k-1)}$$

if $k > p^5 + 1$. We can assume by induction that $\gamma(k-1) < f(k-1)$. There is no problem with the base of induction because one can replace $f(x)$ by $Cf(x)$ if necessary ($C \in \mathbb{R}$). It does not change the asymptotic behaviour and also (6) remains valid. Thus $\Delta\gamma < \frac{1}{2}\Delta f$. Evidently it is sufficient to conclude that $\gamma(m) < f(m)$. Similar arguments prove that $m^{\alpha \log_p m} < \gamma(m)$ if $\alpha < 1/2$. □

The multiplicities of the groups $H_*(G, \mathbb{Z}_p)$ in decompositions (2) and (3) are just coefficients of the polynomials $f^{(p)}_{mp}$. Hence the following corollary gives information on the maximal multiplicities.

Corollary 2. *By $\mu(m)$ denote the maximal coefficient of $f^{(p)}_{mp}$. Then for any $\varepsilon > 0$,*

$$m^{(1/2-\varepsilon)\log_p m} < \mu(m) < m^{(1/2)\log_p m}.$$

Proof. The upper bound is clear because $\mu(m) \le \gamma(m)$. It is easily proved by induction that $f^{(p)}_{mp}$ has degree $2m$. If $\varepsilon > 0$ then

$$\begin{aligned}\mu(m) &\ge \gamma(m)/2m > m^{(1/2-\varepsilon/2)\log_p m}/m \\ &= m^{(1/2-\varepsilon)\log_p m} \cdot \frac{m^{(\varepsilon/2)\log_p m}}{m} \\ &> m^{(1/2-\varepsilon)\log_p m}. \quad \square\end{aligned}$$

Corollary 3. *The functions $\gamma(m)$ and $\mu(m)$ have intermediate growth.* $\square$

Denote by $\|n\|_p$ the p-adic norm, that is, the maximal i such that p^i divides n.

Proposition 4. *Suppose that $n = 0 \mod p$ and let $t = \|n\|_p$, $\rho^{(p)}_n = f^{(p)}_n - \sum_{i=1}^{t} x^{2n/p^i}$. Then $\rho^{(p)}_n(-1) = 0$.*

Proof. For $n = p$ we have $\rho^{(p)}_n = x^2 - x^2 = 0$. For $n > p$ $f^{(p)}_n = x^2 f^{(p)}_{n-p} + f^{(p)}_{n/p}$ and we shall use induction on n. If $n/p \ne 0,1 \mod p$ then $f^{(p)}_{n/p} = 0$, so $f^{(p)}_n = x^2 f^{(p)}_{n-p}$. In this case $\|n\|_p = \|n-p\|_p = 1$ and

$$\rho^{(p)}_n(-1) = f^{(p)}_n(-1) - 1 = f^{(p)}_{n-p}(-1) - 1 = \rho^{(p)}_{n-p}(-1) = 0.$$

If $n/p = 0 \mod p$ then $\|n-p\| = 1$, therefore

$$\begin{aligned}\rho^{(p)}_n(-1) &= f^{(p)}_n(-1) - t = (f^{(p)}_{n-p}(-1) + f^{(p)}_{n/p}(-1)) - t \\ &= (f^{(p)}_{n-p}(-1) - 1) + (f^{(p)}_{n/p} - (t-1)) \\ &= \rho^{(p)}_{n-p}(-1) + \rho^{(p)}_{n/p}(-1) = 0.\end{aligned}$$

At last if $n/p = 1 \mod p$, then $\|n\|_p = 1$, $\|n-p\|_p = \|n/p - 1\|_p$, and $f^{(p)}_n = x^2 f^{(p)}_{n-p} + x f^{(p)}_{n/p+1}$ hence

$$\begin{aligned}\rho^{(p)}_n(-1) &= f^{(p)}_n(-1) - 1 = (f^{(p)}_{n-p}(-1) - f^{(p)}_{n/p-1}(-1) - 1 \\ &= \rho^{(p)}_{n-p}(-1) - \rho^{(p)}_{n/p-1}(-1) = 0. \quad \square\end{aligned}$$

Denote by $h^+(C_n)$ and $h^-(C_n)$ the number of summands of even and odd dimension in decomposition (2) for C_n, and let $h^{\pm}(K_n)$ and $h^{\pm}(T_n)$ be these numbers for K_n and T_n, respectively. The following corollary is an obvious consequence of decompositions (2), (3) and Proposition 4.

Corollary 5. *If* $n = 1 \bmod p$ *then*

$$h^+(C_n) - h^-(C_n) = (-1)^n \|n-1\|_p ,$$
$$h^+(K_n) - h^-(K_n) = (-1)^{n+1} \|n-1\|_p .$$

If $n = 0 \bmod p$ *then*

$$h^+(T_n) - h^-(T_n) = (-1)^n \|n\|_p . \qquad \square$$

By Proposition 4 $1 + x$ divides $\rho_n^{(p)}$ ($n = 0 \bmod p$). It means that summands of $\rho_n^{(p)}$ appear in pairs of even and odd degree; in particular, maximal coefficients of even and odd degree are equal and both have intermediate growth.

In conclusion we write down polynomials $f_n^{(p)}$ for $n \leq p^3$ ($n = 0 \bmod p$).

$$f_{r_1 p}^{(p)} = x^{2r_1} \quad (1 \leq r_1 \leq p-1),$$

$$f_{r_2 p}^{(p)} = x^{2r_2 p} + x^{2r_2} + (1+x) \sum_{r=1}^{r_2 - 1} x^{2((r_2 - r_1)p + r) - 1}$$
$$(1 \leq r_2 \leq p-1),$$

$$f_{r_2 p^2 + r_1 p}^{(p)} = x^{2(r_2 p + r_1)} + (1+x) \sum_{r=1}^{r_2} x^{2((r_2 - r)p + r_1 + r) - 1}$$
$$(1 \leq r_1, r_2 \leq p-1),$$

$$f_{p^3}^{(p)} = x^{2p^2} + x^{2p} + x^2 + (1+x) \sum_{r=1}^{p-1} x^{2((p-r)p + r) - 1} .$$

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