JOURNAL OF ALGEBRA 86, 494–510 (1984)

Conducive Integral Domains

DAVID E. DOBBS

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996

AND

RICHARD FEDDER*

Department of Mathematics, University of Missouri, Columbia, Missouri 65211

Communicated by J. Dieudonné

Received October 12, 1982

This article introduces the concept of a conducive domain, that is, an integral domain each of whose overrings, apart from the quotient field, has nonzero conductor. The principal examples of conducive domains are all $D+M$ constructions and all (pseudo-) valuation domains. Several characterizations are obtained, notably that a domain R is conducive if and only if R has a valuation overring with nonzero conductor. It is proved that if R is a conducive domain but not a field, then Spec(R) is pinched at a prime P such that the set of primes within P is linearly ordered. The converse is shown for R a Prüfer domain, in which case $P = PR_P$. Consequences include pullback characterizations of the seminormal (resp., Prüfer) conducive domains. Special attention is paid to the class of conducive G-domains, with attendant interplay between "conducive" and the property of having a maximum overring.

1. INTRODUCTION

In recent years, one of the most fruitful sources of examples for desired ideal-theoretic behavior in integral domains has been the $D+M$ construction, whose basic properties are conveniently summarized in [3, Theorems 2.1 and 3.1]. The present article initiates the study of a class of domains generalizing the $D+M$ construction. Specifically, we shall say that a (commutative integral) domain R, with quotient field K, is a *conducive domain* in case, for each overring T of R other than K, the conductor $(R:T) = \{t \in K: tT \subset R\}$ is nonzero. Examples of conducive domains include

* This research was supported by a grant from the Research Council of the Graduate School, University of Missouri-Columbia.

0021-8693/84 $3.00

all domains of $D+M$-type (see Proposition 2.2), all valuation domains and, more generally, all pseudo-valuation domains in the sense of [10] (see Proposition 2.1).

Despite the wide range of behavior encompassed within the above types of rings, Section 3 develops extensive information about conducive domains. The study proceeds by introducing several concepts pertaining to the inclusions within the set of all nontrivial valuation overrings of a given domain. The section's main result, Theorem 3.2, establishes, i.a., that a domain R is conducive if (and only if) $(R:V)\neq 0$ for at least one valuation overring V of R. As a consequence (see Corollary 3.3), if R is a conducive domain which is not a field, then Spec(R) is pinched at some nonzero prime P such that the set of primes within P is linearly ordered by inclusion. Another part of Theorem 3.2 asserts that if R is a seminormal domain which is not a field, then R is conducive if and only if some nonzero prime P of R is such that $P=PR_P$ and R_P is a pseudo-valuation domain. In a more concrete vein, Corollary 3.4 asserts that if R is a Prüfer domain which is not a field, then R is conducive if and only if Spec(R) is pinched. Also noteworthy in Section 3 (cf. also Proposition 2.12) are the following pullback characterizations. If R is a seminormal (resp., Prüfer) domain which is not a field, then R is conducive if and only if $R\cong Vx_kA$, where V is a nontrivial valuation domain with residue field k and A is a seminormal (resp., Prüfer) domain contained in (resp., having quotient field) k.

Section 2 foreshadows many of this paper's main themes and, as noted above, provides our initial concrete examples of conducive domains. Moreover, it is shown that the "conducive" property serves to characterize valuation domains of finite rank (resp., discrete rank 1 valuation domains; resp., one-dimensional pseudo-valuation domains; resp., Noetherian pseudo-valuation domains) amongst suitably larger classes of domains: see Proposition 2.11 (resp., Corollary 2.5; resp., Corollary 2.6; resp., Corollary 2.9). A principal tool for these characterizations is Theorem 2.4, the especially tractable case of Corollary 3.3 in which R has a height 1 prime ideal. Also noteworthy in Section 2 is Proposition 2.12(ii), which establishes the "conducive" property for a family of pullbacks including all $D+M$ constructions and all pseudo-valuation domains.

The conducive domains satisfying the tractable condition in Theorem 2.4 are, apart from fields, those which are also G-domains (in the sense of having nonzero pseudo-radical). In Section 4, we pursue the suggested dichotomy, showing in Proposition 4.3 that a conducive domain, other than a field, is a G-domain if and only if R has a maximum nontrivial overring (which is necessarily the complete integral closure). Also noteworthy in Section 4 are Proposition 4.5, describing some of the pathology in a conducive domain which fails to be a G-domain, and Proposition 4.6, a characterization of the conducive Prüfer G-domains.

Throughout the paper, "dimension" refers to Krull dimension, and R denotes a domain with integral closure R', complete integral closure R^* and quotient field K. Any unexplained terminology is standard, as in [11, 15].

2. First Results and Examples

We begin with some elementary observations about the "conducive" property.

Lemma 2.0. (i) *Each overring of a conducive domain is also conducive.*

(ii) *A domain R is conducive if and only if $(R:V) \neq 0$ for each valuation overring V of R other than K.*

Proof. Just the "if" half of (ii) requires any commentary. For this, let T be an overring of R other than K, select any valuation overring V of T whose maximal ideal lies over a preassigned nonzero prime ideal of T (cf. [15, Theorem 56]), and note that $0 \neq (R:V) \subset (R:T)$.

In Theorem 3.2, we shall obtain a substantial improvement of the statement of Lemma 2.0(ii), in which "each" is replaced by "some" and "other than K" is deleted.

Note that it is straightforward to verify that each valuation domain V must be conducive. Indeed, if T is an overring of V other than the quotient field then $T = V_P$ for some nonzero prime P of V (cf. [15, Theorem 65]), whence $(R:T)$ contains $PV_P = P$.

More generally, one may show that certain so-called i-domains are conducive. Specifically, if R is a domain such that R' is a valuation domain which is finitely generated as an R-module, then R is conducive. To see this via Lemma 2.0(ii), first argue as above that if V is a valuation overring of R other than K, then $(R':V)$ contains some nonzero prime P of R'. Next, use the finite-generation of R' to infer that $(R:R')$ is a nonzero ideal, say, I. Finally, observe that $IP \subset (R:V)$ since $IPV \subset IR' \subset R$. Variants of this result, again using the fact that module-finite overrings have nonzero conductor, may be couched in terms of the module-finite pairs introduced recently by Huckaba and Papick [14].

We next give a second generalization of the fact that valuation domains are conducive. First, recall from [13, Theorem 2.7] and [1, Proposition 2.5] that a domain R is said to be a *pseudo-valuation domain* (or, in short, a PVD) in case some valuation overring of R has the same set of prime ideals as R.

Proposition 2.1. *Each pseudo-valuation domain is conducive.*

Proof. Let R be a PVD. If T is an overring of R other than K, select a nonzero prime ideal P of T. By considering numerators, we have $P \cap R \neq 0$. However, [13, Lemma 1.6] yields $P \cap R \subset (R:T)$, which completes the proof.

In [1, pp. 364–366], it was indicated that pseudo-valuation domains and $D+M$ constructions have somewhat similar pullback characterizations. The next result provides another point of similarity.

PROPOSITION 2.2. *Let V be a valuation domain of the form $V = F + M$, where F is a field and M ($\neq 0$) is the maximal ideal of V. Let D be a subring of F. Then $D+M$ is a conducive domain.*

Proof. For convenience, set $A = D + M$. According to [3, Theorem 3.1], the overrings T of A other than the quotient field are of two types: either $T = E + M$ for some ring E contained between D and F or $T = V_P$ for some nonzero prime P of V. In either case, $(A:T) \neq 0$. Indeed, in the former case, $M \subset (A:T)$; while in the latter case, $P = PV_P \subset (A:T)$. The proof is complete.

Remark 2.3. By pooling the preceding two propositions, we see that the class of conducive domains admits great diversity. In particular, a conducive domain need not be integrally closed, it need not satisfy finiteness conditions (such as being Noetherian, coherent or finite-conductor), and it may have infinitely many maximal ideals. Moreover, unlike the case of a pseudo-valuation domain (whose primes are necessarily linearly ordered by inclusion), a conducive domain need not even have depth at most 1. This is a consequence of the following result, whose proof is left to the reader: in the context of Proposition 2.2, if b, a is a regular (R-) sequence in D, then b, a is also a regular sequence in $A = D + M$.

We pause to recall that a domain R is said to be *Archimedean* in case $\cap Rr^n = 0$ for each nonunit r of R; and that, by a result of Ohm [17, Corollary 1.4], each one-dimensional domain is Archimedean.

THEOREM 2.4. *Let R be a conducive domain. Then*:

(i) *If R is not a field, the Jacobson radical of R is nonzero.*

(ii) *R has at most one height* 1 *prime ideal.*

(iii) *If P is a (the) height* 1 *prime of R, then $P \subset N$ for each nonzero prime N of R.*

Proof. (i) Without loss of generality, R is not quasilocal. Let M be a maximal ideal of R and set $I = (R:R_M)$. Then I is an ideal of R which is nonzero (since R is conducive), is contained in $MR_M \cap R = M$, and satisfies $I = IR_M$. It therefore suffices to prove that $I \subset N$ for each maximal ideal N

of R other than M. To this end, choose $r \in N \backslash M$ and infer from $I = IR_M$ that $I \subset rI$, whence $I \subset N$, as desired.

(ii) If the assertion fails, let M and N be distinct height 1 primes of R. By Lemma 2.0(i), we may replace R by $R_{R \backslash (M \cup N)}$; accordingly, we may suppose that 0, M and N are the only primes of R (cf. [15, Theorem 81]). Let I and r be as in the proof of (i). As above, $I = rI = r^2 I \subset \cdots$, whence $I \subset \bigcap R_N r^n$. However, r is a nonunit of R_N, and R_N (being one dimensional) is Archimedean, so that $I = 0$, the desired contradiction.

(iii) If the assertion fails, choose $s \in P \backslash N$. Set $J = (R : R_N)$. As above, one sees that $0 \neq J = JR_N$, entailing $J = sJ$. Select V to be a valuation overring dominating R_P. As R_P has but one nonzero prime, each nonzero prime of V lies over PR_P. Thus, by replacing V with the localization of V at the (prime) intersection of the nonzero primes of V, we may suppose that V is also one dimensional, and hence Archimedean. However, $J = \bigcap s^n J \subset \bigcap V s^n$, which is 0 since s is a nonunit of V; thus $J = 0$, the desired contradiction, to complete the proof.

One cannot hope to improve upon Theorem 2.4(ii), as it is easy to give examples of infinite-dimensional valuation domains having no prime with finite positive height.

Note that, in the terminology of [10], Theorem 2.4(iii) asserts that P coincides with the pseudo-radical of R.

COROLLARY 2.5. *For a domain R, the following conditions are equivalent*:

(1) *R is a conducive Krull domain which is not a field*;

(2) *R is a discrete rank* 1 *valuation domain.*

Proof. We need only discuss why (1) $\Rightarrow$ (2). If R satisfies (1), the very definition of Krull domain (as in [15, p. 82]) gives $R = \bigcap R_P$, where P ranges over the height 1 primes of R and each such R_P is a discrete rank 1 valuation domain. As Theorem 2.4(ii) guarantees that only one such P exists, $R = R_P$, and the proof is complete.

Before stating the next result, we recall that a domain R (with quotient field K) is called *seminormal* in case $u \in K$, $u^2 \in R$, $u^3 \in R$ entail $u \in R$.

COROLLARY 2.6. *For a one-dimensional domain R, the following conditions are equivalent*:

(1) *R is conducive and seminormal*;

(2) *R is a* PVD.

Proof. (2) $\Rightarrow$ (1): Since any pseudo-valuation domain is seminormal (cf. [2, Proposition 3.1(a)]), the assertion follows from Proposition 2.1.

(1) ⇒ (2): Let R satisfy (1). As $\dim(R) = 1$, Theorem 2.4(ii) assures that R is quasilocal, and so an application of [2, Theorem 3.7] completes the proof.

We next give a variant of the preceding technique.

COROLLARY 2.6 (bis). *If R is a one-dimensional conducive domain, then R' is a* PVD.

Proof. By Lemma 2.0(i), R' inherits the conducive property from R (along with one-dimensionality). However, R', being integrally closed, is certainly seminormal, and so an application of Corollary 2.6 completes the proof.

COROLLARY 2.7. *If R is a conducive Noetherian domain, then R is local and* $\dim(R) \leqslant 1$.

Proof. Suppose that R is not local. Choose distinct maximal ideals M and N of R, and set $I = (R : R_M)$. As in the proof of Theorem 2.4, $I = IR_M$ and $I = rI$ for any element $r \in N \backslash M$. Since R_N is Noetherian, a result of Chevalley [4, Lemma 2] supplies a discrete rank 1 valuation overring V which dominates R_N. In particular, V is Archimedean. Moreover, $I = \bigcap r^n I \subset \bigcap Vr^n$ and r is a nonunit of V, whence $I = 0$, contradicting the hypothesis that R is conducive. Thus R is indeed local, say, with maximal ideal M.

Without loss of generality, R is not a field. Since R is Noetherian, the principal ideal theorem of Krull (cf. [15, Theorem 142] then provides a height 1 prime P of R and, in view of Theorem 2.4(ii), further assures that each nonunit of R lies in P. In other words $M = P$, so that $\dim(R) = 1$, completing the proof.

We next sketch an alternate proof of Corollary 2.7. Note that R' is a conducive domain (by Lemma 2.0(i)) and a Krull domain (by the theorem of Mori–Nagata), and so Corollary 2.5 assures that R' is quasilocal, of dimension at most 1. The desired conclusion then follows by integrality (cf. [15, Theorems 44, 47, and 48]).

Remark 2.8. Despite expectations possibly raised by Corollary 2.6, one cannot strengthen the conclusion of Corollary 2.7 to "R is a pseudo-valuation domain." To see this, let $A = F[[X^2, X^3]]$, the ring of those formal power series in the variable X, over a field F, whose coefficient of X is 0. Apart from itself and its quotient field, A has only its integral closure $B = F[[X]]$ as overring. Since $(A : B) = X^2 B \neq 0$, A is conducive. In addition, A is Noetherian. However, A is not a PVD; indeed, A is not even seminormal. This suggests the next result.

COROLLARY 2.9. *For a Noetherian domain R, the following conditions are equivalent*:

(1) *R is conducive and seminormal*;

(2) *R is a* PVD.

Proof. One may argue that (2) $\Rightarrow$ (1) as in the corresponding part of the proof of Corollary 2.6. As for proving that (1) $\Rightarrow$ (2), one need merely combine Corollaries 2.7 and 2.6.

The next lemma delves deeper into the topic in Theorem 2.4(iii). First, recall that the so-called "divided" property of primes P in, e.g., valuation domains V (viz., $PV_P = P$, which has been used above) is stronger than the comparability conclusion of Theorem 2.4(iii). (In particular, the class of divided domains is properly contained in the class of treed domains: cf. [6, Proposition 2.1].)

LEMMA 2.10. *Let R be a seminormal domain. Then*:

(i) *Let S be an overring of R, and set $J = (R:S)$. Then J is a radical ideal of both S and R. If, in addition, the prime ideals of S are linearly ordered by inclusion and $R \neq S$, then J is a prime ideal of both S and R.*

(ii) *If V is any valuation overring of R other than K and if $(R:V) \neq 0$, then there exists a nonzero prime ideal P of R such that $P = PR_P = PV$ is a prime ideal of V.*

(iii) *If R is conducive and has a height 1 prime P, then $P = PR_P$.*

Proof. (i) Let $u \in \mathrm{rad}_S(J)$. Then $u^n \in J$ for some $n \geqslant 1$, so that $u^m \in JS = J \subset R$ for each $m \geqslant n$. As R is seminormal, the criterion in [12, Theorem 1.1] yields $u \in R$. This shows that $\mathrm{rad}_S(J)$ is an ideal of both S and R, so that $J = \mathrm{rad}_S(J)$ is radical in S; then $J = J \cap R$ is also radical in R. For the second assertion of (i), it is enough to note that the hypothesis on S assures that each proper radical ideal of S is prime (cf. [15, Theorem 9]).

(ii) If $R = V$, then any nonzero prime ideal of R suffices as P. If $R \neq V$, apply (i) with $S = V$, denoting J by P. Since $P \subset PR_P \subset PV_P = P$, it follows that $PR_P = P = PV$. Moreover, $P \neq 0$ by hypothesis.

(iii) If P is the unique maximal ideal of R, the assertion is immediate. In the remaining case, apply the second assertion of (i) with $S = R_P$: since $J = (R:R_P)$ is a nonzero prime of both R and R_P, we have $PR_P = J = J \cap R = P$. This completes the proof.

It is convenient to recall next that a domain R is called a *finite-conductor domain* in case, for each $v \in K$, the conductor $\{r \in R : rv \in R\}$ is a finitely generated ideal of R. Examples of finite-conductor domains include all coherent domains (and, a fortiori, Prüfer domains and Noetherian domains), all GCD- (pseudo-Bézout) domains, and certain $D+M$ constructions [7, Proposition 3.9]. The next result is motivated by the observation (cf. [13, Proposition 2.2]) that a GCD-domain whose primes are linearly ordered by inclusion must be a valuation domain.

PROPOSITION 2.11. *For a domain R, the following conditions are equivalent*:

(1) *R is an integrally closed, conducive and finite-conductor domain such that R/P is a* PVD *for some height* 1 *prime P of R*;

(2) *R is a valuation domain with a height* 1 *prime.*

Proof. (2) $\Rightarrow$ (1): This is evident in view of the above remarks.

(1) $\Rightarrow$ (2): We shall give two proofs. First, by appealing to the characterization of valuation domains in [16, Theorem 1], it is enough to prove that the prime ideals of R are linearly ordered by inclusion. Since the primes of (the pseudo-valuation domain) R/P do have this ordering property, one need merely apply Theorem 2.4(iii), completing the first proof.

Here is a more baroque proof. According to the criterion in [8, Theorem 2.3(i)], note that the hypothesis about R/P combines with Lemma 2.10(iii) to yield that $R \subset R_P$ is a strong extension in the sense of [8]. Moreover, as R_P is one-dimensional, integrally closed and finite-conductor, an application of either [16, Theorem 1] or [5, Corollary 4] reveals that R_P is a valuation domain. Thus, according to the criterion in [8, Theorem 2.9], R is a PVD. Another application of either [16, Theorem 1] or [5, Corollary 4] completes the (second) proof.

We next pursue the pullback theme mentioned prior to the statement of Proposition 2.2.

PROPOSITION 2.12. (i) *Let R be a seminormal conducive domain which is not a field. Then R has a nonzero prime ideal P such that R_P is a* PVD *with maximal ideal $P = PR_P$ and R is* (*isomorphic to*) *the pullback* (*in the category of commutative rings with identity*) *of the diagram*

$$\begin{array}{ccc} & & R_P \\ & & \downarrow \\ R/P & \longrightarrow & R_P/P \end{array}$$

in which the vertical (*resp., horizontal*) *map is the canonical surjection* (*resp., injection into the quotient field*).

(ii) *Let R be the pullback of a diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & k \end{array}$$

in which B is a PVD *with nonzero maximal ideal P, k is both the residue field of B and the quotient field of A, and the vertical (resp., horizontal) map is the canonical surjection (resp., injection). Then P is a prime of R, $P = PR_P$, $R_P \cong B$, $R/P \cong A$, and R is conducive. Moreover, R is seminormal (resp., root-closed) if and only if A is seminormal (resp., both A and B are root-closed).*

Proof. (i) Let P be a prime ideal of R of the kind guaranteed by Lemma 2.10(ii). In particular, $P = PR_P$ is a prime ideal of some valuation overring of R, and so, by [13, Theorem 2.7], R_P is a PVD. The final assertion follows since the given diagram's pullback is canonically $R + PR_P$.

(ii) The first four assertions follow by straightforward calculations (cf. [8, Lemma 2.5]). Next, we show that R is conducive, i.e., that $(R:T) \neq 0$ for each overring T of R (other than the quotient field K). To this end, let V denote the valuation overring canonically associated to the pseudo-valuation domain $B (= R_P)$. By aping the proof in [3, Theorem 3.1], one shows that T is comparable to V. If $T \subset V$, then $0 \neq P \subset (R:T)$. If $V \subset T$, then $(V:T) \neq 0$ since V is conducive, $(R:V)$ contains P, and so $(R:T)$ contains the nonzero product $(R:V)(V:T)$, as desired.

In view of the criterion [12, Theorem 1.6] that a domain D is seminormal if and only if the canonical map $\mathrm{Pic}(D) \to \mathrm{Pic}(D[X])$ is an isomorphism, the assertion about seminormality follows easily by a standard argument using Mayer–Vietoris sequences and the five lemma. It may also be proved computationally, by arguing as in the next paragraph.

For the final assertion, assume that both A $(= R/P)$ and B $(= R_P)$ are root-closed. To show that R is root-closed, begin with $u \in K$ such that $u^n \in R$ for some $n \geqslant 1$. It is enough to show that $u \in B$, for then $(u + P)^n \in A$ and, by the hypothesis on A, $u + P \in R/P$, whence $u \in R + P = R$, as desired. However, by the hypothesis on B, $u^n \in R \subset B$ entails $u \in B$, completing the proof of the "if" half. We leave the proof of the "only if" half as a similar, but easier, exercise for the reader.

One consequence of Proposition 2.12(ii) is a new proof of Proposition 2.2. Of course, the main consequence of Proposition 2.12 is a pullback characterization of the conducive seminormal domains. In view of the pullback characterization of pseudo-valuation domains [1, Proposition 2.6], we immediately infer this variant: conducive seminormal domains (other than fields) coincide with the pullbacks Vx_kA, where V is a nontrivial valuation domain with residue field k and A is a seminormal domain contained in k. The next section will present, i.a., additional characterizations of conducive seminormal domains and applications, notably to the Prüfer case.

3. Uniformly Majorized Domains and Funneled Spectra

This section will present applications of a characterization of conducive domains (in Theorem 3.2) and deeper information about the spectra of such rings (Corollary 3.3). It is convenient to begin with the following definitions for a domain R which is properly contained in its quotient field K. We say that R is *uniformly majorized* in case R has a valuation overring $W \neq K$ such that W contains each minimal valuation overring of R. Next, we say that Spec(R) is *pinched* (*at* P) in case R has a nonzero prime ideal P which is comparable under inclusion to each prime of R. Finally, we say that Spec(R) is *funneled* in case Spec(R) is pinched at P and $\{Q \in \text{Spec}(R): Q \subset P\}$ is linearly ordered by inclusion.

LEMMA 3.1. (i) *Let* $\{V_i\}$ *be a nonempty collection of nontrivial valuation domains of a field L. Then there exists a nontrivial valuation ring of L which contains each* V_i *if and only if there exists a subset of L which properly contains* $\{0\}$ *and is a prime ideal of each* V_i.

(ii) *A domain R is uniformly majorized if and only if R has a nontrivial valuation overring W which is comparable under inclusion to each valuation overring of R.*

(iii) *If a domain R is uniformly majorized, then* Spec(R) *is funneled.*

(iv) *If R is a domain such that* R' *is conducive and* $(R:R') \neq 0$, *then R is conducive.*

Proof. (i) Suppose that W is a nontrivial valuation overring of each V_i; let M be the maximal ideal of W. Fix any index j. Now W is the localization of V_j at some prime P_j (cf. [15, Theorem 65]), and so $M = P_j(V_j)_{P_j} = P_j$ is a (nonzero)$'$ of V_j.

Conversely, suppose that M ($\neq 0$) is a prime of each V_i. Fix any index k, and set $W = (V_k)_M$. We shall show, for each index j, that $V_j \subset W$, indeed that $W = (V_j)_M$. This follows from [11, Theorem 17.6(c)] since the maximal ideals coincide: $M(V_K)_M = M = M(V_j)_M$.

(ii) For the "only if" half, let W be a nontrivial valuation overring of R which contains all the minimal valuation overrings of R. To show that W is comparable to any valuation overring T of R, note first that T contains some minimal valuation overring V of R (cf. [11, p. 231]); of course, $W \supset V$

as well. Thus, $T = V_P$ and $W = V_Q$ for suitable prime ideals P and Q of R (cf. [15, Theorem 65]). As V is a valuation domain, P and Q are comparable, and hence so are T and W.

Conversely, let W be a nontrivial valuation overring of R which is comparable to each valuation overring of R. Let $\{V_i\}$ be the set of minimal valuation overrings of R. For each index i, the hypothesis assures that V_i and W are comparable, so that $V_i \subset W$, by the minimality of V_i.

(iii) By (ii), some nontrivial valuation overring W of R is comparable to each valuation overring of R. Let N denote the maximal ideal of W, and set $P = N \cap R$. To see that Spec(R) is pinched at P, consider any prime Q of R, choose a valuation overring V of R whose maximal ideal M satisfies $M \cap R = Q$, and conclude that P and Q are comparable since N and M are comparable (cf. [11, Theorem 17.6(c)]). Moreover, Spec(R) is funneled since $\{Q \in \text{Spec}(R)\colon Q \subset P\} = \{M \cap R\colon M$ is the maximal ideal of a valuation overring of $W\} = \{N_i \cap R\colon N_i \in \text{Spec}(W)\}$ has a linear order induced by that of $\{N_i\}$.

(iv) It is enough to observe that if V is any nontrivial valuation overring of R, then $R' \subset V$, and so $(R:V)$ contains the nonzero product $(R:R')(R':V)$.

THEOREM 3.2. *Let R be a domain which is not a field. Then the following three conditions are equivalent*:

(1) *R is uniformly majorized and $(R:R') \neq 0$*;

(2) *R is conducive*;

(3) *$(R:V) \neq 0$ for some valuation overring V of R.*

Moreover, if R is seminormal, then the above conditions are also equivalent to

(4) *For each nontrivial valuation overring V of R, there exists a nonzero prime ideal P of R such that $P = PR_P = PV$ is a prime ideal of V.*

Proof. (1) ⇒ (2): Since R and R' have the same sets of valuation overrings, it is evident that R' is uniformly majorized if (and only if) R is uniformly majorized. Therefore, by Lemma 3.1(iv), we may suppose that $R = R'$. By Lemma 2.0(ii), it is enough to prove that $(R:V) \neq 0$ for each nontrivial valuation overring V of R. As is well known (cf. [11, p. 231]), $R = \bigcap V_i$, where V_i ranges over the set of minimal valuation overrings of R, and $V \supset V_j$ for some index j. Since R is assumed to be uniformly majorized, Lemma 3.1(i) provides a nonzero set M which is a prime ideal of each V_i. In particular, M is an ideal of $\bigcap V_i = R$, and so $(R:V)$ contains the nonzero product $M(V_j\colon V)$, as desired.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Assume (3). Since $R' \subset V$, (3) assures that $(R' : V) \neq 0$ and $(R : R') \neq 0$. Thus, by earlier comments, we may suppose that $R = R'$. In particular, R is seminormal, and so Lemma 2.10(ii) yields a nonzero prime P such that $P = PR_P = PV$ is a prime of V. We shall show that $W = V_P$ contains each minimal valuation overring T of R. To this end, let N be the maximal ideal of T, and observe from the minimality of T that $Q = N \cap R$ is a maximal ideal of R (cf. [11, p. 231]). As $P = PR_P$, it follows that P is comparable to Q, and so $P \subset Q$, by maximality of Q. Therefore $W \supset T$ by [11, Theorem 17.6(c)], since the corresponding maximal ideals satisfy $PV_P = P \subset N$.

Finally, note that (4) ⇒ (3) trivially; and that (3) ⇒ (4) in case R is seminormal, by Lemma 2.10(ii). This completes the proof.

The next result is a considerable sharpening of Theorem 2.4.

COROLLARY 3.3. *If R is a conducive domain which is not a field, then* Spec(R) *is funneled.*

Proof. Combine Lemma 3.1(iii) and Theorem 3.2.

COROLLARY 3.4. *For a Prüfer domain R which is not a field, the following conditions are equivalent*:

(1) *R is uniformly majorized*;

(2) *R is conducive*;

(3) $(R : R_P) \neq 0$ *for some nonzero prime ideal P of R*;

(4) $P = PR_P$ *for some nonzero prime ideal P of R*;

(5) Spec(R) *is pinched*;

(6) Spec(R) *is funneled.*

Proof. Theorem 3.2 immediately implies the equivalence of conditions (1), (2), (3), and (4). Corollary 3.3 gives (2) ⇒ (6); and it is trivial that (6) ⇒ (5). It will therefore suffice to show that (5) ⇒ (1). For this, suppose that Spec(R) is pinched at P. By Lemma 3.1(ii), it remains only to show that each valuation overring V of R is comparable to R_P. This, in turn, follows since $V = R_Q$ for some prime Q which, by (5), is comparable to P.

By combining Corollary 3.4 with [9, Theorem 2.4(3)] and [8, Lemma 2.5(v)], we easily obtain the following pullback characterization of conducive Prüfer domains (other than fields): they are pullbacks of the form $V x_k A$, where V is a nontrivial valuation domain with residue field k and A is a Prüfer domain with quotient field k.

The remainder of this section is devoted to studying the relation between "conducive" and the following property. A domain R is said to be *majorized* if, for each pair of nontrivial valuation overrings V_1 and V_2 of R, there exists a nontrivial valuation overring W of R which contains both V_1 and V_2. Lemma 3.1(i) easily leads to additional characterizations of majorized domains, which we leave for the reader.

Despite Lemma 3.1(iv) and Theorem 3.2 [(3) ⇒ (2)], one cannot conclude that a domain R is conducive when given only the existence of a conducive overring S such that $(R:S) \neq 0$. The next "descent" result gives a sufficient condition.

PROPOSITION 3.5. *Let R be a majorized domain with an overring S such that $(R:S) \neq 0$. Then R is conducive if (and only if) S is conducive.*

Proof. By Lemma 2.0(ii), it is enough to show that $(R:V) \neq 0$ for each nontrivial valuation overring V of R. Without loss of generality, R is not a field, and so neither is S. Choose a nontrivial valuation overring V_1 of S. Since R is majorized, R has a nontrivial valuation overring W which contains both V and V_1. Then $(R:V)$ contains the nonzero product $(R:S)(S:W)$, as desired. Finally, the parenthetical assertion follows trivially by Lemma 2.0(i). The proof is complete.

The final result of this section should be compared with Theorem 3.2; its proof foreshadows some of the topics to be met in Section 4.

PROPOSITION 3.6. *Each conducive domain is majorized.*

Proof. Consider a conducive domain R which, without loss of generality, is distinct from its quotient field K. Let I denote the pseudo-radical of R. There are two cases.

If $I \neq 0$, select a nonzero element $u \in I$. Since $R[u^{-1}] = K$ by a well-known characterization of the pseudo-radical, $(R:R[u^{-1}]) = 0$, and so u^{-1} is not almost integral over R; in particular, $R^* \neq K$. Let W be a nontrivial valuation overring of R^*. To show that R is majorized, it suffices to show that $V \subset W$ for each nontrivial valuation overring V of R. As R is conducive, $(R:V) \neq 0$, and so [11, Lemma 26.5] assures that R^* coincides with V^*, the complete integral closure of V. Since $V \subset V^*$, the assertion follows.

In the remaining case, $I = 0$. Consider a pair of nontrivial valuation overrings V_1 and V_2 of R; as R is conducive, we may choose nonzero elements $r_i \in (R:V_i)$, for $i = 1, 2$. Since $s = r_1 r_2$ is nonzero, s is not in I, and so $T = R[s^{-1}]$ is unequal to K. Let W be a nontrivial valuation overring of T. It remains only to observe that each $V_i \subset W$, but this follows since $V_i \subset R(r_i)^{-1} \subset R[s^{-1}]$.

By combining Theorem 3.2 with Proposition 3.6, one shows readily that, as the terminology suggests, each uniformly majorized domain is indeed majorized.

4. The Dichotomy and Maximized G-Domains

The dichotomy of the title will be given in Proposition 4.3. First, we introduce a tractable kind of uniformly majorized (and majorized) domain. We shall say that a domain R (with quotient field K) is *maximized* in case R has a *maximum overring*, that is, an overring $T \neq K$ such that T contains each nontrivial overring of R. Any such maximum overring is necessarily a one-dimensional valuation domain. For motivation, we recall that the "maximized" concept has figured in characterizations of pseudo-valuation domains amongst certain classes of quasilocal seminormal domains [2, Theorem 3.7]. One may show that each maximized domain is a G-domain (in the sense of [15], a domain with nonzero pseudo-radical). More precisely, we have

PROPOSITION 4.1. *For a domain R, the following conditions are equivalent*:

(1) *R is maximized*;

(2) *There exist a nonzero prime ideal P of R and a subset M of K such that $M \cap R = P$ and, for each nontrivial valuation overring V of R, one has* $\text{rad}_V(PV) = M$.

Moreover, if the above conditions are satisfied, M must be the maximal ideal of the maximum overring of R, and P is a (the) height 1 prime of R contained in each nonzero prime of R, and so R is a G-domain.

Proof. Most of the assertions follow from the proof of Lemma 3.1(i), by taking $\{V_i\}$ to be the set of nontrivial valuation overrings of R. For the remaining assertions, consider a maximum overring W of R. As noted above, the maximal ideal M of W has height 1. For each index i, write $W = (V_i)_{P_i}$, so that P_i $(= M)$ has height 1 in V_i. Thus, if $P = M \cap R$, then P_i is the minimal prime of V_i containing P, and so $P_i = \text{rad}_{V_i}(PV_i)$. Finally, to show that P is the pseudo-radical of R, let Q be any nonzero prime of R, select an index j so that the maximal ideal N_j of V_j satisfies $N_j \cap R = Q$, and observe from $P_j \subset N_j$ that $P \subset Q$. The proof is complete.

By Theorem 3.2, a maximized domain R is conducive if and only if $(R : R') \neq 0$. Further interplay between these concepts is facilitated by the following definition. A domain R will be called *simply conducive* in case each $u \in K$ such that $R[u] \neq K$ satisfies $(R : R[u]) \neq 0$. Of course, each conducive domain is simply conducive. A partial converse will be given in Lemma 4.2(ii).

LEMMA 4.2. (i) *For a domain R which is not a field, the following three conditions are equivalent*:

(1) $R^* = K$;

(2) *R is simply conducive and R is not a G-domain*;

(3) *Each* $u \in K$ *satisfies* $(R : R[u]) \neq 0$.

(ii) *Let R be an integrally closed domain such that* $R^* \neq K$. *Then R is conducive if (and only if) R is simply conducive.*

Proof. (i) This is straightforward, by using the ideas in the fourth and fifth sentences of the proof of Proposition 3.6. Details are left to the reader.

(ii) By the above remarks, it is enough to prove that R is maximized. Indeed, R^* is a maximum overring of R. To see this, consider any element $u \in K$ such that $R[u] \neq K$. The "simply conducive" hypothesis yields $(R : R[u]) \neq 0$, whence $R^* = R[u]^*$ by [11, Lemma 26.5], and we have

$$u \in R[u] \subset R[u]^* = R^*,$$

completing the proof.

The "dichotomy" is now an easy consequence.

PROPOSITION 4.3. *Let R be a conducive domain which is not a field. Then*:

(i) $R^* = K \Leftrightarrow R$ *is not a G-domain* $\Leftrightarrow$ *the pseudo-radical of R is* 0.

(ii) $R^* \neq K \Leftrightarrow R$ *is a G-domain* $\Leftrightarrow$ *the pseudo-radical of R is a height 1 prime* $\Leftrightarrow$ *R is maximized* $\Leftrightarrow R^*$ *is a maximum overring of R.*

Proof. (i) The first equivalence follows from Lemma 4.2(i); the second is standard (cf. [15, Theorem 19]).

(ii) If $R^* \neq K$, then one may argue as in the proof of Proposition 3.6 to show that R^* is a maximum overring of R. Accordingly, one now needs merely to combine (i) with Proposition 4.1.

In view of Proposition 4.1 and Proposition 4.3(ii), it seems worthwhile to note that a G-domain which is not a field need not be maximized. For an example, consider a one-dimensional Noetherian domain with precisely two maximal ideals. Of course, no such example can be conducive, by Proposition 4.3(ii).

COROLLARY 4.4. *Let R be an integrally closed G-domain which is not a field. Then R is conducive if and only if R is maximized.*

Proof. The "only if" half follows from Proposition 4.3(ii); the "if" half from Theorem 3.2.

Section 1 emphasized the tractable nature of a conducive domain with a height 1 prime. By way of contrast, Proposition 4.3 reveals that any conducive domain which has no height 1 prime and is not a field must exhibit the exotic behavior described in the following result.

PROPOSITION 4.5. *If R is a conducive domain which is not a G-domain, then $\bigcap Rr^n \neq 0$ for each nonzero element r of R and each nonzero prime ideal of R has infinite height.*

Proof. For the first assertion, consider a nonzero nonunit r of R and set $T = R[r^{-1}]$. As R is not a G-domain, $T \neq K$, and so the (simply) conducive condition assures that $I = (R:T)$ is nonzero. Moreover, I coincides with the localization of I at the multiplicatively closed set generated by r. In particular $I = rI$, so that $I \subset \bigcap Rr^n$, as desired.

For the second assertion, note via Theorems 2.4(ii), (iii) (or Corollary 3.3) that the presence in R of a nonzero prime of finite height would entail that the pseudo-radical of R is a nonzero (height 1) prime, contradicting the hypothesis that R is not a G-domain. This completes the proof.

Finally, it seems natural to ask for a global analogue of Proposition 2.11, that is, a characterization of the conducive Prüfer G-domains. Such rings abound: consider $\mathbb{Z} + X\mathbb{Q}[[X]]$. More generally, if $F + M$ is a one-dimensional valuation domain and D is a Prüfer domain with quotient field F, then $D + M$ is a conducive Prüfer G-domain.

PROPOSITION 4.6. *For a Prüfer domain R which is not a field, the following conditions are equivalent:*

(1) *$P = PR_P$ for some height 1 prime ideal P of R;*

(2) *The pseudo-radical of R is a nonzero prime ideal;*

(3) *R is a conducive G-domain.*

Proof. (1) $\Rightarrow$ (2): If P is as in (1), then P is comparable to each principal ideal of R, and hence P is the pseudo-radical of R.

(2) $\Rightarrow$ (3): Let I be the pseudo-radical of R. Since $I \neq 0$ by (2), Zorn's lemma assures that each prime containing I must contain a (nonzero) prime P minimal over I. By the definitions of I and P, we see that P has height 1; and since I is prime, $I = P$.

As R is integrally closed, Theorem 3.2 (or Corollary 4.4) shows that it suffices to prove that R_P is a maximum overring of R. To this end, let T be any nontrivial overring of R, expand T to a nontrivial valuation domain V, observe that $V = R_Q$ for some nonzero prime Q since R is a Prüfer domain (cf. [15, Theorem 65]), and note $P \subset Q$ since P is the pseudo-radical. Thus $T \subset V \subset R_P$, as desired.

(3) $\Rightarrow$ (1): Assume (3). Since R is a G-domain, it follows as above that R has a height 1 prime ideal P. Since R is seminormal and conducive,

we may apply Lemma 2.10(ii) with $V = R_P$, to show that $P = PR_P$. The proof is complete.

Acknowledgments

The authors are grateful to David Anderson, Marco Fontana, Jim Huckaba, and Ira Papick for stimulating conversations on some of the questions discussed herein.

Note added in proof. We have noticed recently that the equivalence $(2) \Leftrightarrow (3)$ in our Theorem 3.2 has been obtained also in [3, Theorem 4.5], and are pleased to record the priority of Bastida-Gilmer in this regard.

References

1. D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, *Canad. J. Math.* **32** (1980), 362–384.
2. D. F. Anderson, D. E. Dobbs, and J. A. Huckaba, On seminormal overrings, *Comm. Algebra* **10** (1982), 1421–1448.
3. E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$, *Michigan Math. J.* **20** (1973), 79–95.
4. C. Chevalley, La notion d'anneau de décomposition, *Nagoya Math. J.* **7** (1954), 21–33.
5. D. E. Dobbs, On going down for simple overrings, *Proc. Amer. Math. Soc.* **39** (1973), 515–519.
6. D. E. Dobbs, Divided rings and going-down, *Pacific J. Math.* **67** (1976), 353–363.
7. D. E. Dobbs, Coherence, ascent of going-down and pseudovaluation domains, *Houston J. Math.* **4** (1978), 551–567.
8. D. E. Dobbs, M. Fontana, J. A. Huckaba, and I. J. Papick, Strong ring extensions and pseudo-valuation domains, *Houston J. Math.* **8** (1982), 167–184.
9. M. Fontana, Topologically defined classes of commutative rings, *Ann. Mat. Pura Appl.* **123** (1980), 331–355.
10. R. Gilmer, The pseudo-radical of a commutative ring, *Pacific J. Math.* **19** (1966), 275–284.
11. R. Gilmer, "Multiplicative Ideal Theory," Dekker, New York, 1972.
12. R. Gilmer and R. C. Heitmann, On Pic($R[X]$) for R seminormal, *J. Pure Appl. Algebra* **16** (1980), 251–257.
13. J. R. Hedstrom and E. G. Houston, Pseudovaluation domains, *Pacific J. Math.* **75** (1978), 137–147.
14. J. A. Huckaba and I. J. Papick, Remarks on module-finite pairs, *Proc. Edinburgh Math. Soc.* **24** (1981), 137–139.
15. I. Kaplansky, "Commutative Rings," revised ed., Univ. of Chicago Press, Chicago, 1974.
16. S. McAdam, Two conductor theorems, *J. Algebra* **23** (1972), 239–240.
17. J Ohm, Some counterexamples related to integral closure in $D[[x]]$, *Trans. Amer. Math. Soc.* **122** (1966), 321–333.