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The structure relation for Askey–Wilson polynomials

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This paper is dedicated to Nico Temme on the occasion of his 65th birthday

Abstract

An explicit structure relation for Askey–Wilson polynomials is given. This involves a divided q-difference operator which is skew symmetric with respect to the Askey–Wilson inner product and which sends polynomials of degree n to polynomials of degree $n+1$. By specialization of parameters and by taking limits, similar structure relations, as well as lowering and raising relations, can be obtained for other families in the q-Askey scheme and the Askey scheme. This is explicitly discussed for Jacobi polynomials, continuous q-Jacobi polynomials, continuous q-ultraspherical polynomials, and for big q-Jacobi polynomials. An already known structure relation for this last family can be obtained from the new structure relation by using the three-term recurrence relation and the second order q-difference formula. The results are also put in the framework of a more general theory. Their relationship with earlier work by Zhedanov and Bangerezako is discussed. There is also a connection with the string equation in discrete matrix models and with the Sklyanin algebra.

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1. Introduction

One of the many ways to characterize a family $\{p_n(x)\}$ of classical orthogonal polynomials (Jacobi, Laguerre and Hermite polynomials) is by a *structure relation*

$$\pi(x)\,p_n'(x) = a_n\,p_{n+1}(x) + b_n\,p_n(x) + c_n\,p_{n-1}(x), \tag{1.1}$$

where $\pi(x)$ is a fixed polynomial (necessarily of degree $\leqslant 2$). In view of the three-term recurrence relation this is equivalent to a characterization by a *lowering relation*

$$\pi(x)\,p_n'(x) = (\alpha_n x + \beta_n)\,p_n(x) + \gamma_n\,p_{n-1}(x) \tag{1.2}$$

or to a characterization by a *raising relation*

$$\pi(x)\,p_n'(x) = (\tilde\alpha_n x + \tilde\beta_n)\,p_n(x) + \tilde\gamma_n\,p_{n+1}(x). \tag{1.3}$$

The lowering relation (1.2) was given in [7, 10.7(4)] (in [7, 10.8(15)] explicitly for Jacobi polynomials), where it was attributed to Tricomi (1948). The characterization of classical orthogonal polynomials by their property (1.2) (and thus

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equivalently by their property (1.1)) was first given by Al-Salam and Chihara [2]. See also [16] for a characterization by (1.1). Note that the lowering and raising relations (1.2), (1.3) are different from the more familiar *shift operator relations*, where not only the degree is lowered or raised, but also the parameters are shifted. These formulas are well-known for all orthogonal polynomials in the (q-)Askey scheme, see for instance [14].

Orthogonal polynomials satisfying the more general structure relation

$$\pi(x)p_n'(x) = \sum_{j=n-s}^{n+t} a_{n,j}\,p_j(x) \qquad (\pi(x) \text{ a polynomial; } s,t \text{ independent of } n). \tag{1.4}$$

are called *semi-classical*, see Maroni [17]. According to [2] the question to characterize orthogonal polynomials satisfying (1.4) was first posed by Askey.

García et al. [8] characterized discrete classical orthogonal polynomials in the Hahn class (Hahn, Krawtchouk, Meixner and Charlier polynomials) by a structure relation similar to (1.1), with the derivative replaced by the difference operator $(\Delta f)(x) := f(x+1) - f(x)$. Next Medem et al. [18] (see also [3]) characterized the orthogonal polynomials in the q-Hahn class by a structure relation obtained from (1.1) by replacing the derivative by the q-derivative D_q, where

$$(D_q f)(x) = D_{q,x}(f(x)) := \frac{f(x) - f(qx)}{(1-q)x}. \tag{1.5}$$

Here a family of orthogonal polynomials $p_n(x)$ is in the *q-Hahn class* if the polynomials $(D_q p_n)(x)$ are again orthogonal.

Variants of lowering and raising relations (1.2), (1.3) are scattered over the literature. See a brief survey in [15]. Note in particular the lowering and raising relations for A_n type Macdonald polynomials given by Kirillov and Noumi [13]. The A_1 case yields lowering and raising relations for continuous q-ultraspherical polynomials. A lowering relation for continuous q-Jacobi polynomials was given by Ismail [12, Theorem 15.5.2].

For Askey–Wilson polynomials, a structure relation was given in a very implicit way by Zhedanov [24] (see my discussion in Remark 2.6), while a lowering and raising relation was given by Bangerezako [5] (see my discussion after (4.13)). The main result of the present paper gives a structure relation for Askey–Wilson polynomials in the form

$$Lp_n = a_n p_{n+1} + c_n p_{n-1}, \tag{1.6}$$

where L is a linear operator which is skew symmetric with respect to the inner product $\langle .,. \rangle$ for which the Askey–Wilson polynomials are orthogonal:

$$\langle Lf, g\rangle = -\langle f, Lg\rangle. \tag{1.7}$$

This operator is explicitly given by

$$\begin{aligned}(Lf)[z] := ((1-az)(1-bz)(1-cz)(1-dz)z^{-2}f[qz] \\ - (1-a/z)(1-b/z)(1-c/z)(1-d/z)z^2 f[q^{-1}z])(z-z^{-1})^{-1}.\end{aligned} \tag{1.8}$$

It sends symmetric Laurent polynomials of degree n to symmetric Laurent polynomials of degree $n+1$. By specialization and limit transition a structure relation of the form (1.6) with L satisfying (1.7) can be obtained for all families of orthogonal polynomials in the q-Askey scheme and the Askey scheme (a list of these families is given in [14]). For instance, for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ we get for the operator L:

$$(Lf)(x) := (1-x^2)f'(x) - \tfrac{1}{2}(\alpha - \beta + (\alpha+\beta+2)x)f(x) \tag{1.9}$$

$$= (1-x)^{-\frac{1}{2}\alpha+\frac{1}{2}}(1+x)^{-\frac{1}{2}\beta+\frac{1}{2}}\frac{d}{dx}((1-x)^{\frac{1}{2}\alpha+\frac{1}{2}}(1+x)^{\frac{1}{2}\beta+\frac{1}{2}}f(x)) \tag{1.10}$$

$$= ((DX - XD)f)(x), \tag{1.11}$$

where X is multiplication by x and D is a second order differential operator having the Jacobi polynomials as eigenfunctions (see (3.6)). It will turn out that the form (1.10) of L, involving something close to the square root of the weight function, can also be realized higher up in the (q-)Askey scheme, notably in the Askey–Wilson case (1.8). It will also turn out that the form (1.11) of L, i.e., as a commutator of X and a second order differential or (q-) difference operator D having the orthogonal polynomials as eigenfunctions, persists in the (q-)Askey scheme. In fact, there is an essentially one-to-one relationship between operators L and D.

For those families where a structure relation had been given earlier, one can relate that formula to (1.6) by use of the three-term recurrence relation, and sometimes also of the second-order (q-) difference equation.

The general theory of the structure relation of the form (1.6) with skew symmetric L will be given in Section 2. This theory is easy and elegant. The coefficients in the resulting structure relation and in the lowering and raising relations are very close to the coefficients in the three-term recurrence relation. In Section 2 I will also discuss the relationship with bispectral problems (Grünbaum and Haine), Zhedanov's algebra, and the string equation in the context of discrete matrix models. The case of Jacobi polynomials will be discussed in Section 3. The main result, the Askey–Wilson case, is the topic of Section 4. Here also a connection with the Sklyanin algebra will be made. In Section 5 this is specialized to the case of continuous q-Jacobi polynomials and we show that it has the results for Jacobi polynomials as a limit case. A further specialization to continuous q-ultraspherical polynomials is given in Section 6, and the resulting structure relation is related to another one obtained from results in [15]. Finally, in Section 7, we take the limit of the Askey–Wilson case to the case of big q-Jacobi polynomials, and we relate the resulting structure relation to the one in [18].

Conventions:

Throughout assume that $0<q<1$. For (q-)Pochhammer symbols and (q-)hypergeometric series use the notation of Gasper and Rahman [9]. Symmetric Laurent polynomials $f[z]=\sum_{k=-n}^{n} c_k z^k$ (where $c_k = c_{-k}$) are related to ordinary polynomials $f(x)$ in $x=\frac{1}{2}(z+z^{-1})$ by $f(\frac{1}{2}(z+z^{-1}))=f[z]$.

2. The general form of the structure relation

Suppose we have a family of orthogonal polynomials $p_n(x)$ with respect to an orthogonality measure μ on $\mathbb{R}$:

$$p_n(x)=k_n x^n+\cdots,\quad \langle p_n,p_m\rangle = h_n\delta_{n,m}, \tag{2.1}$$

where

$$\langle f,g\rangle := \int_{\mathbb{R}} f(x)g(x)\,d\mu(x). \tag{2.2}$$

Write the three-term recurrence relation as

$$xp_n(x)=A_n p_{n+1}(x)+B_n p_n(x)+C_n p_{n-1}(x). \tag{2.3}$$

Then

$$A_n=\frac{k_n}{k_{n+1}},\quad C_n=\frac{k_{n-1}}{k_n}\frac{h_n}{h_{n-1}}=A_{n-1}\frac{h_n}{h_{n-1}}. \tag{2.4}$$

The proof of the following proposition is straightforward.

Proposition 2.1. *Let L be a linear operator acting on the space $\mathbb{R}[x]$ of polynomials in one variable with real coefficients such that L is skew symmetric with respect to the inner product* (2.2) (*i.e.*, (1.7) *holds*) *and such that*

$$L(x^n)=\gamma_n x^{n+1}+\textit{terms of lower degree}, \tag{2.5}$$

where $\gamma_n\ne 0$. Then the following structure relation and lowering and raising relations hold:

$$(Lp_n)(x)=\gamma_n A_n p_{n+1}(x)-\gamma_{n-1}C_n p_{n-1}(x), \tag{2.6}$$

$$-\gamma_n(x-B_n)p_n(x)+(Lp_n)(x)=-(\gamma_n+\gamma_{n-1})C_n p_{n-1}(x), \tag{2.7}$$

$$\gamma_{n-1}(x-B_n)p_n(x)+(Lp_n)(x)=(\gamma_n+\gamma_{n-1})A_n p_{n+1}(x). \tag{2.8}$$

Skew symmetric operators L as above can be produced from symmetric operators D which have the p_n as eigenfunctions. First note that

$$(Xf)(x) := xf(x) \tag{2.9}$$

defines a symmetric operator X on $\mathbb{R}[x]$ with respect to the inner product (2.2), i.e., $\langle Xf, g\rangle = \langle f, Xg\rangle$. Now the following proposition can be shown immediately.

Proposition 2.2. *Let D be a linear operator acting on $\mathbb{R}[x]$ which is symmetric with respect to the inner product* (2.2), *i.e., $\langle Df, g\rangle = \langle f, Dg\rangle$, and which satisfies*

$$D(x^n) = \lambda_n x^n + \textit{terms of lower degree}, \tag{2.10}$$

where $\lambda_n \neq \lambda_{n-1}$, so p_n is an eigenfunction of D:

$$Dp_n = \lambda_n p_n. \tag{2.11}$$

Then the commutator

$$L := [D, X] = DX - XD \tag{2.12}$$

is skew symmetric with respect to the inner product (2.2) *and satisfies* (2.5) *with*

$$\gamma_n = \lambda_{n+1} - \lambda_n \neq 0. \tag{2.13}$$

So L also satisfies the structure relation (2.6).

In fact, we can reverse Proposition 2.2: From skew symmetric L as in Proposition 2.1 we can produce symmetric D as in Proposition 2.2, and D is uniquely determined by L up to a term which is constant times identity.

Proposition 2.3. *Let L be as in Proposition* 2.1. *Define a linear operator D on $\mathbb{R}[x]$ by its action on monomials*:

$$D(1) = 0, \quad D(x^n) = \sum_{k=0}^{n-1} X^k L(x^{n-k-1}). \tag{2.14}$$

Then D satisfies the properties of Proposition 2.2 *with $\lambda_0 = 0$. Any other operator D satisfying* (2.12) *and having* 1 *as eigenfunction differs from D given by* (2.14) *by a constant times identity.*

Proof. Formula (2.12) acting on x^n follows directly from (2.14). From (2.5) together with $DX = L + XD$ acting on x^n we see by induction that (2.10) holds with λ_n satisfying (2.13). In order to prove that D is symmetric, observe that, for $n > 0$,

$$\begin{aligned}\langle Dx^n, x^m\rangle - \langle x^n, Dx^m\rangle &= \langle DXx^{n-1}, x^m\rangle - \langle x^{n-1}, XDx^m\rangle \\ &= \langle (L + XD)x^{n-1}, x^m\rangle + \langle x^{n-1}, (L - DX)x^m\rangle \\ &= \langle Dx^{n-1}, x^{m+1}\rangle - \langle x^{n-1}, Dx^{m+1}\rangle.\end{aligned}$$

Hence

$$\langle Dx^{n+m}, 1\rangle = \langle Dx^n, x^m\rangle - \langle x^n, Dx^m\rangle = -\langle 1, Dx^{n+m}\rangle.$$

So $\langle Dx^{n+m}, 1\rangle = 0$ and $\langle Dx^n, x^m\rangle = \langle x^n, Dx^m\rangle$.

Finally, for the uniqueness, let D_1 and D_2 satisfy (2.14) and let them have 1 as eigenfunction. Then $D_1 - D_2$ commutes with X. Hence $(D_1 - D_2)(x^n) = X^n(D_1 - D_2)(1) = X^n(c1) = cx^n$. $\square$

Remark 2.4. If L satisfies (2.12) and D satisfies (2.11) then the lowering and raising relations (2.7) and (2.8) are preserved if we add to their left-hand sides a term $g(x)(D - \lambda_n)p_n(x)$, where $g(x)$ is any function.

Remark 2.5. Van Moerbeke [22, Section 7] (also jointly with Adler in [1, Section 5]) gives explicit skew symmetric first order differential operators L (Q in his notation) satisfying Proposition 2.1 for the case of Jacobi, Laguerre and Hermite polynomials. More generally he gives these operators if the weight function is perturbed by multiplying the weight function with $\exp(\sum_{j=1}^{\infty} t_j x^j)$ (only finitely many t_j nonzero). Then (2.5) and (2.6) are no longer valid, but the right-hand side of (2.6) has to be replaced by some linear combination of orthogonal polynomials $p_k(x)$ with coefficients depending on the t_j. In all these cases L satisfies the so-called *string equation*

$$[X, L] = f_0(X) \tag{2.15}$$

with f_0 a polynomial given by $f_0(X) = 1 - X^2$, X, 1 for Jacobi, Laguerre and Hermite polynomials, respectively, and for their deformations. This is inspired by the matrix models of the physicists, where the Hermite case occurs, see Witten [23, Section 4c], in particular (4.43), (4.53), (4.65), (4.66).

Remark 2.6. The structure relation (2.6) can be written in a form which is symmetric with respect to the variables n and x. Write $p(n, x)$ instead of $p_n(x)$. Define operators J and Λ acting on functions on $\mathbb{Z}_{\geqslant 0}$ by

$$(J\phi)(n) := A_n\phi(n+1) + B_n\phi(n) + C_n\phi(n-1), \tag{2.16}$$

$$(\Lambda\phi)(n) := \lambda_n\phi(n). \tag{2.17}$$

Now use (2.9), (2.12), (2.3) and (2.11) in order to rewrite (2.6) as

$$([D, X]p(n, .))(x) = ([J, \Lambda]p(., x))(n). \tag{2.18}$$

This equation essentially occurs in Duistermaat and Grünbaum [6, (1.7)] and Grünbaum and Haine [10, (3)], where they study the bispectral problem with D a differential operator and n continuous, respectively, discrete.

For p_n, D and λ_n q-dependent, a q-analogue of (2.18) has been considered in the literature which involves q-commutators instead of ordinary commutators:

$$((q^{\frac{1}{2}}DX - q^{-\frac{1}{2}}XD)p(n, .))(x) = ((q^{\frac{1}{2}}J\Lambda - q^{-\frac{1}{2}}\Lambda J)p(., x))(n). \tag{2.19}$$

In fact, in Zhedanov [24] formulas (1.4) and (1.8a) may be interpreted as formulas (2.11) and (2.3) in the present paper, respectively. Then formula (1.8b) together with (1.1a) in [24] can be interpreted as (2.19) above. See also Grünbaum and Haine [11] on the bispectral problem in the q-case, where some formulas in Section 3 may be close to (2.19) above.

Formula (1.1c) in [24] may be interpreted as the q-commutator $q^{\frac{1}{2}}XL - q^{-\frac{1}{2}}LX$ being equal to a linear combination of X, D and L. A similar formula is observed in [11, (2.4)], where it is called the q-string equation as a q-analogue of the string equation (2.15).

3. Jacobi polynomials

Jacobi polynomials $P_n(x) = P_n^{(\alpha,\beta)}(x)$ (see [7, Section 10.8]) are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta\,dx \quad (\alpha, \beta > -1).$$

The coefficients in the three-term recurrence relation (2.3) are

$$A_n = \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \tag{3.1}$$

$$B_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{3.2}$$

$$C_n = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{3.3}$$

For the operator L given by (1.9) we see immediately that (2.5) holds with

$$\gamma_n = -\tfrac{1}{2}(2n+\alpha+\beta+2) \tag{3.4}$$

and that the skew symmetry (1.7) holds. Hence, by Proposition 2.1, the structure relation (2.6) is valid. Explicitly it reads as follows:

$$\begin{aligned}&\left((1-x^2)\frac{d}{dx} - \tfrac{1}{2}(\alpha-\beta+(\alpha+\beta+2)x)\right)P_n^{(\alpha,\beta)}(x)\\ &\quad = -\frac{(n+1)(n+\alpha+\beta+1)}{2n+\alpha+\beta+1}P_{n+1}^{(\alpha,\beta)}(x) + \frac{(n+\alpha)(n+\beta)}{2n+\alpha+\beta+1}P_{n-1}^{(\alpha,\beta)}(x).\end{aligned} \tag{3.5}$$

We can also make explicit formulas (1.11) and (2.13) with

$$D = \tfrac{1}{2}(1-x^2)\frac{d^2}{dx^2} + \tfrac{1}{2}(\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}, \quad \lambda_n = -\tfrac{1}{2}n(n+\alpha+\beta+1). \tag{3.6}$$

Formula (3.5) can be combined with the three-term recurrence relation in order to obtain the structure relation of the form (1.1)

$$\begin{aligned}(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = &-\frac{2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x)\\ &+\frac{2n(n+\alpha+\beta+1)(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x)\\ &+\frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x).\end{aligned} \tag{3.7}$$

4. Askey–Wilson polynomials

Askey–Wilson polynomials (see [4,9, Section 7.5; [14, Section 3.1]) are defined by

$$\begin{aligned}p_n[z] &= p_n\left(\frac{1}{2}(z+z^{-1})\right) = p_n\left(\frac{1}{2}(z+z^{-1}); a,b,c,d \mid q\right)\\ &:= \frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n}, q^{n-1}abcd, az, az^{-1}\\ ab, ac, ad\end{matrix}; q, q\right).\end{aligned} \tag{4.1}$$

If $a,b,c,d \in \mathbb{C}$ satisfy

$$a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-k} \mid k = 0,1,2,\ldots\} \tag{4.2}$$

then these polynomials satisfy the orthogonality relation

$$\langle f, g\rangle := \frac{1}{4\pi i}\oint_C f[z]g[z]w(z)\frac{dz}{z}, \tag{4.3}$$

$$w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty}, \tag{4.4}$$

where C is the unit circle traversed in positive direction with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to ∞. If a,b,c,d are four reals, or two reals and one pair of complex conjugates, or two pairs of complex conjugates such that $|ab|$, $|ac|$, $|ad|$, $|bc|$, $|bd|$, $|cd| < 1$, then the Askey–Wilson polynomials are real-valued and their orthogonality can be rewritten as an integral over $x = \frac{1}{2}(z+z^{-1}) \in [-1,1]$ plus a finite sum over real x-values outside $[-1,1]$. This finite sum does not occur if $|a|$, $|b|$, $|c|$, $|d| < 1$.

Now k_n, h_n, B_n, D and λ_n in Section 2 can be specified for the Askey–Wilson case as follows:

$$k_n=2^n(abcdq^{n-1};q)_n,\quad \frac{h_n}{h_0}=\frac{1-abcdq^{-1}}{1-abcdq^{2n-1}}\frac{(q,ab,ac,ad,bc,bd,cd;q)_n}{(abcdq^{-1};q)_n},$$

$$\begin{aligned}B_n&=((a+b+c+d)(q-abcdq^{n-1}-abcdq^n+abcdq^{2n})\\&\quad+(bcd+abd+acd+abc)(1-q^n-q^{n+1}+abcdq^{2n-1}))\\&\quad\times\frac{q^{n-1}}{2(1-abcdq^{2n-2})(1-abcdq^{2n})},\end{aligned}\tag{4.5}$$

$Dp_n=\lambda_n p_n$, where

$$\begin{aligned}\tfrac12(1-q^{-1})(Df)[z]&=v(z)f[qz]-(v(z)+v(z^{-1}))f[z]+v(z^{-1})f[q^{-1}z],\\ v(z)&=\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)},\\ \tfrac12(1-q^{-1})\lambda_n&=(q^{-n}-1)(1-abcdq^{n-1}).\end{aligned}\tag{4.6}$$

Define an operator L acting on symmetric Laurent polynomials

$$\begin{aligned}(Lf)[z]:=&((1-az)(1-bz)(1-cz)(1-dz)z^{-2}f[qz]\\&-(1-a/z)(1-b/z)(1-c/z)(1-d/z)z^2f[q^{-1}z])(z-z^{-1})^{-1}\end{aligned}\tag{4.7}$$

$$\begin{aligned}&=\tfrac12(1-q^2)\frac{(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q^2)_\infty}{(qz^2,qz^{-2};q^2)_\infty}\\&\quad\times\frac{\delta_{q^2}}{\delta_{q^2}x}\left(\frac{(qz^2,qz^{-2};q^2)_\infty f[z]}{(qaz,qaz^{-1},qbz,qbz^{-1},qcz,qcz^{-1},qdz,qdz^{-1};q^2)_\infty}\right).\end{aligned}\tag{4.8}$$

Here

$$\frac{\delta_q}{\delta_q x}g[z]:=\frac{2(g[q^{\frac12}z]-g[q^{-\frac12}z])}{(q^{\frac12}-q^{-\frac12})(z-z^{-1})}\tag{4.9}$$

is a divided q-difference operator (see [4, Section 5]). It tends to $d/dx\,g(x)$ as $q\uparrow 1$.

Then L sends symmetric Laurent polynomials of degree n to symmetric Laurent polynomials of degree $n+1$,

$$\gamma_n=2(abcdq^n-q^{-n})\tag{4.10}$$

and L is skew symmetric with respect to the inner product (4.3), so (1.7) holds. For the proof of (1.7) note that

$$\begin{aligned}&\oint_C\frac{(1-az)(1-bz)(1-cz)(1-dz)f[qz]g[z]}{z^2(z-z^{-1})}w(z)\frac{dz}{z}\\&=\oint_C\frac{f[qz]g[z]}{(qz)^{-2}(qz-(qz)^{-1})}\frac{(q^2z^2,q^{-2}z^{-2};q)_\infty}{(qaz,az^{-1},qbz,bz^{-1},qcz,cz^{-1},qdz,dz^{-1};q)_\infty}\frac{dz}{z}\\&=\oint_C\frac{f[z]g[q^{-1}z]}{z^{-2}(z-z^{-1})}\frac{(z^2,z^{-2};q)_\infty}{(az,qaz^{-1},bz,qbz^{-1},cz,qcz^{-1},dz,qdz^{-1};q)_\infty}\frac{dz}{z}\\&=\oint_C\frac{f[z](1-az)(1-bz)(1-cz)(1-dz)g[q^{-1}z]}{z^{-2}(z-z^{-1})}w(z)\frac{dz}{z}.\end{aligned}$$

(Actually, the contour deformation above can be done with avoidance of poles in the generic case of complex a, b, c, d such that the four line segments connecting a, b, c, d with 0 avoid the four halflines $\{ta^{-1}, tb^{-1}, tc^{-1}, td^{-1} \mid t \geqslant 1.\}$ Alternatively, we can observe that $L = [D, X]$, $\gamma_n = \lambda_{n+1} - \lambda_n$ with D, λ_n given by (4.6), and apply Proposition 2.2.

By Proposition 2.1 we have the structure relation (2.6), which can be more explicitly written (with usage of (4.7)) as

$$(Lf)[z] = -\frac{(1-abcdq^{n-1})p_{n+1}[z]}{q^n(1-abcdq^{2n-1})} + (1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1}) \times (1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})\frac{(1-q^n)p_{n-1}[z]}{q^{n-1}(1-abcdq^{2n-1})}. \tag{4.11}$$

We can also write the lowering and raising relations (2.7), (2.8) more explicitly (with usage of (4.7) and (4.5))

$$(Lp_n)[z]-(abcdq^n-q^{-n})(z+z^{-1}-2B_n)p_n[z] = (1-abq^{n-1})(1-acq^{n-1}) \times (1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1}) \times \frac{(1+q)(1-q^n)p_{n-1}[z]}{q^n(1-abcdq^{2n-2})}, \tag{4.12}$$

$$(Lp_n)[z] + (abcdq^{n-1} - q^{1-n})(z+z^{-1}-2B_n)p_n[z] = -\frac{(1+q)(1-abcdq^{n-1})}{q^n(1-abcdq^{2n})}p_{n+1}[z]. \tag{4.13}$$

Lowering and raising relations for Askey–Wilson polynomials were earlier obtained by Bangerezako [5, (41)]. He obtained his lowering and raising operator by the factorization method, see Proposition 1 in [5]. His lowering and raising relation can be obtained from (4.12) and (4.13), respectively, by adding $\frac{1}{2}(1-q^{-1})(z-qz^{-1})(Dp_n)[z]$ to the left-hand sides of these relations (cf. Remark 2.4).

Remark 4.1. It turns out that the operator (4.7), which occurs in the structure relation for Askey–Wilson polynomials, is the trigonometric case $p=0$ of the difference operator $\Delta(a, b, c, d)$ with elliptic coefficients given by Rosengren [20, Section 6] (also replace q by q^2 in Rosengren's operator in order to arrive at (4.7)). Rosengren observes, following Rains [19], that the operators $\Delta(a, b, c, d)$ with $abcd = q^{-N}$ $(N = 0, 1, 2, \ldots)$ generate a representation of the Sklyanin algebra as in Sklyanin [21, Theorem 4]. Indeed, if we write $L = L_{a,b,c,d}$ for the operator in (4.7) then

$$L_{a,b,ce,de^{-1}}L_{qa,qb,q^{-1}c,q^{-1}d} = L_{a,b,c,d}L_{qa,qb,q^{-1}ce,q^{-1}de^{-1}}. \tag{4.14}$$

This quasi-commutation relation, which can be proved in a straightforward way, is a specialization ($p=0$, $n=1$) of Rains' relation [19, (3.27)] and it is a way to implement the relations in the Sklyanin algebra.

5. Continuous q-Jacobi polynomials

Continuous q-Jacobi polynomials

$$P_n[z] = P_n^{(\alpha,\beta)}(\tfrac{1}{2}(z+z^{-1}) \mid q)$$

(see [9, Section 7.5; 14, Section 3.10]) can be obtained as restrictions of Askey–Wilson polynomials in two different ways (see [14, Section 4.10]):

$$P_n^{(\alpha,\beta)}(x \mid q) = \frac{q^{(\frac{1}{2}\alpha+\frac{1}{4})n}p_n(x; q^{\frac{1}{2}\alpha+\frac{1}{4}}, -q^{\frac{1}{2}\beta+\frac{1}{4}}, q^{\frac{1}{4}}, -q^{\frac{1}{4}} \mid q^{\frac{1}{2}})}{(-q^{\frac{1}{2}(\alpha+\beta+1)}; q^{\frac{1}{2}})_n(q;q)_n} \tag{5.1}$$

$$= \frac{q^{(\frac{1}{2}\alpha+\frac{1}{4})n}p_n(x; q^{\frac{1}{2}\alpha+\frac{1}{4}}, q^{\frac{1}{2}\alpha+\frac{3}{4}}, -q^{\frac{1}{2}\beta+\frac{1}{4}}, -q^{\frac{1}{2}\beta+\frac{3}{4}} \mid q)}{(-q^{\frac{1}{2}(\alpha+\beta+1)}; q^{\frac{1}{2}})_{2n}(q;q)_n}. \tag{5.2}$$

These polynomials are orthogonal with respect to the inner product

$$\langle f, g\rangle := \oint f[z]g[z]\frac{(z^2, z^{-2}; q)_\infty}{(q^{\frac12\alpha+\frac14}z, q^{\frac12\alpha+\frac14}z^{-1}, -q^{\frac12\beta+\frac14}z, -q^{\frac12\beta+\frac14}z^{-1}; q^{\frac12})_\infty}\frac{dz}{z}. \tag{5.3}$$

The coefficients A_n and C_n in (2.3) here become:

$$A_n = \frac{(1-q^{n+1})(1-q^{n+\alpha+\beta+1})}{2q^{\frac12\alpha+\frac14}(1-q^{n+\frac12(\alpha+\beta+1)})(1-q^{n+\frac12(\alpha+\beta+2)})}, \tag{5.4}$$

$$C_n = \frac{q^{\frac12\alpha+\frac14}(1-q^{n+\alpha})(1-q^{n+\beta})}{2(1-q^{n+\frac12(\alpha+\beta)})(1-q^{n+\frac12(\alpha+\beta+1)})}. \tag{5.5}$$

Corresponding to (5.1), (5.2) we can obtain two versions of the operators L and D for continuous q-Jacobi polynomials by specialization of (4.7) and (4.6). For L this becomes:

$$(Lf)[z] = \frac{v(z)f[q^{\frac12}z] - v(z^{-1})f[q^{-\frac12}z]}{z-z^{-1}},$$
$$v(z) = (1-q^{\frac12\alpha+\frac14}z)(1+q^{\frac12\beta+\frac14}z)(1-q^{\frac12}z^2)z^{-2}, \tag{5.6}$$

$$(\tilde{L}f)[z] = \frac{\tilde{v}(z)f[qz] - \tilde{v}(z^{-1})f[q^{-1}z]}{z-z^{-1}},$$
$$\tilde{v}(z) = (1-q^{\frac12\alpha+\frac14}z)(1-q^{\frac12\alpha+\frac34}z)(1+q^{\frac12\beta+\frac14}z)(1+q^{\frac12\beta+\frac34}z)z^{-2}. \tag{5.7}$$

Related to these operators are coefficients γ_n, respectively, $\tilde{\gamma}_n$ (see (2.5)):

$$\gamma_n = 2(q^{\frac12(n+\alpha+\beta+2)} - q^{-\frac12 n}), \quad \tilde{\gamma}_n = 2(q^{n+\alpha+\beta+2} - q^{-n}). \tag{5.8}$$

The form (4.8) of the operator L can also be specialized corresponding to (5.1) and (5.2). In particular, corresponding to (5.1) we obtain:

$$(Lf)[z] = \tfrac12(1-q)\frac{(q^{\frac12\alpha+\frac14}z, q^{\frac12\alpha+\frac14}/z, -q^{\frac12\beta+\frac14}z, -q^{\frac12\beta+\frac14}/z; q)_\infty}{(q^{\frac32}z^2, q^{\frac32}z^{-2}; q^2)_\infty}$$
$$\times \frac{\delta_q}{\delta_q x}\left(\frac{(q^{\frac12}z^2, q^{\frac12}/z^2; q^2)_\infty f[z]}{(q^{\frac12\alpha+\frac34}z, q^{\frac12\alpha+\frac34}/z, -q^{\frac12\beta+\frac34}z, -q^{\frac12\beta+\frac34}/z; q)_\infty}\right). \tag{5.9}$$

By (2.6) and the formulas above we get two structure relations for continuous q-Jacobi polynomials:

$$\frac{v(z)P_n[q^{\frac12}z] - v(z^{-1})P_n[q^{-\frac12}z]}{z-z^{-1}} = \gamma_n A_n P_{n+1}[z] - \gamma_{n-1}C_n P_{n-1}[z]. \tag{5.10}$$

$$\frac{\tilde{v}(z)P_n[qz] - \tilde{v}(z^{-1})P_n[q^{-1}z]}{z-z^{-1}} = \tilde{\gamma}_n A_n P_{n+1}[z] - \tilde{\gamma}_{n-1}C_n P_{n-1}[z]. \tag{5.11}$$

As in (2.7), (2.8), or more generally in Remark 2.4, lowering and raising relations can also be written explicitly here (I will skip this).

Ismail [12, Theorem 15.5.2] gives an explicit lowering relation for continuous q-Jacobi polynomials which is proved by using a determinant formula due to Christoffel. This expresses orthogonal polynomials with respect to an orthogonality measure $\Phi(x)\,d\mu(x)$ in terms of polynomials with orthogonality measure $d\mu(x)$ ($\Phi(x)$ a nonnegative polynomial on supp(μ)), see [12, Theorem 2.7.1].

The structure relation (3.5) for Jacobi polynomials is a limit case of both structure relations above. This can be most easily seen in a formal way by using the expressions for L and $\tilde{L}$ involving the divided q-difference. Let us give the details for L as given by (5.9). Rewrite (5.10) as

$$\tfrac{2}{1-q}(L_q P_n)[z;q] = \tfrac{2}{1-q}\gamma_n(q)A_n(q)P_{n+1}[z;q] - \tfrac{2}{1-q}\gamma_{n-1}(q)C_n P_{n-1}[z;q], \tag{5.12}$$

emphasizing the q-dependence. As $q \uparrow 1$, $P_n[z;q] \to P_n[z] = P_n^{(\alpha,\beta)}((z+z^{-1})/2)$ (see [14, (5.10.1)]). Also, $A_n(q)$, $C_n(q)$ and $2/(1-q)\gamma_n(q)$ tend to A_n, C_n and $4\gamma_n$ for Jacobi polynomials as given in Section 3. Thus the right-hand side of (5.12) tends to the right-hand side of (2.6) for Jacobi polynomials, i.e., to the right-hand side of (3.5). Now consider the left-hand side of (5.12) with (5.9) substituted. Use that

$$\lim_{q\uparrow 1} \frac{(q^a z;q)_\infty}{(z;q)_\infty} = (1-z)^{-a}$$

(see [9, (1.3.19)]). Hence

$$\frac{(q^{\frac12\alpha+\frac14}z, \frac{q^{\frac12\alpha+\frac14}}{z}, -q^{\frac12\beta+\frac14}z, -\frac{q^{\frac12\beta+\frac14}}{z};q)_\infty}{(q^{\frac34}z, \frac{q^{\frac34}}{z}, -q^{\frac34}z, -\frac{q^{\frac34}}{z};q)_\infty} \to (2-z-\tfrac{1}{z})^{\frac12-\frac12\alpha}(2+z+\tfrac{1}{z})^{\frac12-\frac12\beta},$$

$$\frac{(q^{\frac14}z, \frac{q^{\frac14}}{z}, -q^{\frac14}z, -\frac{q^{\frac14}}{z};q)_\infty}{(q^{\frac12\alpha+\frac34}z, \frac{q^{\frac12\alpha+\frac34}}{z}, -q^{\frac12\beta+\frac34}z, -\frac{q^{\frac12\beta+\frac34}}{z};q)_\infty} \to (2-z-\tfrac{1}{z})^{\frac12+\frac12\alpha}(2+z+\tfrac{1}{z})^{\frac12+\frac12\beta}.$$

Hence $2/(1-q)(L_q P_n)[z;q] \to 4(LP_n)[z]$ with L given by (1.10). So (5.12) tends to 4 times (3.5) as $q \uparrow 1$.

6. Continuous q-ultraspherical polynomials

In view of [9, (7.5.34)] and (5.1) *continuous q-ultraspherical polynomials* are specializations of Askey–Wilson polynomials as follows:

$$\begin{aligned} C_n[z] &= C_n(\tfrac12(z+z^{-1});t \mid q) \\ &= \frac{(t;q^{\frac12})_n}{(q^{\frac12}t;q)_n(q;q)_n} p_n(\tfrac12(z+z^{-1}); t^{\frac12}, -t^{\frac12}, q^{\frac14}, -q^{\frac14} \mid q^{\frac12}). \end{aligned} \tag{6.1}$$

By specialization of (4.7) and (4.8) we obtain the two ways of writing the operator L corresponding to (6.1):

$$(Lf)[z] = \frac{(1-tz^2)(z^{-2}-q^{\frac12})}{z-z^{-1}} f[q^{\frac12}z] - \frac{(1-tz^{-2})(z^2-q^{\frac12})}{z-z^{-1}} f[q^{-\frac12}z] \tag{6.2}$$

$$= \tfrac12(1-q)\frac{(tz^2,tz^{-2};q^2)_\infty}{(q^{\frac32}z^2,q^{\frac32}z^{-2};q^2)_\infty}\frac{\delta_q}{\delta_q x}\left(\frac{(q^{\frac12}z^2,q^{\frac12}z^{-2};q^2)_\infty}{(qtz^2,qtz^{-2};q^2)_\infty} f[z]\right). \tag{6.3}$$

Then the structure relation (4.11) for Askey–Wilson polynomials specializes to the following structure relation for the polynomials C_n:

$$\begin{aligned} &\frac{(1-tz^2)(z^{-2}-q^{\frac12})}{z-z^{-1}} C_n[q^{\frac12}z] - \frac{(1-tz^{-2})(z^2-q^{\frac12})}{z-z^{-1}} C_n[q^{-\frac12}z] \\ &= -\frac{(1-tq^{n+\frac12})(1-q^{n+1})}{q^{\frac12 n}(1-tq^n)} C_{n+1}[z] + \frac{(1-tq^{n-\frac12})(1-t^2q^{n-1})}{q^{\frac12 n-\frac12}(1-tq^n)} C_{n-1}[z]. \end{aligned} \tag{6.4}$$

In [15, Section 2] some lowering and raising relations for continuous q-ultraspherical polynomials were given which were obtained as A_1 cases of the lowering and raising relations for the A_n Macdonald polynomials in [13]. For instance:

$$
\begin{aligned}
&-\frac{t-z^{-2}}{z-z^{-1}}C_n[q^{\frac12}z]+\frac{t-z^2}{z-z^{-1}}C_n[q^{-\frac12}z]+q^{-\frac12 n}(z+z^{-1})C_n[z]\\
&\quad=(q^{-\frac12 n}-t^2q^{\frac12 n-1})C_{n-1}[z],
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
&-\frac{tz^2-1}{z-z^{-1}}C_n[q^{\frac12}z]+\frac{tz^{-2}-1}{z-z^{-1}}C_n[q^{-\frac12}z]+q^{-\frac12 n}(z+z^{-1})C_n[z]\\
&\quad=(q^{-\frac12 n}-q^{\frac12 n+1})C_{n+1}[z].
\end{aligned}
\tag{6.6}
$$

By subtracting (6.5) from (6.6) we obtain another structure relation for continuous q-ultraspherical polynomials of the form (1.6), but with an operator L which is not skew symmetric:

$$
\begin{aligned}
&z^{-1}(1-tz^2)C_n[q^{\frac12}z]+z(1-tz^{-2})C_n[q^{-\frac12}z]\\
&\quad=q^{-\frac12 n}(1-q^{n+1})C_{n+1}[z]-q^{-\frac12 n}(1-t^2q^{n-1})C_{n-1}[z].
\end{aligned}
\tag{6.7}
$$

Relation (6.7) is connected with relation (6.4) by yet another relation of the form (1.6):

$$
\begin{aligned}
&-\frac{(1-tz^2)((1+z^{-2})}{z-z^{-1}}C_n[q^{\frac12}z]+\frac{(1-tz^{-2})(1+z^2)}{z-z^{-1}}C_n[q^{-\frac12}z]\\
&\quad=\frac{(q^{-\frac12 n}+tq^{\frac12 n})(1-q^{n+1})}{1-tq^n}C_{n+1}[z]+\frac{(q^{-\frac12 n}+tq^{\frac12 n})(1-t^2q^{n-1})}{1-tq^n}C_{n-1}[z].
\end{aligned}
\tag{6.8}
$$

Indeed, (6.4) equals $\frac12(q-1)$ times (6.8) plus $\frac12(q+1)$ times (6.7).

Relation (6.8) is a consequence of the three-term recurrence relation

$$
(z+z^{-1})C_n[z]=\frac{1-q^{n+1}}{1-tq^n}C_{n+1}[z]+\frac{1-t^2q^{n-1}}{1-tq^n}C_{n-1}[z]
\tag{6.9}
$$

(see [14, (3.10.17)]) and the second order q-difference formula

$$
\frac{1-tz^2}{1-z^2}C_n[q^{\frac12}z]+\frac{1-tz^{-2}}{1-z^{-2}}C_n[q^{-\frac12}z]=(q^{-\frac12 n}+q^{\frac12 n}t)C_n[z],
\tag{6.10}
$$

which last formula is the specialization by (6.1) of the second order q-difference formula (4.6) for Askey–Wilson polynomials.

7. Big q-Jacobi polynomials

Big q-Jacobi polynomials (see [9, Section 7.3; 14, Section 3.5]) are given by

$$
P_n(x)=P_n(x;a,b,-c;q):={}_3\phi_2\left(\begin{matrix}q^{-n},abq^{n+1},x\\ aq,-cq\end{matrix};q,q\right).
\tag{7.1}
$$

They can be obtained as limits of Askey–Wilson polynomials (see [14, (4.1.3)]):

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon^n}{(aq,-cq,-\varepsilon^2bc^{-1};q)_n}p_n(\tfrac12(a^{-1}x+ax^{-1});\varepsilon,\varepsilon^{-1}aq,-\varepsilon^{-1}cq,-\varepsilon bc^{-1}\mid q)\\
&\quad=P_n(x;a,b,-c;q).
\end{aligned}
\tag{7.2}
$$

Make in (4.11) together with (1.8) the substitutions corresponding to the left-hand side of (7.2) and let $\varepsilon \to 0$. Then we obtain a structure relation for big q-Jacobi polynomials:

$$(LP_n)(x) = \frac{(1-aq^{n+1})(1+cq^{n+1})(1-abq^{n+1})}{q^{n+2}ac(1-abq^{2n+1})} P_{n+1}(x) - \frac{(1-q^n)(1-bq^n)(1+abc^{-1}q^n)}{1-abq^{2n+1}} P_{n-1}(x), \tag{7.3}$$

where

$$(Lf)(x) := \frac{(1-x)(1+bc^{-1}x)f(qx) - (1-a^{-1}q^{-1}x)(1+c^{-1}q^{-1}x)f(q^{-1}x)}{x}. \tag{7.4}$$

Note that we can rewrite (7.4) as

$$(Lf)(x) = (q-q^{-1}) \frac{(x, -bc^{-1}x; q^2)_\infty}{(qa^{-1}x, -qc^{-1}x; q^2)_\infty} d_{q,x} \left(\frac{(a^{-1}x, -c^{-1}x; q^2)_\infty}{(qx, -qbc^{-1}x; q^2)_\infty} f[x] \right), \tag{7.5}$$

where $d_{q,x}$ is a central q-derivative:

$$d_{q,x}(g(x)) := \frac{g(qx) - g(q^{-1}x)}{(q-q^{-1})x}. \tag{7.6}$$

We can rewrite (7.3) in two steps as the explicit structure relation of the form

$$(x-1)(bx+c)D_{q,x}(P_n(x)) = \tilde{a}_n P_{n+1}(x) + \tilde{b}_n P_n(x) + \tilde{c}_n P_{n-1}(x) \tag{7.7}$$

given in [18], where the q-derivative D_q is defined by (1.5). First eliminate $P_n(q^{-1}x)$ from (7.3) by means of the second order q-difference equation for big q-Jacobi polynomials. Then we obtain from (7.3) a relation of the form

$$(x-1)(bx+c)D_{q,x}(P_n(x)) = \alpha_n P_{n+1}(x) + (\delta_n x + \beta_n) P_n(x) + \gamma_n P_{n-1}(x). \tag{7.8}$$

Next eliminate $xP_n(x)$ from (7.8) by means of the three-term recurrence relation in order to obtain a relation of the form (7.7).

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References

[1] M. Adler, P. van Moerbeke, Matrix integrals, Toda symmetries, Virasoro constraints, and orthogonal polynomials, Duke Math. J. 80 (1995) 863–911.
[2] W.A. Al-Salam, T.S. Chihara, Another characterization of the classical orthogonal polynomials, SIAM J. Math. Anal. 3 (1972) 65–70.
[3] R. Álvarez-Nodarse, J. Arvesú, On the q-polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$, Integral Transform. Spec. Funct. 8 (1999) 299–324.
[4] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc. 319 (1985).
[5] G. Bangerezako, The factorization method for the Askey–Wilson polynomials, J. Comput. Appl. Math. 107 (1999) 219–232.
[6] J.J. Duistermaat, F.A. Grünbaum, Differential equations in the eigenvalue parameter, Comm. Math. Phys. 103 (1986) 177–240.
[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. II, McGraw-Hill, New York, 1953.
[8] A.G. García, F. Marcellán, L. Salto, A distributional study of discrete classical orthogonal polynomials, J. Comput. Appl. Math. 57 (1995) 147–162.
[9] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge University Press, Cambridge, 2004.
[10] F.A. Grünbaum, L. Haine, A theorem of Bochner, revisited, in: A.S. Fokas, I.M. Gelfand (Eds.), Algebraic Aspects of Integrable Systems, Progress in Nonlinear Differential Equations and their Applications, vol. 26, Birkhäuser, Basel, 1997, pp. 143–172.
[11] F.A. Grünbaum, L. Haine, On a q-analogue of the string equation and a of generalization the classical orthogonal polynomials, in: J.F. van Diejen, L. Vinet (Eds.), Algebraic methods and q-special functions, CRM Proceedings and Lecture Notes, vol. 22, American Mathematical Society, 1999, pp. 171–181.

[12] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge University Press, Cambridge, 2005.
[13] A.N. Kirillov, M. Noumi, q-Difference raising operators for Macdonald polynomials and the integrality of transition coefficients, in: Algebraic Methods and q-Special Functions, CRM Proceedings and Lecture Notes, vol. 22, American Mathematical Society, 1999, pp. 227–243.
[14] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998; ⟨http://aw.twi.tudelft.nl/~koekoek/askey/⟩.
[15] T.H. Koornwinder, Lowering and raising operators for some special orthogonal polynomials, in: Jack, Hall-Littlewood and Macdonald Polynomials, Contemporary Mathematics, vol. 417, 2006.
[16] F. Marcellán, A. Branquinho, J. Petronilho, Classical orthogonal polynomials: a functional approach, Acta Appl. Math. 34 (1994) 283–303.
[17] P. Maroni, Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, Ann. Mat. Pura Appl. 149 (4) (1987) 165–184.
[18] J.C. Medem, R. Álvarez-Nodarse, F. Marcellán, On the q-polynomials: a distributional study, J. Comput. Appl. Math. 135 (2001) 157–196.
[19] E.M. Rains, BC_n-symmetric abelian functions, Duke Math. J. 135 (2006) 99–180.
[20] H. Rosengren, An elementary approach to $6j$-symbols (classical, quantum, rational, trigonometric, and elliptic), Ramanjan J., to appear, Preprint, arXiv:math.CA/0312310.
[21] E.K. Sklyanin, Some algebraic structures connected with the Yang–Baxter equation, representations of a quantum algebra, Funct. Anal. Appl. 17 (1983) 273–284.
[22] P. van Moerbeke, Integrable Foundations of String Theory, in: Lectures on Integrable Systems, World Scientific, Singapore, 1994, pp. 163–267.
[23] E. Witten, Two-dimensional Gravity and Intersection Theory on Moduli Space, in: Surveys in Differential Geometry, Lehigh University, Bethlehem, PA, 1991, pp. 243–310.
[24] A.S. Zhedanov, "Hidden symmetry" of Askey–Wilson polynomials, Theoret. Math. Phys. 89 (1991) 1146–1157.