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Several identities for the generalized Apostol–Bernoulli polynomials

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ABSTRACT

The purpose of this paper is to give several symmetric identities on the generalized Apostol–Bernoulli polynomials by applying the generating functions. These results extend some known identities.

1. Introduction

For a real or complex parameter α, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$, are defined by means of the following exponential generating function (see [1–4]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < 2\pi). \tag{1}$$

The classical Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n are

$$B_n(x) = B_n^{(1)}(x), \qquad B_n = B_n(0),$$

respectively.

As a natural generalization of the Bernoulli polynomials and numbers, the Apostol–Bernoulli polynomials and numbers were first defined by Apostol [5] when he studied the Lipschitz–Lerch zeta functions. Luo and Srivastava introduced the generalized Apostol–Bernoulli polynomials which are defined as follows (see [6–9]).

Definition 1.1. For arbitrary real or complex parameters α and λ, the generalized Apostol–Bernoulli polynomials $\mathscr{B}_n^{(\alpha)}(x;\lambda)$ are defined by the following generating functions:

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathscr{B}_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} \quad (|t+\log\lambda| < 2\pi). \tag{2}$$

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The Apostol–Bernoulli polynomials $\mathscr{B}_n(x;\lambda)$ and the Apostol–Bernoulli numbers $\mathscr{B}_n(\lambda)$ are given by

$$\mathscr{B}_n(x;\lambda)=\mathscr{B}_n^{(1)}(x;\lambda),\qquad \mathscr{B}_n(\lambda)=\mathscr{B}_n(0;\lambda)\quad (n\in\mathbb{N}_0), \tag{3}$$

respectively.

The Bernoulli polynomials and numbers have numerous important applications in combinatorics, number theory and numerical analysis. As imitations of important properties of the Bernoulli polynomials and numbers, Luo and Srivastava studied systematically these polynomials [6–9]. Recently, Wang, Jia and Wang [10] also established two relationships between the generalized Apostol–Bernoulli and Apostol–Euler polynomials.

For each integer $k\geq 0, S_k(n)=\sum_{i=0}^{n} i^k$ is called sum of integer powers, or simply power sum. The exponential generating function for $S_k(n)$ is

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!}=1+\mathrm{e}^t+\mathrm{e}^{2t}+\cdots+\mathrm{e}^{nt}=\frac{\mathrm{e}^{(n+1)t}-1}{\mathrm{e}^t-1}. \tag{4}$$

Now, we define the generalized sum of integer powers as follows:

Definition 1.2. For an arbitrary real or complex parameter λ, the generalized sum of integer powers $\mathscr{S}_k(n;\lambda)$ is defined by the following generating functions:

$$\sum_{k=0}^{\infty} \mathscr{S}_k(n;\lambda)\frac{t^k}{k!}=\frac{\lambda\mathrm{e}^{(n+1)t}-1}{\lambda\mathrm{e}^t-1}. \tag{5}$$

It is easy to see that

$$\mathscr{S}_k(n;1)=S_k(n).$$

Similarly, for each integer $k\geq 0$, $M_k(n)=\sum_{i=0}^{n}(-1)^i i^k$ is called sum of alternative integer powers. The exponential generating function for $M_k(n)$ is

$$\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!}=1-\mathrm{e}^t+\mathrm{e}^{2t}+\cdots+(-1)^n\mathrm{e}^{nt}=\frac{1-(-\mathrm{e}^t)^{(n+1)}}{\mathrm{e}^t+1}, \tag{6}$$

and we can define the generalized sum of alternative integer powers as follows:

Definition 1.3. For an arbitrary real or complex parameter λ, the generalized sum of alternative integer powers $\mathscr{S}_k(n;\lambda)$ is defined by the following generating functions:

$$\sum_{k=0}^{\infty} \mathscr{M}_k(n;\lambda)\frac{t^k}{k!}=\frac{1-\lambda(-\mathrm{e}^t)^{(n+1)}}{\lambda\mathrm{e}^t+1}. \tag{7}$$

It is easy to see that

$$\mathscr{M}_k(n;1)=M_k(n).$$

2. Some symmetric identities on the Apostol–Bernoulli polynomials

In 2006, Garg, Jain and Srivastava [11] derived an explicit representation of these generalized Apostol–Bernoulli polynomials and proceeded to establishing a functional relationship between the generalized Apostol–Bernoulli polynomials and the Hurwitz zeta function. Following closely, Lin, Srivastava and Wang [12] presented a systematic investigation of expansion and transformation formulas for several general families of the Hurwitz–Lerch zeta functions.

The purpose of this paper is to give several symmetric identities on the generalized Apostol–Bernoulli polynomials by applying the generating functions. These results extend some known identities in [13–16].

Theorem 2.1. *For all integers $a>0$, $b>0$, and $n\geq 0$, $m\geq 1$, $\lambda\in\mathbb{C}$, we have the following identity:*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\mathscr{B}_{n-k}^{(m)}(bx;\lambda)\sum_{i=0}^{k}\binom{k}{i}\mathscr{S}_i(a-1;\lambda)\mathscr{B}_{k-i}^{(m-1)}(ay;\lambda)\\ &\quad=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k+1}\mathscr{B}_{n-k}^{(m)}(ax;\lambda)\sum_{i=0}^{k}\binom{k}{i}\mathscr{S}_i(b-1;\lambda)\mathscr{B}_{k-i}^{(m-1)}(by;\lambda).\end{aligned} \tag{8}$$

Proof. Let $g(t) = t^{2m-1}e^{abxt}(\lambda e^{abt} - 1)e^{abyt}/(\lambda e^{at} - 1)^m(\lambda e^{bt} - 1)^m$. Note that this expression for $g(t)$ is symmetric in a and b. In order to prove the theorem we expanded $g(t)$ into series in two ways.

$$\begin{aligned} g(t) &= \frac{t^{2m-1}e^{abxt}(\lambda e^{abt} - 1)e^{abyt}}{(\lambda e^{at} - 1)^m(\lambda e^{bt} - 1)^m} \\ &= \frac{1}{a^m b^{m-1}}\left(\frac{at}{\lambda e^{at} - 1}\right)^m e^{abxt}\left(\frac{\lambda e^{abt} - 1}{\lambda e^{bt} - 1}\right)\left(\frac{bt}{\lambda e^{bt} - 1}\right)^{m-1} e^{abyt} \\ &= \frac{1}{a^m b^{m-1}}\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(m)}(bx;\lambda)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{S}_n(a-1;\lambda)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(m-1)}(ay;\lambda)\frac{(bt)^n}{n!}\right) \\ &= \frac{1}{a^m b^m}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\mathscr{B}_{n-k}^{(m)}(bx;\lambda)\sum_{i=0}^{k}\binom{k}{i}\mathscr{S}_i(a-1;\lambda)\mathscr{B}_{k-i}^{(m-1)}(ay;\lambda)\right)\frac{t^n}{n!}. \end{aligned}$$

Using a similar plan, we have

$$g(t) = \frac{1}{a^m b^m}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k+1}\mathscr{B}_{n-k}^{(m)}(ax;\lambda)\sum_{i=0}^{k}\binom{k}{i}\mathscr{S}_i(b-1;\lambda)\mathscr{B}_{k-i}^{(m-1)}(by;\lambda)\right)\frac{t^n}{n!}.$$

Equating coefficients of $(t^n/n!)$ on the right-hand sides of the last two equations gives us (8). □

By setting $\lambda = 1$ in Theorem 2.1, we have a common special case of the identity which is one of the main results of Yang [16, Eqs. (9)]:

Corollary 2.2. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $m \geq 1$,*

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}B_{n-k}^{(m)}(bx)\sum_{i=0}^{k}\binom{k}{i}S_i(a-1)B_{k-i}^{(m-1)}(ay) \\ &\quad = \sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k+1}B_{n-k}^{(m)}(ax)\sum_{i=0}^{k}\binom{k}{i}S_i(b-1)B_{k-i}^{(m-1)}(by). \end{aligned} \tag{9}$$

Setting $y = 0$ and $m = 1$ in Theorem 2.1, we obtain the relation:

Corollary 2.3. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $\lambda \in \mathbb{C}$,*

$$\sum_{i=0}^{n}\binom{n}{i}a^{i-1}b^{n-i}\mathscr{B}_i(bx;\lambda)\mathscr{S}_{n-i}(a-1;\lambda) = \sum_{i=0}^{n}\binom{n}{i}b^{i-1}a^{n-i}\mathscr{B}_i(ax;\lambda)\mathscr{S}_{n-i}(b-1;\lambda). \tag{10}$$

Setting $x = 0$ in (10), we have the relation:

Corollary 2.4. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $\lambda \in \mathbb{C}$, we have the following relation:*

$$\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k(\lambda)a^{k-1}b^{n-k}\mathscr{S}_{n-k}(a-1;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k(\lambda)b^{k-1}a^{n-k}\mathscr{S}_{n-k}(b-1;\lambda). \tag{11}$$

When $\lambda = 1$ in (11), we get the relation of symmetry between the power sum polynomials and the Bernoulli polynomials in Tuenter [14]. It is given by

$$\sum_{k=0}^{n}\binom{n}{k}a^{k-1}B_k b^{n-k}S_{n-k}(a-1) = \sum_{k=0}^{n}\binom{n}{k}b^{k-1}B_k a^{n-k}S_{n-k}(b-1). \tag{12}$$

Setting $b = 1$ in (10), we have:

$$\mathscr{B}_n(ax;\lambda) = \sum_{i=0}^{n}a^{i-1}\binom{n}{i}\mathscr{B}_i(x;\lambda)\mathscr{S}_{n-i}(a-1;\lambda). \tag{13}$$

On the one hand, by setting $\lambda = 1$ in (13), a common special case of the identity is

$$B_n(ax) = \sum_{i=0}^{n}a^{i-1}\binom{n}{i}B_i(x)S_{n-i}(a-1), \tag{14}$$

which can be found in Yang [16, Eqs.(11)].

On the other hand, by setting $x = 0$ in (13), we have a recurrence on the Apostol–Bernoulli polynomials:

Corollary 2.5. *For $\lambda \in \mathbb{C}$ and $n \in N_0$, the Apostol–Bernoulli numbers $\mathscr{B}_n(\lambda)$ satisfy*

$$\mathscr{B}_n(\lambda) = \sum_{i=0}^{n} \binom{n}{i} a^{i-1} \mathscr{B}_i(\lambda) \mathscr{S}_{n-i}(a-1;\lambda). \tag{15}$$

Corollary 2.6. *For any positive integer n and any positive integer $a > 1$, $\lambda \in \mathbb{C}$,*

$$\mathscr{B}_n(\lambda) = \frac{1}{a\left(1 - a^{n-1}\sum\limits_{k=0}^{a-1}\lambda^k\right)} \sum_{i=0}^{n-1} \binom{n}{i} a^i \mathscr{B}_i(\lambda) \mathscr{S}_{n-i}(a-1;\lambda). \tag{16}$$

Proof. Since $\mathscr{S}_0(a-1;\lambda) = \sum_{k=0}^{a-1}\lambda^k$, by simple computing in (15), we have the recurrence relation. □

Setting $\lambda = 1$ in Corollary 2.6, we have

$$B_n = \frac{1}{a(1-a^n)} \sum_{i=0}^{n-1} a^i \binom{n}{i} B_i S_{n-i}(a-1). \tag{17}$$

Recurrence relation (17) for the Bernoulli numbers have been proved in Deeba and Rodriguez [13] and Gessel [17].

Theorem 2.7. *For each pair of positive integers a and b, and all integers $n \geq 0$, $\lambda \in \mathbb{C}$ and $m \geq 1$, we have the following identity:*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} \lambda^{i+j} a^k b^{n-k} \mathscr{B}_k^{(m)}\left(bx + \frac{b}{a}i;\lambda\right) \mathscr{B}_{n-k}^{(m)}\left(ay + \frac{a}{b}j;\lambda\right)\\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1}\sum_{j=0}^{a-1} \lambda^{i+j} b^k a^{n-k} \mathscr{B}_k^{(m)}\left(ax + \frac{a}{b}i;\lambda\right) \mathscr{B}_{n-k}^{(m)}\left(by + \frac{b}{a}j;\lambda\right).
\end{aligned} \tag{18}$$

Proof. Let $h(t) = t^{2m} e^{abxt}(\lambda^a e^{abt} - 1)(\lambda^b e^{abt} - 1)e^{abyt}/(\lambda e^{at} - 1)^{m+1}(\lambda e^{bt} - 1)^{m+1}$, then, the expression for $h(t)$ is symmetric in a and b, and we can expand $h(t)$ into series in two ways to prove the theorem.

$$\begin{aligned}
h(t) &= \frac{t^{2m} e^{abxt}(\lambda^a e^{abt} - 1)(\lambda^b e^{abt} - 1)e^{abyt}}{(\lambda e^{at} - 1)^{m+1}(\lambda e^{bt} - 1)^{m+1}}\\
&= \frac{1}{a^m b^m}\left(\frac{at}{\lambda e^{at} - 1}\right)^m e^{abxt}\left(\frac{\lambda^a e^{abt} - 1}{\lambda e^{bt} - 1}\right)\left(\frac{bt}{\lambda e^{bt} - 1}\right)^m e^{abyt}\left(\frac{\lambda^b e^{abt} - 1}{\lambda e^{at} - 1}\right)\\
&= \frac{1}{a^m b^m}\left(\frac{at}{\lambda e^{at} - 1}\right)^m e^{abxt}\sum_{i=0}^{a-1}\lambda^i e^{bti}\left(\frac{bt}{\lambda e^{bt} - 1}\right)^m e^{abyt}\sum_{j=0}^{b-1}\lambda^j e^{atj}\\
&= \frac{1}{a^m b^m}\sum_{i=0}^{a-1}\lambda^i\left(\frac{at}{\lambda e^{at} - 1}\right)^m e^{\left(bx+\frac{b}{a}i\right)at}\sum_{j=0}^{b-1}\lambda^j\left(\frac{bt}{\lambda e^{bt} - 1}\right)^m e^{\left(ay+\frac{a}{b}j\right)bt}\\
&= \frac{1}{a^m b^m}\left(\sum_{i=0}^{a-1}\lambda^i\sum_{n=0}^{\infty}\mathscr{B}_n^{(m)}\left(bx + \frac{b}{a}i;\lambda\right)\frac{(at)^n}{n!}\right)\left(\sum_{j=0}^{b-1}\lambda^j\sum_{n=0}^{\infty}\mathscr{B}_n^{(m)}\left(ay + \frac{a}{b}j;\lambda\right)\frac{(bt)^n}{n!}\right)\\
&= \frac{1}{a^m b^m}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}a^k b^{n-k}\mathscr{B}_k^{(m)}\left(bx + \frac{b}{a}i;\lambda\right)\mathscr{B}_{n-k}^{(m)}\left(ay + \frac{a}{b}j;\lambda\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Separately,

$$\begin{aligned}
h(t) &= \frac{t^{2m} e^{abxt}(\lambda^a e^{abt} - 1)(\lambda^b e^{abt} - 1)e^{abyt}}{(\lambda e^{at} - 1)^{m+1}(\lambda e^{bt} - 1)^{m+1}}\\
&= \frac{1}{a^m b^m}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\lambda^{i+j}b^k a^{n-k}\mathscr{B}_k^{(m)}\left(ax + \frac{a}{b}i;\lambda\right)\mathscr{B}_{n-k}^{(m)}\left(by + \frac{b}{a}j;\lambda\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $t^n/n!$ on the right-hand sides of the last two equations we arrive at the desired results (18). □

By setting $\lambda = 1$ in Theorem 2.7, we have a common special case of the identity:

Corollary 2.8. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $m \geq 1$,*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}a^k b^{n-k}B_k^{(m)}\left(bx+\frac{b}{a}i\right)B_{n-k}^{(m)}\left(ay+\frac{a}{b}i\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}b^k a^{n-k}B_k^{(m)}\left(ax+\frac{a}{b}i\right)B_{n-k}^{(m)}\left(by+\frac{b}{a}j\right). \tag{19}$$

Setting $y = 0$, $m = 1$ in Theorem 2.7, we have:

Corollary 2.9. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $\lambda \in \mathbb{C}$,*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}a^k b^{n-k}\mathscr{B}_k\left(bx+\frac{b}{a}i;\lambda\right)\mathscr{B}_{n-k}\left(\frac{a}{b}j;\lambda\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\lambda^{i+j}b^k a^{n-k}\mathscr{B}_k\left(ax+\frac{a}{b}i;\lambda\right)\mathscr{B}_{n-k}\left(\frac{b}{a}j;\lambda\right). \tag{20}$$

When $b = 1$ in (20), we have the relationship:

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\lambda^i a^k\mathscr{B}_k\left(x+\frac{i}{a};\lambda\right)\mathscr{B}_{n-k}(\lambda)=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\lambda^j a^{n-k}\mathscr{B}_k(ax;\lambda)\mathscr{B}_{n-k}(\frac{j}{a};\lambda). \tag{21}$$

Substituting $\lambda = 1$ in (21), we have the relationship:

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}a^k B_k\left(x+\frac{i}{a}\right)B_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}a^{n-k}B_k(ax)B_{n-k}\left(\frac{j}{a}\right). \tag{22}$$

Theorem 2.10. *For each pair of positive integers a and b, and all integers $n \geq 0$, $\lambda \in \mathbb{C}$ and $m \geq 1$, we have the following identity:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}a^k b^{n-k}\mathscr{B}_k^{(m)}\left(bx+\frac{b}{a}i+j;\lambda\right)\mathscr{B}_{n-k}^{(m)}(ay;\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}\lambda^{i+j}b^k a^{n-k}\mathscr{B}_k^{(m)}\left(ax+\frac{a}{b}i+j;\lambda\right)\mathscr{B}_{n-k}^{(m)}(by;\lambda). \tag{23}$$

Proof. The proof is analogous to Theorem 2.7, but we need to change the order of the summation of series. On the one hand,

$$\begin{aligned}
h(t) &= \frac{t^{2m}e^{abxt}(\lambda^a e^{abt}-1)(\lambda^b e^{abt}-1)e^{abyt}}{(\lambda e^{at}-1)^{m+1}(\lambda e^{bt}-1)^{m+1}}\\
&= \frac{1}{a^m b^m}\left(\frac{at}{\lambda e^{at}-1}\right)^m e^{abxt}\left(\frac{\lambda^a e^{abt}-1}{\lambda e^{bt}-1}\right)\left(\frac{bt}{\lambda e^{bt}-1}\right)^m e^{abyt}\left(\frac{\lambda^b e^{abt}-1}{\lambda e^{at}-1}\right)\\
&= \frac{1}{a^m b^m}\left(\frac{at}{\lambda e^{at}-1}\right)^m e^{abxt}\sum_{i=0}^{a-1}\lambda^i e^{bti}\left(\frac{bt}{\lambda e^{bt}-1}\right)^m e^{abyt}\sum_{j=0}^{b-1}\lambda^j e^{atj}\\
&= \frac{1}{a^m b^m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\left(\frac{at}{\lambda e^{at}-1}\right)^m e^{\left(bx+\frac{b}{a}i+j\right)at}\left(\frac{bt}{\lambda e^{bt}-1}\right)^m e^{(abyt)}\\
&= \frac{1}{a^m b^m}\left(\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}\sum_{n=0}^{\infty}\mathscr{B}_n^{(m)}\left(bx+\frac{b}{a}i+j;\lambda\right)\frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_n^{(m)}(ay;\lambda)\frac{(bt)^n}{n!}\right)\\
&= \frac{1}{a^m b^m}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\lambda^{i+j}a^k b^{n-k}\mathscr{B}_k^{(m)}\left(bx+\frac{b}{a}i+j;\lambda\right)\mathscr{B}_{n-k}^{(m)}(ay;\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

On the other hand,

$$h(t) = \frac{t^{2m} e^{abxt} (\lambda^a e^{abt} - 1)(\lambda^b e^{abt} - 1) e^{abyt}}{(\lambda e^{at} - 1)^{m+1} (\lambda e^{bt} - 1)^{m+1}}$$
$$= \frac{1}{a^m b^m} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} \lambda^{i+j} b^k a^{n-k} \mathscr{B}_k^{(m)} \left(ax + \frac{a}{b} i + j; \lambda \right) \mathscr{B}_{n-k}^{(m)} (by; \lambda) \right) \frac{t^n}{n!}.$$

Equating coefficients of $(t^n/n!)$ on the right-hand sides of the last two equations gives us (23). □

By setting $\lambda = 1$ in Theorem 2.10, we have a common special case of the identity:

Corollary 2.11. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $m \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^k b^{n-k} B_k^{(m)} \left(bx + \frac{b}{a} i + j \right) B_{n-k}^{(m)} (ay) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} b^k a^{n-k} B_k^{(m)} \left(ax + \frac{a}{b} i + j \right) B_{n-k}^{(m)} (by). \quad (24)$$

Setting $y = 0$, $m = 1$ in Theorem 2.10, we have:

Corollary 2.12. *For all integers $a > 0$, $b > 0$, and $n \geq 0$, $\lambda \in \mathbb{C}$,*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} \lambda^{i+j} a^k b^{n-k} \mathscr{B}_k \left(bx + \frac{b}{a} i + j; \lambda \right) \mathscr{B}_{n-k}(\lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} \lambda^{i+j} b^k a^{n-k} \mathscr{B}_k \left(ax + \frac{a}{b} i + j; \lambda \right) \mathscr{B}_{n-k}(\lambda). \quad (25)$$

When $b = 1$ in (25), we have the relationship:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \lambda^i a^k \mathscr{B}_k \left(x + \frac{i}{a}; \lambda \right) \mathscr{B}_{n-k}(\lambda) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} \lambda^j a^{n-k} \mathscr{B}_k (ax + j; \lambda) \mathscr{B}_{n-k}(\lambda). \quad (26)$$

Substituting $\lambda = 1$ in (26), we have the relationship:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} a^k B_k \left(x + \frac{i}{a} \right) B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} a^k B_{n-k} (ax + j) B_k. \quad (27)$$

3. A remark

For arbitrary real or complex parameters α and λ, the generalized Apostol–Euler polynomials $\mathscr{E}_n^{(\alpha)}(x; \lambda)$ are defined by the following generating functions: $\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathscr{E}_n^{(\alpha)}(x, \lambda) \frac{t^n}{n!}$ (see [7,8,10]). We can also establish the similar symmetric identities for the generalized Apostol–Euler polynomials. However, the condition is that the integers a and b are both even integers or odd integers. For example, for each pair of positive even integers a and b, or each pair of positive odd integers a and b, and all integers $n \geq 0$, $\lambda \in \mathbb{C}$ and $m \geq 1$, completely analogous to the proof of Theorem 2.1, we can establish the following relation between the generalized Apostol–Euler polynomials and Definition 1.3:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \mathscr{E}_{n-k}^{(m)}(bx; \lambda) \sum_{i=0}^{k} \binom{k}{i} \mathscr{M}_i(a-1; \lambda) \mathscr{E}_{k-i}^{(m-1)}(ay; \lambda)$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k+1} \mathscr{E}_{n-k}^{(m)}(ax; \lambda) \sum_{i=0}^{k} \binom{k}{i} \mathscr{M}_i(b-1; \lambda) \mathscr{E}_{k-i}^{(m-1)}(by; \lambda).$$

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