Note
The generating function of irreducible coverings by edges of complete $k$-partite graphs

Virgil Domocoș*, Ş. N. Buzeanu

Faculty of Mathematics, University of Bucharest, P.O. Box 13-115, Bucharest, Romania

Received 28 January 1994

Abstract
A generalization of a recent result of Tomescu (1993) is presented. The method is purely combinatorial and is based on the theory of species of several variables.

1. Introduction

Our goal is to present an application of the theory of species in the enumeration of a certain class of $k$-partite graphs. More precisely, we shall use some operations on species of several variables in order to obtain the exponential generating function of the number of irreducible coverings by edges of the complete $k$-partite graphs.

Let us first introduce the necessary definitions related to the combinatorial object that we shall enumerate.

We shall denote by $[k]$ the set $\{1, \ldots, k\}$. We denote by $K(V_1, \ldots, V_k)$ the complete $k$-partite graph having the set $V$ of vertices partitioned in non-empty classes $V_1, \ldots, V_k$.

Definition 1.1. A subgraph $C = (V_1, \ldots, V_k; \mathcal{A})$ of $K(V_1, \ldots, V_k)$, (where $\mathcal{A} \subseteq E(K(V_1, \ldots, V_k))$), is an irreducible covering by edges of the graph $K(V_1, \ldots, V_k)$ if the following conditions hold:

(i) $\forall v \in \bigcup_{i \in [k]} V_i$, $\deg_C v \geq 1$;

(ii) $\forall e \in \mathcal{A}$, $\exists v \in \bigcup_{i \in [k]} V_i$, such that $\deg_{C-\{e\}} v = 0$.

* Corresponding author.

0012-365X/95/$09.50 © 1995—Elsevier Science B.V. All rights reserved
SSDI 0012-365X(94)00176-6
(we have denoted by $C - \{e\}$ the subgraph $(V_1, \ldots, V_k; \mathcal{A} - \{e\})$ of $K(V_1, \ldots, V_k)$). If $C$ verifies only the condition (i) above, we say that it is a covering by edges of $K(V_1, \ldots, V_k)$.

For the sake of brevity, we shall simply call a (irreducible) covering by edges of a complete $k$-partite graph as a (irreducible) covering.

Let $N(n_1, \ldots, n_k)$ denote the number of irreducible coverings of a complete $k$-partite graph $K(V_1, \ldots, V_k)$ having $n_i$ vertices in the class $V_i$, for all $i \in [k]$. We shall determine the exponential generating function $F(x_1, \ldots, x_k)$ defined by:

$$F(x_1, \ldots, x_k) = \sum_{n_i > 1} N(n_1, \ldots, n_k) \cdot \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}. \quad (1)$$

This function was recently determined by Tomescu [2] for the particular case $k = 2$. He used some recurrence relations which hold between the numbers $N(n_1, n_2)$ in order to obtain a partial differential equation verified by $F(x_1, x_2)$. Solving this equation for the natural initial conditions, he obtained the expression

$$F(x_1, x_2) = \exp(x_1 e^{x_2} + x_2 e^{x_1} - x_1 - x_2 - x_1 x_2) - 1. \quad (2)$$

We shall use here another method which is more combinatorial. In the spirit of many applications of the theory of species, we shall split the objects we are counting (irreducible coverings) into a (ordered) sequence of simpler objects (graphs of some simpler types).

2. A decomposition of irreducible coverings

The following obvious property characterizes the irreducible coverings of $K(V_1, \ldots, V_k)$.

**Lemma 2.1.** A subgraph $C = (V_1, \ldots, V_k; \mathcal{A})$ of $K(V_1, \ldots, V_k)$ is an irreducible covering if and only if:

(i) $C$ is a covering of $K(V_1, \ldots, V_k)$ and

(ii) every edge from $\mathcal{A}$ has at least one extremity of degree 1.

Based on this lemma we can observe that all connected components of an irreducible covering are stars (with at least two rays) or edges.

We now fix an arbitrary irreducible covering that we denote by $C$. For every $i \in [k]$, we define the subgraph $S_i$ to be the union of those connected components of $C$ which are stars with center in $V_i$. If there are no such stars, we denote by $S_i$ the empty graph. Then, for every $1 \leq i < j \leq k$, we define the subgraph $B_{i,j}$ to be the union of all isolated edges of $C$ which have the endpoints in $V_i$ and $V_j$ respectively. As above, if there are no such edges in $C$, we denote by $B_{i,j}$ the empty graph.
The sequence of subgraphs \( \tau(C) = ((S_i)_{i \in [k]}, (B_{i,j})_{1 \leq i < j \leq k}) \) is uniquely determined by \( C \).

Conversely, a sequence of pairwise disjoint subgraphs of \( K(V_1, \ldots, V_k), ((S_i)_{i \in [k]}, (B_{i,j})_{1 \leq i < j \leq k}) \), which verifies:

- \( S_i \) is a union of stars (with at least two rays) having centers in \( V_i \),
- \( B_{i,j} \) is a union of disjoint edges having extremities in \( V_i \) and \( V_j \), respectively,
- the graphs of the sequence include all the vertices of \( K(V_1, \ldots, V_k) \),

determines, by union, an irreducible covering of \( K(V_1, \ldots, V_k) \) (this fact is also based on Lemma 2.1 above).

Hence, the above correspondence is the inverse of \( \tau \), i.e. \( \tau \) is a bijection.

Moreover, let us observe that each nonempty \( S_i \) can be interpreted as the graph of a surjection \( f_i \), having as domain the subset of \( \bigcup_{j \in [k]-\{i\}} V_j \) formed by extremities of (all) the stars with centers in \( V_i \), and as codomain the set \( P_i \) of the centers of stars in \( S_i \) (which is a subset of \( V_i \)). In addition, for every \( p \in P_i \), the set \( f_i^{-1}(p) \) has at least two elements.

In the same spirit, we can interpret \( B_{i,j} \) as being the graph of a bijection between two subsets of \( V_i \) and \( V_j \) respectively.

These remarks can be expressed in a more rigorous way as follows.

**Proposition 2.1.** Let \( K(V_1, \ldots, V_k) \) be a complete \( k \)-partite graph. There exists a bijection between the set of irreducible coverings of this graph and the set of sequences

\[
(P_i, (R_i^{(j)})_{j \in [k]-\{i\}}, (T_i^{(j)})_{j \in [k]-\{i\}}, f_i, (\sigma_{i,j})_{j > i})_{i \in [k]},
\]

where

(i) for every \( i \in [k] \), the sets (which can be empty)

\[
P_i, R_i^{(1)}, \ldots, R_i^{(k)}, T_i^{(1)}, \ldots, T_i^{(k)}
\]

produce a partition of \( V_i \);

(ii) for every \( i \in [k] \), we have the inequality:

\[
2 |P_i| \leq |R_i^{(1)}| + \cdots + |R_i^{(i-1)}| + |R_i^{(i+1)}| + \cdots + |R_i^{(k)}|;
\]

(iii) for every \( i \in [k] \), \( f_i \) is a surjection from \( \bigcup_{j \in [k]-\{i\}} R_i^{(j)} \) to \( P_i \), such that for any \( p \in P_i \), the set \( f_i^{-1}(p) \) contains at least two elements (if \( P_i = \emptyset \), we take the void function as \( f_i \));

(iv) for every \( 1 \leq i < j \leq k \), we have the relation:

\[
|T_i^{(j)}| = |T_j^{(i)}|;
\]

(v) for every \( 1 \leq i < j \leq k \), \( \sigma_{i,j} \) is a bijection from \( T_i^{(j)} \) to \( T_j^{(i)} \) (if both sets are empty then \( \sigma_{i,j} \) is the void function).
3. Determination of $F$

We shall present a sequence of relations verified by some species (which will be defined below), leading us to the determination of the generating function $F$. One of these relations (namely (9)) is a rewriting (using the language of the theory of species) of Proposition 2.1. The others immediately follow from the definitions of the operations of sum, product, and substitution of species of several variables. For these reasons we shall not present any proof of them. All notions about species used in this paper have been defined by Joyal [1].

In what follows we shall define some species useful for our method. (The exponential generating function of a species $S$ in $t$ variables will be denoted by $S(x_1, \ldots, x_t)$.)

(1) The species

$$U_{>2} : B \to E,$$

defined by

$$U_{>2}[E] = \begin{cases} \{ E \} & \text{if } |E| \geq 2, \\ \emptyset & \text{otherwise}. \end{cases}$$

From this definition we obtain:

$$U_{>2}(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x - x - 1; \quad (3)$$

(2) The species

$$\text{Perm} : B^2 \to E,$$

defined by

$$\text{Perm}[E_1, E_2] = \{ f : E_1 \to E_2 ; f = \text{bijection} \}.$$

This definition implies that at the level of generating functions,

$$\text{Perm}(x_1, x_2) = 1 + \frac{x_1 x_2}{1! 1!} + \frac{x_1^2 x_2^2}{2! 2!} + \cdots = e^{x_1 x_2}. \quad (4)$$

(3) For every $1 \leq i < j \leq k$, we define the species

$$\text{Perm}^{(i,j)} : B^k \to E,$$

by

$$\text{Perm}^{(i,j)}[E_1, \ldots, E_k] = \begin{cases} \{ f : E_i \to E_j ; f = \text{bijection} \} & \text{if } (\forall t \notin \{ i, j \}, \ E_t = \emptyset), \\ \emptyset & \text{otherwise}. \end{cases}$$

We have

**Proposition 3.1.**

$$\text{Perm}^{(i,j)} = \text{Perm}(X_i, X_j). \quad (5)$$
As a corollary, we obtain

\[ \text{Perm}^{(i,j)}(x_1, \ldots, x_k) = e^{x_i x_j}. \]  

(6)

(4) For every \( i \in [k] \), the species

\[ \text{Surj}^{(i)}_{\geq 2} : B^k \to E, \]

defined by

\[ \text{Surj}^{(i)}_{\geq 2}[E_1, \ldots, E_k] = \{ f : \bigcup_{i \in [k]-\{i\}} E_i \to E_i; \]

\[ f = \text{surjection with card}(f^{-1}(t)) \geq 2, \forall t \in E_i \}. \]

It leads to the following proposition.

**Proposition 3.2.**

\[ \text{Surj}^{(i)}_{\geq 2} = \text{Perm}(U \geq 2(X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_k), X_i). \]  

(7)

As a consequence, we obtain the relation

\[ \text{Surj}^{(i)}_{\geq 2}(x_1, \ldots, x_k) = \exp(e^{x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k} x_i)

- (x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k) x_i - x_i). \]  

(8)

(5) The species \( \text{F'} : B^k \to E \), defined as follows:

(a) if \( |E_i| > 0 \), \( \forall i \in [k] \), then \( \text{F'}[E_1, \ldots, E_k] \) is the set of irreducible coverings of the complete \( k \)-partite graph \( K(E_1, \ldots, E_k) \);

(b) if \( E_i = \emptyset \), \( \forall i \in [k] \), then \( \text{F'}[E_1, \ldots, E_k] = \{ \text{the empty graph} \} \);

(c) for all other cases, \( \text{F'}[E_1, \ldots, E_k] = \emptyset \).

Proposition 2.1 can be now rewritten in the following way.

**Proposition 3.3.** We have the relation

\[ \text{F'} = \prod_{i=1}^{k} \text{Surj}^{(i)}_{\geq 2} \cdot \prod_{1 \leq i < j \leq k} \text{Perm}^{(i,j)}. \]  

(9)

At the level of generating function we obtain the following corollary.

**Corollary 3.1.** The generating function \( \text{F'}(x_1, \ldots, x_k) \) verifies

\[ \text{F'}(x_1, \ldots, x_k) = \exp \left( \sum_{i=1}^{k} (e^{x_1 + \cdots + x_i-1 + x_{i+1} + \cdots + x_k} x_i - x_i) - \sum_{1 \leq i < j \leq k} x_i x_j \right). \]  

(10)

**Hint:** Apply (8) and (6) in (9).

From the definition of the species \( \text{F'} \), we immediately obtain \( F(x_1, \ldots, x_k) = F'(x_1, \ldots, x_k) - 1 \) hence we have
Corollary 3.2. The generating function $F(x_1, \ldots, x_k)$ verifies the relation:

$$F(x_1, \ldots, x_k) = \exp \left( \sum_{i=1}^{k} (e^{x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k} - x_i) - \sum_{1 \leq i < j \leq k} x_i x_j \right) - 1. \quad (11)$$

For $k = 2$ one obtains relation (2).

Acknowledgements

We want to express many thanks to Professor Gian-Carlo Rota, who taught one of us how to use effectively the modern theory of species. We also thank Professor Ioan Tomescu for suggesting this problem and for his advices and encouragement.

References