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An initial boundary-value problem for model electromagnetoelasticity system

Viatcheslav Priimenko [a,b,*], Mikhail Vishnevskii [a,b]

[a] *North Fluminense State University, Rod. Amaral Peixoto, km 163, Imboacica, 27.925-310 Macaé, RJ, Brazil*
[b] *Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Koptyug prosp., 4, 630090 Novosibirsk, Russia*

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Abstract

In this paper we consider an initial boundary-value problem related to the electrodynamics of vibrating elastic media. The aim is to prove an existence and uniqueness result for a model describing the nonlinear interactions of the electromagnetic and elastic waves. We assume that the motion of the continuum occurs at velocities that are much smaller than the propagation velocity of the electromagnetic waves through the elastic medium. The model under study consists of two coupled differential equations, one of them is the hyperbolic equation (an analog of the Lamé system) and another one is the parabolic equation (an analog of the diffusion Maxwell system). One stability result is proved too.

MSC: 35R15; 78Q12

Keywords: Electromagnetoelastic coupling; Nonlinear model; First initial boundary-value problem; Uniqueness and existence theorems; Stability estimate

1. Introduction

In this work we consider an initial boundary-value problem for a model nonlinear system of differential equations related to the electrodynamics of vibrating elastic media. In order to introduce the model we need some preliminary discussions. If an elastic electroconductive medium is embedded in an electromagnetic field, then the elastic waves propagating through the medium

* Corresponding author.
E-mail addresses: slava@lenep.uenf.br (V. Priimenko), mikhail@uenf.br (M. Vishnevskii).

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will excite oscillations of the electromagnetic field and themselves will change under influence of the latter. The waves arising as a result of such an interaction are called as *electromagnetoelastic waves*. The first attempts to apply the theory of electromagnetoelasticity to the investigation of the wave propagation process in electroconductive media were made by Knopoff [10], Chadwick [5], Dunkin and Eringen [6]. For a profound acquaintance with the modern state of the theory of electromagnetoelastic interactions the reader is referred to, e.g., [7,17].

Consider an isotropic electromagnetoelastic medium from the viewpoint of linear elasticity connected with the process of diffusion of electromagnetic waves through slow movement of the medium only, i.e. we neglect by influence of the electrokinetic and piezo effects on the coupling mechanism of the elastic and electromagnetic waves. Equations governing the electrodynamic process are the following ones

$$\nabla \times \vec{H} = \sigma \vec{E} + \sigma \mu_e \vec{U}_t \times \vec{H} + \vec{J}, \tag{1}$$

$$\nabla \times \vec{E} = -\mu_e \vec{H}_t, \qquad \nabla \cdot (\mu_e \vec{H}) = 0. \tag{2}$$

To describe of the elastic waves propagation we use the system

$$\rho \vec{U}_{tt} = \nabla \cdot T(\vec{U}) + \mu_e (\nabla \times \vec{H}) \times \vec{H} + \vec{F}, \tag{3}$$

where the stress tensor $T(\vec{U})$ is defined by

$$T_{i,j} = \lambda \nabla \cdot \vec{U} \delta_{i,j} + \mu (U_{i,x_j} + U_{j,x_i}), \quad 1 \leqslant i, j \leqslant 3. \tag{4}$$

Here $\vec{E} = (E_1, E_2, E_3)$ and $\vec{H} = (H_1, H_2, H_3)$ are the electrical and magnetic components of electromagnetic field, $\vec{U} = (U_1, U_2, U_3)$ is the displacement vector of medium, σ, μ_e, ρ, λ, μ are the electroconductivity, the magnetic permeability, the density of medium and the Lamé coefficients, respectively, $\vec{J}$ is a source of electromagnetic field, $\vec{F}$ is a source of elastic field; and $\delta_{i,j}$ is the Kronecker symbol. Consider 1D case of Eqs. (1)–(4) when all functions depend of (z, t) variables and $\vec{J}$, $\vec{F}$ have the representations

$$\vec{J} = (0, 1, 0) J(z, t), \qquad \vec{F} = (0, 0, 1) F(z, t), \tag{5}$$

where J, F are scalar functions and z stands for the variable x_3. In the case $\rho = \text{const}$, $\mu = \text{const}$ these assumptions allow us to form the following nonlinear model system

$$H_t = \left(\frac{1}{\sigma \mu_e} H_z \right)_z - (H U_t)_z - \left(\frac{1}{\sigma \mu_e} J \right)_z, \tag{6}$$

$$U_{tt} = \left(v_p^2 U_z \right)_z - \frac{\mu_e}{\rho} H H_z + \frac{1}{\rho} F, \tag{7}$$

$$E = \frac{1}{\sigma} H_z - \mu_e H U_t. \tag{8}$$

Here H, E, U, and v_p denote the first component of magnetic field, the second component of electrical field and the third component of elastic field, respectively, $v_p = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of longitudinal elastic wave. For our further purposes it is convenient to have the nondimensional case of Eqs. (6)–(8). After simple transformations we obtain, see [1]

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \tag{9}$$

$$u_{tt} = \left(\nu^2 u_z\right)_z - phh_z + f, \tag{10}$$

$$e = rh_z - hu_t, \tag{11}$$

where h, e, u, j, f are the dimensionless analogues of H, E, U, J, F, respectively; $r^{-1} = \mu_e L V_0 \sigma$ is the magnetic Reynolds number, $p = \mu_e H_0^2 \rho^{-1} V_0^{-2}$, $\nu = v_p / V_0$ is the dimensionless velocity of the elastic waves propagation; and L, V_0, H_0 are the characteristic values of length, velocity and magnetic field, respectively.

Finally, we stress that mathematical problems of the propagation of elastic and electromagnetic waves with account of the interaction of the two fields have been dealt with in the recent years by Avdeev, Goryunov, and Priimenko [2], Avdeev, Goryunov, Soboleva, and Priimenko [3], Burdakova and Yakhno [4], Imomnazarov [9], Merazhov and Yakhno [16], Lavrent'ev (Jr.) and Priimenko [11], Lorenzi and Priimenko [14], Lorenzi and Romanov [15], Priimenko and Vishnevskii [18,19], Romanov [20,21], Yakhno [23], Yakhno and Merazhov [24].

2. Definition of the basic functional spaces

Let Q_T be the bounded set $\Omega \times (0, T)$, i.e. the set of points (z, t) of $\mathbb{R}^2$ with $z \in \Omega = (-l, l)$, $t \in (0, T)$.

The Banach space $L_q(\Omega)$ consists of all measurable functions on Ω that are qth-power ($q \geqslant 1$) summable on Ω provided with the norm $\|v\|_{q,\Omega} = (\int_\Omega |v(z)|^q \,\mathrm{d}z)^{1/q}$. Measurability and summability are to be understood everywhere in the sense of Lebesgue.

The Banach space $L_{q,\gamma}(Q_T)$, $q, \gamma \geqslant 1$, consists of all measurable functions on Q_T with a finite norm $\|v\|_{q,\gamma,Q_T} = (\int_0^T (\int_\Omega |u(z,t)|^q \,\mathrm{d}z)^{\frac{\gamma}{q}} \,\mathrm{d}t)^{1/\gamma}$. In the case $q = \gamma$, the Banach space $L_{q,q}(Q_T)$ will be denoted by $L_q(Q_T)$, and the norm $\|v\|_{q,q,Q_T}$—by $\|v\|_{q,Q_T}$.

Weak (generalized) derivatives are to be understood in the way that is now customary in the majority of books on differential equations, see, for example, [8,22].

$W_q^l(\Omega)$ for l integral is the Banach space consisting of all functions of $L_q(\Omega)$ having weak derivatives of all forms up to order l inclusively, that are qth-power summable on Ω. The norm in $W_q^l(\Omega)$ is defined by the equality

$$\|v\|_{q,\Omega}^{(l)} = \sum_{s=0}^{l} \left\| D_z^s v \right\|_{q,\Omega}.$$

$\mathring{W}_q^l(\Omega)$ is the closure in $W_q^l(\Omega)$ of all functions that are infinitely differentiable and finite in Ω.

$W_q^{2l,l}(Q_T)$ for l integral ($q \geqslant 1$) is the Banach space consisting of the $L_q(Q_T)$-elements having weak derivatives of the form $D_t^r D_z^s$ with any r, s satisfying the inequality $2r + s \leqslant 2l$. The norm in it is defined by the equality

$$\|v\|_{q,Q_T}^{(2l)} = \sum_{j=0}^{2l} \sum_{2r+s=j} \left\| D_t^r D_z^s v \right\|_{q,Q_T}.$$

The summation $\sum_{2r+s=j}$ is taken over all nonnegative integers r and s satisfying the condition $2r + s = j$.

$W_2^{1,k}(Q_T)$, $k = 0, 1$, is the Hilbert space with scalar product

$$(u, v)_{W_2^{1,k}(Q_T)} = \int\limits_{Q_T} (uv + u_z v_z + k u_t v_t)\, dz\, dt.$$

$V_2(Q_T)$ is the Banach space consisting of all $W_2^{1,0}(Q_T)$-elements having a finite norm

$$|v|_{Q_T} = \operatorname{vrai} \max_{t\in[0,T]} \|v\|_{2,\Omega} + \|v_z\|_{2,Q_T},$$

where here and below

$$\|v_z\|_{2,Q_T} = \Big(\int\limits_{Q_T} v_z^2\, dz\, dt \Big)^{1/2}.$$

$V_2^{1,0}(Q_T)$ is the Banach space obtained by completing the set $W_2^{1,1}(Q_T)$ in the norm of $V_2(Q_T)$.

$V_2^{1,1/2}(Q_T)$ is the subset of those elements $v(z,t) \in V_2^{1,0}(Q_T)$ for which

$$\int\limits_0^{T-\tau} \int\limits_\Omega \tau^{-1} \big(v(z, t+\tau) - v(z,t) \big)^2\, dz\, dt \to 0 \quad \text{as } \tau \to 0.$$

A zero over $W_2^{1,0}(Q_T)$, $W_2^{1,1}(Q_T)$, $V_2(Q_T)$, $V_2^{1,0}(Q_T)$, $V_2^{1,1/2}(Q_T)$ means that only those elements of these spaces are taken, which vanish on $S_T = \partial\Omega \times (0,T)$.

$C^{\alpha,\alpha/2}(Q_T)$ is the set of all continuous functions in $\overline{Q}_T$ satisfying Hölder conditions in z with exponent α and in t with exponent $\alpha/2$.

3. Statement of the problem. Weak solutions

We can now state our problem. Consider in Q_T the equations

$$h_t = (r h_z)_z - (h u_t)_z - (rj)_z, \tag{12}$$

$$u_{tt} = \big(\nu^2 u_z\big)_z - p h h_z + f, \tag{13}$$

where $r(z)$, $\nu(z)$ are positive piecewise smooth functions, discontinuous at the points $z = z_k$, $k = 1, 2, \ldots, m$, $-l < z_1 < z_2 < \cdots < z_m < l$; p is a positive number.

The following first initial boundary-value problem is considered for Eqs. (12)–(13) with the initial data

$$h(z,0) = h_0(z), \quad z \in \Omega, \tag{14}$$

$$u(z,0) = u_0(z), \qquad u_t(z,0) = u_1(z), \quad z \in \Omega, \tag{15}$$

and the boundary conditions

$$h(-l,t)=h(l,t)=0, \quad t\in(0,T), \tag{16}$$

$$u(-l,t)=u(l,t)=0, \quad t\in(0,T). \tag{17}$$

Problem (12)–(17) can be considered as a diffraction problem, i.e., as the problem with Q_T partitioned into several domains $Q_T^{(k)}$, $Q_T^{(k)}=\Omega^{(k)}\times(0,T)$, $\Omega^{(k)}=(z_k,z_{k+1})$, $k=0,1,\dots,m$, $z_0=-l$, $z_{m+1}=l$, in each of which there is given parabolic–hyperbolic system (12)–(13) with smooth coefficients and free terms. We wish to find in $\overline{Q}_T$ a solution of this system satisfying:

- in $Q_T^{(k)}$, $k=1,2,\dots,m$, the corresponding equations (12)–(13);
- on the lower base of Q_T the initial condition (14)–(15);
- on the lateral surface of Q_T the boundary conditions (16)–(17);
- at the jump points z_k, $k=1,2,\dots,m$, the following compatibility conditions

$$[h]=[u]=0, \tag{18}$$

$$\big[r(h_z-j)\big]=\big[\nu^2u_z\big]=0. \tag{19}$$

The symbol $[v]$ denotes the jump of the function v as it passes through z_k.

Problems of this type can be reduced by means of a simple technique to problems for the determination of weak (generalized) solutions of ordinary initial boundary-value problems with discontinuous coefficients, see [12, pp. 224–232]. This fact will be used for the analysis of problem (12)–(19).

3.1. *Weak solutions of problem* (12)–(19)

Diffraction problem (12)–(19) can be formulated in weak terms defining its solutions as follows

Definition 3.1. Functions $h(z,t)\in\mathring{V}_2(Q_T)$, $u(z,t)\in\mathring{W}_2^{1,1}(Q_T)$ are called a weak solution of the initial boundary-value problem (12)–(19) if they satisfy the identities

$$-\int\limits_{Q_T} h\eta_t\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_T} rh_z\eta_z\,\mathrm{d}z\,\mathrm{d}t-\int\limits_{Q_T} hu_t\eta_z\,\mathrm{d}z\,\mathrm{d}t$$
$$=\int\limits_{Q_T} rj\eta_z\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{\Omega} h_0(z)\eta(z,0)\,\mathrm{d}z, \tag{20}$$

$$-\int\limits_{Q_T} u_t\zeta_t\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_T} \nu^2u_z\zeta_z\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_T} phh_z\zeta\,\mathrm{d}z\,\mathrm{d}t$$
$$=\int\limits_{Q_T} f\zeta\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{\Omega} u_1(z)\zeta(z,0)\,\mathrm{d}z, \quad u(z,0)=u_0(z),\ z\in\Omega, \tag{21}$$

for all $\eta(z,t)$, $\zeta(z,t)$ from $\mathring{W}_2^{1,1}(Q_T)$ that are equal to zero for $t=T$.

It is possible to define the weak solution of (12)–(19) somewhat differently.

Definition 3.2. Functions $h(z,t) \in \overset{\circ}{V}_2(Q_T), u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ are called a weak solution of problem (12)–(19) if they satisfy for almost all $t_1 \in [0,T]$ the identities

$$\begin{aligned} -\int\limits_{Q_{t_1}} h\eta_t \,\mathrm{d}z\,\mathrm{d}t + \int\limits_{Q_{t_1}} rh_z\eta_z \,\mathrm{d}z\,\mathrm{d}t - \int\limits_{Q_{t_1}} hu_t\eta_z \,\mathrm{d}z\,\mathrm{d}t \\ = \int\limits_{Q_{t_1}} rj\eta_z \,\mathrm{d}z\,\mathrm{d}t + \int\limits_{\Omega} h_0(z)\eta(z,0)\,\mathrm{d}z - \int\limits_{\Omega} h(z,t_1)\eta(z,t_1)\,\mathrm{d}z, \end{aligned} \tag{22}$$

$$\begin{aligned} -\int\limits_{Q_{t_1}} u_t\zeta_t \,\mathrm{d}z\,\mathrm{d}t + \int\limits_{Q_{t_1}} \nu^2u_z\zeta_z \,\mathrm{d}z\,\mathrm{d}t + \int\limits_{Q_{t_1}} phh_z\zeta \,\mathrm{d}z\,\mathrm{d}t \\ = \int\limits_{Q_{t_1}} f\zeta \,\mathrm{d}z\,\mathrm{d}t + \int\limits_{\Omega} u_1(z)\zeta(z,0)\,\mathrm{d}z - \int\limits_{\Omega} u_t(z,t_1)\zeta(z,t_1)\,\mathrm{d}z, \end{aligned} \tag{23}$$

$u(z,0) = u_0(z)$, $z \in \Omega$, where $Q_{t_1} = \Omega \times (0,t_1)$, and $\eta(z,t), \zeta(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$.

Both of the definitions are equivalent to each other. The fulfillment of transmission conditions (19) is understand in the sense of the identities considered in Definition 3.1. For more details we refer of the reader to [18].

4. Main results

4.1. Existence of weak solutions

Suppose that the functions r, ν, the free members j, f, the constant p and the initial data h_0, u_0, u_1 in problem (12)–(19) enjoy the properties

(a) r, ν, j, f are supposed to be piecewise smooth functions with jumps at the points z_m: $-l < z_1 < z_2 < \cdots < z_m < l$; $0 < r_0 \leqslant r(z) \leqslant r_1 < \infty$, $0 < \nu_0 \leqslant \nu(z) \leqslant \nu_1 < \infty$ and p is a positive number;
(b) $h_0 \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, $h_0(\pm l) = 0$, and $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, $u_1 \in L_2(\Omega)$.

Let us show that problem (12)–(19) is solvable; namely, let us prove the following proposition:

Theorem 4.1 *(Existence of weak solution). If conditions* (a)–(b) *are fulfilled, then problem* (12)–(19) *has a weak solution*

$$h \in \overset{\circ}{V}_2(Q_T), \qquad u \in \overset{\circ}{W}{}_2^{1,1}(Q_T).$$

Proof. For a proof we make use of Galerkin's method. Let us check that the functions h, u satisfy identities (22)–(23). Consider in $\overset{\circ}{W}{}_2^1(\Omega)$ a fundamental system of functions $\{\psi_k\}$ and assume it to be orthonormalized in $L_2(\Omega)$ and to be orthogonalized in $W_2^1(\Omega)$. We will seek an approximating solution in the form

$$h^N(z,t) = \sum_{k=1}^{N} a_k^N(t)\psi_k(z), \qquad u^N(z,t) = \sum_{k=1}^{N} b_k^N(t)\psi_k(z), \tag{24}$$

where

$$a_k^N = \left(h^N, \psi_k\right), \qquad b_k^N = \left(u^N, \psi_k\right), \quad k = 1, \dots, N,$$

are determined from the equations

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(h^N, \psi_k\right) = -\left(rh_z^N - u_t^N h^N - rj, \psi_{kz}\right),\\
&\left(h^N(\cdot,0), \psi_k\right) = h_{0k},
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(u^N, \psi_k\right) = -\left(\nu^2 u_z^N - \frac{p}{2}\left(h^N\right)^2, \psi_{kz}\right) + (f, \psi_k),\\
&\left(u^N(\cdot,0), \psi_k\right) = u_{0k}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(u^N(\cdot,0), \psi_k\right) = u_{1k},
\end{aligned} \tag{26}$$

and h_{0k}, u_{0k}, u_{1k} are the Fourier coefficients in $L_2(\Omega)$ of initial data with respect to the system of the functions ψ_k. Equations (25)–(26) are a system of nonlinear ordinary differential equations. Its solution exists on a maximal interval $[0,\tau)$ if $\tau \leqslant T$, and

$$\lim_{t\to\tau} \max_k \left(\left|a_k^N\right|, \left|b_k^N\right|\right) \to \infty.$$

We will prove that $|a_k^N|$, $|b_k^N|$, $k = 1, \dots, N$, are bounded functions for $t \in [0,T]$, and therefore system (25)–(26) will have a unique solution on $[0,T]$ for any $T > 0$. For this purpose we multiply the differential equation in (25) by pa_k^N and the differential equation in (26)—by b_{kt}^N, then sum the obtained equalities over all k from 1 to N; and integrate the result with respect to t from 0 to t_1. The sum of the results obtained gives

$$\begin{aligned}
&\frac{p}{2}\left\|h^N(\cdot,t)\right\|_{2,\Omega}^2\Big|_{t=0}^{t=t_1} + \frac{1}{2}\left\|u_t^N(\cdot,t)\right\|_{2,\Omega}^2\Big|_{t=0}^{t=t_1} + \frac{1}{2}\left\|\nu u_z^N(\cdot,t)\right\|_{2,\Omega}^2\Big|_{t=0}^{t=t_1} + \left\|\sqrt{pr}h_z^N\right\|_{2,Q_{t_1}}^2\\
&\quad = \int_{Q_{t_1}} prjh_z^N \,\mathrm{d}z\,\mathrm{d}t + \int_{Q_{t_1}} fu_t^N \,\mathrm{d}z\,\mathrm{d}t.
\end{aligned} \tag{27}$$

Note that

$$\left\|h^N(\cdot,0)\right\|_{2,\Omega}^2 = \sum_{k=1}^{N} a_k^2(0) \leqslant \|h_0\|_{2,\Omega}^2; \qquad \left\|u_t^N(\cdot,0)\right\|_{2,\Omega}^2 = \sum_{k=1}^{N} b_{kt}^2(0) \leqslant \|u_1\|_{2,\Omega}^2;$$

$$\left\|\nu u_z^N(\cdot,0)\right\|_{2,\Omega}^2 \leqslant \mu_0 \nu_1^2 \|u_{0z}\|_{2,\Omega}^2; \qquad -\frac{1}{2}\int_{Q_{t_1}} pr\left[\left(h_z^N\right)^2 - 2h_z^N j + j^2\right]\mathrm{d}z\,\mathrm{d}t \leqslant 0,$$

where the positive constant μ_0 does not depend on N. Therefore the following inequality is valid

$$\frac{p}{2}\left\|h^N(\cdot,t_1)\right\|_{2,\Omega}^2+\frac{1}{2}\left\|u_t^N(\cdot,t_1)\right\|_{2,\Omega}^2+\frac{1}{2}\left\|\nu u_z^N(\cdot,t_1)\right\|_{2,\Omega}^2+\frac{1}{2}\left\|\sqrt{pr}h_z^N\right\|_{2,Q_{t_1}}^2 \leqslant \mu_1+\int\limits_{Q_{t_1}}\left|fu_t^N\right|\mathrm{d}z\,\mathrm{d}t,$$

where

$$\mu_1=\frac{p}{2}\|h_0\|_{2,\Omega}^2+\frac{1}{2}\|u_1\|_{2,\Omega}^2+\frac{\mu_0\nu_1^2}{2}\|u_{0z}\|_{2,\Omega}^2+\frac{1}{2}\int\limits_{Q_{t_1}} prj^2\,\mathrm{d}z\,\mathrm{d}t$$

does not depend on N. In particular, we have

$$\frac{1}{2}\left\|u_t^N(\cdot,t_1)\right\|_{2,\Omega}^2 \leqslant \mu_1+\left|\int\limits_{Q_{t_1}} fu_t^N\,\mathrm{d}z\,\mathrm{d}t\right|. \tag{28}$$

Integrate the latter inequality with respect to t_1 from 0 to T. This gives

$$\frac{1}{2}\left\|u_t^N\right\|_{2,Q_T}^2 \leqslant \mu_1 T+\int\limits_0^T\left|\int\limits_{Q_{t_1}} fu_t^N\,\mathrm{d}z\,\mathrm{d}t\right|\mathrm{d}t_1. \tag{29}$$

Note that

$$\left|\int\limits_{Q_{t_1}} fu_t^N\,\mathrm{d}z\,\mathrm{d}t\right| \leqslant \frac{\epsilon}{2}\int\limits_{Q_T}\left(u_t^N\right)^2\mathrm{d}z\,\mathrm{d}t+\frac{1}{2\epsilon}\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t. \tag{30}$$

Setting $\epsilon=1/2T$ we have

$$\int\limits_0^T\left|\int\limits_{Q_{t_1}} fu_t^N\,\mathrm{d}z\,\mathrm{d}t\right|\mathrm{d}t_1 \leqslant \frac{1}{4}\int\limits_{Q_T}\left(u_t^N\right)^2\mathrm{d}z\,\mathrm{d}t+T^2\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t. \tag{31}$$

Then inequalities (29), (31) yield

$$\frac{1}{4}\left\|u_t^N\right\|_{2,Q_T}^2 \leqslant \mu_1 T+T^2\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t.$$

Using the latter and inequality (30) with $\epsilon=1/2T$ gives us

$$\left|\int\limits_{Q_{t_1}} fu_t^N\,\mathrm{d}z\,\mathrm{d}t\right| \leqslant \frac{1}{4T}\int\limits_{Q_T}\left(u_t^N\right)^2\mathrm{d}z\,\mathrm{d}t+T\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t \leqslant \mu_1+2T\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t.$$

Therefore from (27) we have for any $t_1 \in (0, T]$

$$\frac{p}{2}\|h^N(\cdot,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \frac{1}{2}\|u_t^N(\cdot,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \frac{1}{2}\|\nu u_z^N(\cdot,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \frac{1}{2}\|\sqrt{pr}h_z^N\|^2_{2,Q_{t_1}} \leqslant 2\mu_1 + 2T\int\limits_{Q_T} f^2\,\mathrm{d}z\,\mathrm{d}t \equiv \mu_2, \tag{32}$$

where the positive constant μ_2 does not depend on N. It follows from (32) that a_k^N, b_k^N, b_{kt}^N are uniformly bounded functions for $t \in [0, t_1]$, $t_1 \leqslant T$. Let us show that for fixed k and arbitrary $N \geqslant k$ they are equicontinuous on $[0, T]$. Indeed, from (25) we have

$$a_k^N(t+\Delta t) - a_k^N(t) = -\int\limits_{Q_{t,t+\Delta t}} \big(rh_z^N - u_t^N h^N - rj\big)\psi_{kz}\,\mathrm{d}z\,\mathrm{d}t,$$

where $Q_{t,t+\Delta t} = \Omega \times (t, t+\Delta t)$. For an estimate of the right-hand side we use the following inequalities

$$\int\limits_{Q_{t,t+\Delta t}} |v_1 v_2 v_3|\,\mathrm{d}z\,\mathrm{d}t \leqslant \|v_1\|_{q_1,\gamma_1,Q_{t,t+\Delta t}} \cdot \|v_2\|_{q_2,\gamma_2,Q_{t,t+\Delta t}} \cdot \|v_3\|_{q_3,\gamma_3,Q_{t,t+\Delta t}},$$
$$q_i, \gamma_i \in [1,\infty),\ i = 1,2,3, \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} = 1. \tag{33}$$

$$\int\limits_{Q_{t,t+\Delta t}} \big|rh_z^N \psi_{kz}\big|\,\mathrm{d}z\,\mathrm{d}t \leqslant r_1\|\psi_{kz}\|_{2,Q_{t,t+\Delta t}} \cdot \big\|h_z^N\big\|_{2,Q_{t,t+\Delta t}}, \tag{34}$$

$$\int\limits_{Q_{t,t+\Delta t}} \big|u_t^N h^N \psi_{kz}\big|\,\mathrm{d}z\,\mathrm{d}t \leqslant \|\psi_{kz}\|_{2,Q_{t,t+\Delta t}} \cdot \big\|u_t^N\big\|_{2,4,Q_{t,t+\Delta t}} \cdot \big\|h^N\big\|_{\infty,4,Q_{t,t+\Delta t}}. \tag{35}$$

Using Theorem 2.2 in [12, pp. 62–63] gives us

$$\|v\|_{q,\Omega} \leqslant \beta\|v_z\|_{2,\Omega}^{\frac{2}{\gamma}} \cdot \|v\|_{2,\Omega}^{1-\frac{2}{\gamma}},$$

where

$$\beta = 2^{\frac{2}{\gamma}}, \quad q \in [2,+\infty],\ \gamma \in [4,+\infty], \quad \frac{1}{\gamma} + \frac{1}{2q} = \frac{1}{4}.$$

After an integration of the latter inequality with respect to the time variable from t to $t + \Delta t$ we obtain

$$\|v\|_{q,\gamma,Q_{t,t+\Delta t}} \leqslant \beta\|v_z\|_{2,Q_{t,t+\Delta t}}^{\frac{2}{\gamma}} \cdot \operatorname{vrai}\max_{\tau\in[t,t+\Delta t]} \|v\|_{2,\Omega}^{1-\frac{2}{\gamma}}.$$

Using Young's inequality allows us to rewrite the latter as

$$\|v\|_{q,\gamma,Q_{t,t+\Delta t}} \leqslant \beta\frac{2}{\gamma}\|v_z\|_{2,Q_{t,t+\Delta t}} + \beta\left(1-\frac{2}{\gamma}\right)\operatorname{vrai}\max_{\tau\in[t,t+\Delta t]}\|v\|_{2,\Omega}. \tag{36}$$

The fact that integrals (34)–(35) tend to zero as $\Delta t \to 0$ gives

$$\left|a_k^N(t+\Delta t) - a_k^N(t)\right| \leqslant \epsilon(\Delta t)\|\psi_{kz}\|_{2,\Omega}$$

with $\epsilon(\Delta t)$ not depending on N and tending to zero as $\Delta t \to 0$, i.e., the equicontinuity of the a_k^N, $N = k, k+1, \ldots,$ in t.

Equicontinuity of functions b_k^N follows from the boundedness of their derivatives. The fact that b_{kt}^N are equicontinuous functions for $N \geqslant k$ is proved in same way as it was done for the functions a_k^N, $N \geqslant k$. By the usual diagonal process we select a subsequence $\{N_m\}$, $m = 1, 2, \ldots,$ such that the function sequences $\{a_k^{N_m}\}$, $\{b_k^{N_m}\}$ be converging uniformly on $[0, T]$ to some continuous functions $a_k(t)$, $b_k(t)$. The functions a_k, b_k define two functions

$$h = \sum_{k=1}^{\infty} a_k\psi_k, \qquad u = \sum_{k=1}^{\infty} b_k\psi_k.$$

To the function $h(\cdot, t)$ the sequence $\{h^{N_m}\}$ converges weakly in $L_2(\Omega)$ and uniformly with respect to t in $[0, T]$. Indeed, for any function $\psi(z)$ from $L_2(\Omega)$ we have

$$\left(h^{N_m} - h, \psi\right) = \sum_{k=1}^{s}(\psi, \psi_k)\left(h^{N_m} - h, \psi_k\right) + \left(\sum_{k=s+1}^{\infty}\left(h^{N_m} - h, (\psi, \psi_k)\psi_k\right)\right) \tag{37}$$

with

$$\left|\left(h^{N_m} - h, \sum_{k=s+1}^{\infty}(\psi, \psi_k)\psi_k\right)\right| \leqslant C_1\left(\sum_{k=s+1}^{\infty}(\psi, \psi_k)^2\right)^{1/2} \equiv C_1 R(s),$$

where the positive constant C_1 does not depend on N_m, s. We choose s so large that $C_1 R(s)$ becomes less than a preassigned $\epsilon > 0$. On the other hand, for fixed s and large enough N_m, the first sum in (37) will be less than ϵ for all t in $[0, T]$. Thus $|(h^{N_m} - h, \psi)|$ can be made less than ϵ for all t in $[0, T]$. It is shown that the sequence $\{h^{N_m}\}$ converges to h weakly in $L_2(\Omega)$, uniformly with respect to $t \in [0, T]$.

The sequence $\{u^{N_m}\}$ is bounded in $L_\infty(0, T; \mathring{W}_2^1(\Omega))$ and the sequence $\{u_t^{N_m}\}$ is bounded in $L_\infty(0, T; L_2(\Omega))$. For this reason the sequence $\{u^{N_m}\}$ converges to u $*$-weakly in $L_\infty(0, T;$ $\mathring{W}_2^1(\Omega))$ and the sequence $\{u_t^{N_m}\}$ converges to u_t $*$-weakly in $L_\infty(0, T; L_2(\Omega))$. The functions u^{N_m} belong to $\mathring{W}_2^{1,1}(Q_T)$ and in virtue of $\mathring{W}_2^{1,1}(Q_T) \hookrightarrow L_2(Q_T)$ the sequence $\{u^{N_m}\}$ converges to the function u strongly in $L_2(Q_T)$ and pointwise a.e. in Q_T. Let us show that the sequence $\{u_{tt}^N\}$ is bounded in $L_2(0, T; H^{-1}(\Omega))$, where $H^{-1}(\Omega)$ is the dual space to $H_0^1(\Omega) = \mathring{W}_2^1(\Omega)$. For this purpose consider a function Ψ from $\mathring{W}_2^1(\Omega)$, such that $\|\Psi\|_{\mathring{W}_2^1(\Omega)} = 1$, $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 \in \operatorname{span}\{\psi_k\}_{k=1}^{k=N}$, and $(\Psi_2, \psi_k) = 0$, $k = 1, \dots, N$. Let us denote by $\langle a, b\rangle$ the pairing between $a \in H^{-1}(\Omega)$ and $b \in \mathring{W}_2^1(\Omega)$. From (26) we have

$$\langle u_{tt}^N, \Psi\rangle = (u_{tt}^N, \Psi_1) = -(\nu^2 u_z^N, \Psi_{1z}) + \frac{p}{2}\big((h^N)^2, \Psi_{1z}\big) + (f, \Psi_1).$$

It is easy to check that $\|\Psi_1\|_{\mathring{W}_2^1(\Omega)} \leqslant 1$. In view of this and estimates obtained above, we get

$$\big|\langle u_{tt}^N, \Psi\rangle\big| \leqslant C_2\mu_2,$$

where the positive constant C_2 does not depend on N. It proves that $\{u_{tt}^N\}$ is bounded in $L_2(0, T; H^{-1}(\Omega))$. Thus we can take for granted

$$u_{tt}^N \rightharpoonup u_{tt} \quad \text{weakly in } L_2\big(0, T; H^{-1}(\Omega)\big).$$

It follows from (32) that one can extract from the sequence $\{h^{N_m}\}$ a subsequence converging to h weakly in $L_2(Q_T)$ together with $\{h_z^{N_m}\}$. Let us show the sequence $\{h_t^N\}$ is bounded in $L_2(0, T; H^{-1}(\Omega))$. For this purpose consider a function $\Phi \in \mathring{W}_2^1(\Omega)$ such that $\|\Phi\|_{\mathring{W}_2^1(\Omega)} = 1$, $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 \in \operatorname{span}\{\psi_k\}_{k=1}^{k=N}$, and $(\Phi_2, \psi_k) = 0$, $k = 1, \dots, N$. Because of

$$h_t^N = \sum_{k=1}^{N} a_{kt}^N \psi_k \in \operatorname{span}\{\psi_k\}_{k=1}^{k=N}$$

the following equality is valid

$$\langle h_t^N, \Phi\rangle = \langle h_t^N, \Phi_1\rangle = -\big(r h_z^N - u_t^N h^N - rj, \Phi_{1z}\big).$$

Using $\|\Phi_1\|_{W_2^1(\Omega)} \leqslant 1$ and (32)–(36) we obtain the inequality

$$\big|\langle h_t^N, \Phi\rangle\big| \leqslant C_3\mu_2,$$

where the positive constant C_3 does not depend on N. The latter proves that $\{h_t^N\}$ is bounded in $L_2(0, T; H^{-1}(\Omega))$. From (32) follows that $\{h^N\}$ is bounded in $L_2(0, T; \mathring{W}_2^1(\Omega))$ too. A subsequence of $\{h^N\}$ converges strongly in $L_2(Q_T)$, which means the convergence of a subsequence of the previous one a.e. in Q_T, see Theorem 5.1 in [13, p. 58]. Without loss of generality we can assume that the sequences $\{h^{N_m}\}$, $\{u^{N_m}\}$ converge to the limit functions h, u in the sense mentioned above. By this reason the sequence $\{h^N\}$ converges a.e. in Q_T.

Let us prove now that the limit functions h, u satisfy equalities (20)–(21). First, we will show that the function h satisfies equality (20). For this purpose we multiply each equation of (25) by a smooth function $\alpha_k(t)$ that is equal to zero for $t = T$, then sum over all k from 1 to $N' \leqslant N$, and integrate the result with respect to t from 0 to T. After an integration by parts we obtain

$$\int_0^T \big(h^N, \Upsilon_t^{N'}\big)\,\mathrm{d}t = \int_0^T \big[\big(r h_z^N, \Upsilon_z^{N'}\big) - \big(u_t^N h^N, \Upsilon_z^{N'}\big) - \big(rj, \Upsilon_z^{N'}\big)\big]\,\mathrm{d}t + \big(h_0^N, \Upsilon_z^{N'}(\cdot, 0)\big), \tag{38}$$

where $\Upsilon^{N'}(z,t)=\sum_{k=1}^{N'}\alpha_k(t)\psi_k(z)$ belongs to $L_\infty(0,T;H_0^1(\Omega))\hookrightarrow L_\infty(Q_T)$. We claim that we can pass to the limit in (38) along the subsequence $\{N_m\}$ selected above, assuming $\Upsilon^{N'}$ fixed, and thereby arrive at (38) with h^{N_m}, u^{N_m} being replaced by h, u. Indeed, let us prove that $\int_0^T (u_t^{N_m}h^{N_m}, \Upsilon_z^{N'})\,\mathrm{d}t$ tends to $\int_0^T (u_t h, \Upsilon_z^{N'})\,\mathrm{d}t$, i.e.,

$$\int_0^T \big(u_t^{N_m}h^{N_m}-u_t h, \Upsilon_z^{N'}\big)\,\mathrm{d}t\to 0, \quad \text{as } m\to\infty.$$

Observe that the integral can be represented in the equivalent form

$$\int_0^T \big(u_t^{N_m}h^{N_m}-u_t h, \Upsilon_z^{N'}\big)\,\mathrm{d}t=\int_0^T \big(u_t^{N_m}\big(h^{N_m}-h\big), \Upsilon_z^{N'}\big)\,\mathrm{d}t+\int_0^T \big(u_t^{N_m}-u_t, h\Upsilon_z^{N'}\big)\,\mathrm{d}t.$$

But the first term in the right-hand side of the latter equality tends to zero for $m\to\infty$ according to the estimates

$$\begin{aligned}
&\int_0^T \big|u_t^{N_m}(\cdot,t)\big(h^{N_m}-h\big)(\cdot,t), \Upsilon_z^{N'}(\cdot,t)\big|\,\mathrm{d}t\\
&\quad\leqslant \big\|\Upsilon_z^{N'}\big\|_{L_\infty(Q_T)}\int_0^T \big\|u_t^{N_m}(\cdot,t)\big\|_{L_2(\Omega)}\big\|\big(h^{N_m}-h\big)(\cdot,t)\big\|_{L_2(\Omega)}\,\mathrm{d}t\\
&\quad\leqslant 2\mu_2 T\big\|\Upsilon_z^{N'}\big\|_{L_\infty(Q_T)}\big\|\big(h^{N_m}-h\big)\big\|_{L_\infty(0,T;L_2(\Omega))}\to 0, \quad \text{as } m\to\infty.
\end{aligned}$$

Observe then that also the last term in (38) tends to 0, since we have proved that $u_t^{N_m}$ converges to u_t $*$-weakly in $L_\infty(0,T;L_2(\Omega))$ and $h\Upsilon_z^{N'}\in L_2(0,T;L_2(\Omega))\hookrightarrow L_1(0,T;L_2(\Omega))$ due to the estimate

$$\big\|h\Upsilon_z^{N'}\big\|_{L_2(0,T;L_2(\Omega))}\leqslant \|h\|_{L_2(0,T;L_\infty(\Omega))}\big\|\Upsilon_z^{N'}\big\|_{L_\infty(0,T;L_2(\Omega))}.$$

But the $\Upsilon^{N'}$ are dense in the space of all functions required in the first definition of a weak solution, see Lemma 4.12 in [12, p. 89]. In view of this h satisfies equality (20) and belongs to be the space $\overset{\circ}{V}_2(Q_T)$. Since

$$u_t^2\in L_{1,\frac{2}{1-2\kappa}}(Q_T), \quad \kappa\in\left(0,\frac{1}{2}\right),$$

we deduce from Theorems 7.1 and 10.1 of [12, pp. 181, 204] that

$$\max_{Q_T}|h|\leqslant C_4, \quad h\in C^{\alpha,\alpha/2}(\overline{Q}_T)$$

for some positive constant C_4. Let us show now that the function u satisfies equality (21). For this purpose we multiply each equation of (26) by the $\alpha_k(t)$, then sum over all k from 1 to $N' \leqslant N$, and integrate the result with respect to t from 0 to T. After an integration by parts we arrive at

$$\int_0^T \left(u_t^N, \Upsilon_t^{N'}\right) \mathrm{d}t = \int_0^T \left[\left(\nu^2 u_z^N, \Upsilon_z^{N'}\right) - \frac{p}{2}\left(\left(h^N\right)^2, \Upsilon_z^{N'}\right) + \left(f, \Upsilon^{N'}\right)\right] \mathrm{d}t$$
$$- \left(u_1^N, \Upsilon^{N'}(\cdot, 0)\right), \quad u^N(z,0) = u_0^N(z), \tag{39}$$

where the $\Upsilon^{N'}$ were defined in (38). In this equality one can pass to the limit with respect to the subsequence $\{N_m\}$ selected above, assuming $\Upsilon^{N'}$ fixed, and thereby arrive at (39) with h^N, u^N being replaced by h, u. Lemma 1.3 of [13, p. 12] shows that the sequence $\{(h^N)^2\}$ converges to h^2 weakly and it allows to pass to the limit in the nonlinear term in (39). Since $\max_{Q_T} |h| \leqslant C_4$ then $\{\int_0^T (h^2, \Upsilon_z^{N'})\, \mathrm{d}t\}$ is bounded for any $\Upsilon^{N'} \in \mathring{W}_2^{1,1}(Q_T)$ and, as the $\Upsilon^{N'}$ are dense in the space considered in the definition of weak solution, we deduce that the function u satisfies equality (21) and is the weak solution from $\mathring{W}_2^{1,1}(Q_T)$, see Lemma 4.12 in [12, p. 89]. Theorem 4.1 is proved. □

From Lemma 4.1 of [12, p. 158] and Theorem 4.1 we get

Corollary 4.1. *Any weak solution $h(z,t)$ of problem* (12)–(19) *from $\mathring{V}_2(Q_T)$ belongs to $\mathring{V}_2^{1,1/2}(Q_T)$.*

Corollary 4.2. *For any function $\phi(z,t) \in L_2(0,T; \mathring{W}_2^1(\Omega))$ is valid the following equality*

$$\int_0^T \langle u_{tt}, \phi\rangle \, \mathrm{d}t = \int_0^T \left\{(f - phh_z, \phi) - \left(\nu^2 u_z, \phi_z\right)\right\} \mathrm{d}t.$$

The latter allows us to conclude that the following equation holds for any function $\xi(z) \in \mathring{W}_2^1(\Omega)$ and for almost all $t \in [0,T]$

$$\langle u_{tt}, \xi\rangle = (f - phh_z, \xi) - \left(\nu^2 u_z, \xi_z\right).$$

Moreover notice that

$$u \in C\left([0,T]; L_2(\Omega)\right), \qquad u_t \in C\left([0,T]; H^{-1}(\Omega)\right).$$

4.2. Uniqueness of weak solution

We need the following lemma to prove uniqueness theorem.

Lemma 4.1. *Suppose $h(z,t) \in \mathring{V}_2^{1,1/2}(Q_T)$, $u(z,t) \in \mathring{W}_2^{1,1}(Q_T)$ are a weak solution of problem* (12)–(19)*. Then the following inequality is valid for almost all $t_1 \in [0,T]$*

$$\frac{1}{2}\int\limits_{\Omega}\big(ph^2(z,t_1)+u_t^2(z,t_1)+\nu^2u_z^2(z,t_1)\big)\,\mathrm{d}z+\frac{1}{2}\int\limits_{Q_{t_1}}prh_z^2\,\mathrm{d}z\,\mathrm{d}t$$

$$\leqslant\int\limits_{\Omega}\big(ph_0^2+u_1^2+\nu^2u_{0,z}^2\big)\,\mathrm{d}z+\int\limits_{Q_{t_1}}prj^2\,\mathrm{d}z\,\mathrm{d}t+2t_1\int\limits_{Q_{t_1}}f^2\,\mathrm{d}z\,\mathrm{d}t, \tag{40}$$

where $Q_{t_1}=\Omega\times(0,t_1)$.

Proof. Suppose h,u are a weak solution of problem (12)–(19). We take as η and ζ in (20)–(21) the functions

$$\hat\eta_{\bar k}(z,t)=\frac{1}{k}\int\limits_{t-k}^{t}\hat\eta(z,\tau)\,\mathrm{d}\tau,\qquad \hat\zeta_{\bar k}(z,t)=\frac{1}{k}\int\limits_{t-k}^{t}\hat\zeta(z,\tau)\,\mathrm{d}\tau,$$

where $\hat\eta(z,\tau),\hat\zeta(z,\tau)\in W_2^{1,1}(Q_{k,T})$, and $\hat\eta(z,\tau)=0$, $\hat\zeta(z,\tau)=0$ for $\tau\in[-k,0]\cup[T-k,T]$, $Q_{k,T}=\Omega\times(-k,T)$, $k\in(0,T)$. In view of $D_t(\hat\eta_{\bar k})=(D_t\hat\eta)_{\bar k}$ we transform the first term in (20) in the following manner

$$-\int\limits_{Q_{T-k}}h\hat\eta_{\bar kt}\,\mathrm{d}z\,\mathrm{d}t=-\int\limits_{Q_{T-k}}h_k\hat\eta_t\,\mathrm{d}z\,\mathrm{d}t=\int\limits_{Q_{T-k}}h_{kt}\hat\eta\,\mathrm{d}z\,\mathrm{d}t.$$

Here we have used the notation

$$h_k(z,t)=\frac{1}{k}\int\limits_{t}^{t+k}h(z,\tau)\,\mathrm{d}\tau$$

and the relation

$$\int\limits_0^T f(t)\hat g_{\bar k}(t)\,\mathrm{d}t=\int\limits_0^{T-k}f_k(t)\hat g(t)\,\mathrm{d}t,$$

valid for any piecewise summable functions $f(t),\hat g(t)$ on $[-k,T]$, one of each is equal to zero on the intervals $[-k,0]$ and $[T-k,T]$. The latter equality is the result of interchanging the order of integration with respect to t and τ. In a similar way we obtain

$$-\int\limits_{Q_{T-k}}u_t\hat\zeta_{\bar kt}\,\mathrm{d}z\,\mathrm{d}t=\int\limits_{Q_{T-k}}u_{ktt}\hat\zeta\,\mathrm{d}z\,\mathrm{d}t.$$

In all other terms of (20)–(21) we also transfer the averaging $()_{\bar k}$ from $\hat\eta,\hat\zeta$ to theirs coefficients. Taking into account the permutability of this averaging with differentiation with respect to z we obtain identities

$$\int_{Q_{T-k}} \left\{h_{kt}\hat{\eta} + (rh_z - hu_t - rj)_k\hat{\eta}_z\right\} \mathrm{d}z\,\mathrm{d}t = 0, \tag{41}$$

$$\int_{Q_{T-k}} \left\{\nu^2 u_{kz}\hat{\zeta}_z + (u_{tt} + phh_z - f)_k\hat{\zeta}\right\} \mathrm{d}z\,\mathrm{d}t = 0. \tag{42}$$

These identities are actually valid for a class of functions $\hat{\eta}$, $\hat{\zeta}$ that is more extended than the class just considered; namely, they are valid for any functions $\hat{\eta}$, $\hat{\zeta}$ that are equal to zero for $t \geqslant t_1$ $(0 \leqslant t_1 \leqslant T - k)$ and are equal to some functions $\eta \in \mathring{V}_2^{1,0}(Q_{t_1})$, $\zeta \in \mathring{W}_2^{1,1}(Q_{t_1})$ for $t \in [0, t_1]$. This property was proved in [18]. Thereby we have

$$\int_{Q_{t_1}} \left\{h_{kt}\eta + (rh_z - hu_t - rj)_k\eta_z\right\} \mathrm{d}z\,\mathrm{d}t = 0,$$

$$\int_{Q_{t_1}} \left\{\nu^2 u_{kz}\zeta_z + (u_{tt} + phh_z - f)_k\zeta\right\} \mathrm{d}z\,\mathrm{d}t = 0.$$

In the latter formulas we take $\eta = ph_k$, $\zeta = u_{kt}$ and represent the corresponding terms in the form

$$\int_{Q_{t_1}} h_{kt}h_k\,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int_{\Omega} h_k^2(z,t)\,\mathrm{d}z\Big|_{t=0}^{t=t_1}, \qquad \int_{Q_{t_1}} u_{ktt}u_{kt}\,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int_{\Omega} u_{kt}^2(z,t)\,\mathrm{d}z\Big|_{t=0}^{t=t_1},$$

$$\int_{Q_{t_1}} \nu^2 u_{kz}u_{kzt}\,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int_{\Omega} \nu^2(z)u_{kz}^2(z,t)\,\mathrm{d}z\Big|_{t=0}^{t=t_1},$$

after which let k tend to zero. By analogy with (27) we obtain

$$\frac{1}{2}\int_{\Omega}\left(ph^2(z,t_1) + u_t^2(z,t_1) + \nu^2 u_z^2(z,t_1)\right)\mathrm{d}z - \frac{1}{2}\int_{\Omega}\left(ph_0^2 + u_1^2 + \nu^2 u_{0,z}^2\right)\mathrm{d}z + \int_{Q_{t_1}} prh_z^2\,\mathrm{d}z\,\mathrm{d}t$$
$$= \int_{Q_{t_1}} (prjh_z + fu_t)\,\mathrm{d}z\,\mathrm{d}t.$$

Note that

$$\int_{Q_{t_1}} pr\left(\frac{1}{2}h_z^2 - jh_z + \frac{1}{2}j^2\right)\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int_{Q_{t_1}} pr(h_z - j)^2\,\mathrm{d}z\,\mathrm{d}t \geqslant 0.$$

In view of this we have

$$\frac{1}{2}\int\limits_{\Omega}\big(ph^2(z,t_1)+u_t^2(z,t_1)+\nu^2u_z^2(z,t_1)\big)\,\mathrm{d}z-\frac{1}{2}\int\limits_{\Omega}\big(ph_0^2+u_1^2+\nu^2u_{0,z}^2\big)\,\mathrm{d}z+\frac{1}{2}\int\limits_{Q_{t_1}}prh_z^2\,\mathrm{d}z\,\mathrm{d}t$$

$$\leqslant\frac{1}{2}\int\limits_{Q_{t_1}}prj^2\,\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_{t_1}}|fu_t|\,\mathrm{d}z\,\mathrm{d}t. \tag{43}$$

The latter gives us

$$\int\limits_{\Omega}u_t^2(z,t_1)\,\mathrm{d}z\leqslant\int\limits_{\Omega}\big(ph_0^2+u_1^2+\nu^2u_{0,z}^2\big)\,\mathrm{d}z+\int\limits_{Q_{t_1}}prj^2\,\mathrm{d}z\,\mathrm{d}t+2\int\limits_{Q_{t_1}}|fu_t|\,\mathrm{d}z\,\mathrm{d}t.$$

Integrating this inequality with respect to the time variable from 0 to t_1 and applying Cauchy's inequality with $\epsilon=1/2t_1$, we obtain

$$\int\limits_{Q_{t_1}}u_t^2\,\mathrm{d}z\,\mathrm{d}t\leqslant4t_1\int\limits_{\Omega}\big(ph_0^2+u_1^2+\nu^2u_{0,z}^2\big)\,\mathrm{d}z+2t_1\int\limits_{Q_{t_1}}prj^2\,\mathrm{d}z\,\mathrm{d}t+4t_1^2\int\limits_{Q_{t_1}}f^2\,\mathrm{d}z\,\mathrm{d}t.$$

Using the latter and Cauchy's inequality with $\epsilon=1/2t_1$ in (43) gives us

$$\frac{1}{2}\int\limits_{\Omega}\big(ph^2(z,t_1)+u_t^2(z,t_1)+\nu^2u_z^2(z,t_1)\big)\,\mathrm{d}z+\frac{1}{2}\int\limits_{Q_{t_1}}prh_z^2\,\mathrm{d}z\,\mathrm{d}t$$

$$\leqslant\int\limits_{\Omega}\big(ph_0^2+u_1^2+\nu^2u_{0,z}^2\big)\,\mathrm{d}z+\int\limits_{Q_{t_1}}prj^2\,\mathrm{d}z\,\mathrm{d}t+2t_1\int\limits_{Q_{t_1}}f^2\,\mathrm{d}z\,\mathrm{d}t.$$

Thus (40) is proved. □

From (40) we obtain (see [18] for details)

$$\max_{Q_T}|h|+\|h_z\|_{2,Q_T}+\|u\|_{\mathring{W}_2^{1,1}(Q_T)}\leqslant C_5,\quad h\in C^{\alpha,\alpha/2}(\overline{Q}_T), \tag{44}$$

where C_5 is a positive constant. Fix now a positive constant T_0. Let us prove that the constant C_5 depends only on T_0 whenever $T\in[0,T_0]$: we will write $C_5=C_5(T_0)$. From inequality (40) it follows that $\|h_z\|_{2,Q_T}$, $\|u\|_{\mathring{W}_2^{1,1}(Q_T)}$ are well defined from above with a constant not depending on T. To estimate $\max_{Q_T}|h|$ we consider (20) as the linear parabolic equation with coefficients satisfying additionally to conditions (7.1)–(7.2) in [12, p. 181] with constants independent of T. It follows from Theorem 7.1 of [12, p. 181] that $\operatorname{vrai}\max_{Q_T}|h|$ is estimated from above with a constant not depending on T. Let us use now Theorem 10.1 of [12, p. 204] taking into account that on the parabolic boundary h is a function from the Hölder space with exponent α in z. Function h is equal to zero in $z=\pm l$ and by this reason has any smoothness on the boundary with respect to t. So, $h\in C^{\alpha,\alpha/2}(\overline{Q}_T)$ and the exponent α is independent of $T\in[0,T_0]$. By this reason $\operatorname{vrai}\max_{Q_T}|h|=\max_{Q_T}|h|$.

Let us now prove the uniqueness result about solvability of problem (12)–(19).

Theorem 4.2 *(Uniqueness of weak solution). A weak solution of* (12)–(19) *is unique.*

Proof. Let $h_k, u_k, k = 1, 2,$ be two weak solutions of problem (12)–(19). Introduce functions v, w by the formulas

$$v(z,t) = h_2(z,t) - h_1(z,t), \qquad w(z,t) = u_2(z,t) - u_1(z,t). \tag{45}$$

The functions v, w are a weak solution of the homogeneous problem

$$v_t = (rv_z)_z - (h_2 w_t + u_{1t} v)_z, \tag{46}$$

$$w_{tt} = \left(\nu^2 w_z\right)_z - p(h_2 v_z + h_{1z} v), \tag{47}$$

$$v(\pm l, t) = w(\pm l, t) = 0, \tag{48}$$

$$v(z,0) = w(z,0) = w_t(z,0) = 0, \tag{49}$$

satisfying at the jump points $z_k, k = 1, 2, \ldots, m,$ the compatibility conditions (18)–(19). Therefore the functions v, w satisfy the integral equalities

$$\int_{Q_T} \left\{-v\eta_t + (rv_z - h_2 w_t - u_{1t} v)\eta_z\right\} \mathrm{d}z\,\mathrm{d}t = 0,$$

$$\int_{Q_T} \left\{-w_t\zeta_t + \nu^2 w_z\zeta_z + p(h_2 v_z + h_{1z} v)\zeta\right\} \mathrm{d}z\,\mathrm{d}t = 0.$$

By analogy with (41)–(42) we obtain

$$\int_{Q_{T-k}} \left\{v_{kt}\hat{\eta} + (rv_z - h_2 w_t - u_{1t} v)_k \hat{\eta}_z\right\} \mathrm{d}z\,\mathrm{d}t = 0,$$

$$\int_{Q_{T-k}} \left\{\nu^2 w_{kz}\hat{\zeta}_z + (w_{tt} + ph_2 v_z + ph_{1z} v)_k \hat{\zeta}\right\} \mathrm{d}z\,\mathrm{d}t = 0. \tag{50}$$

Multiply the first equality in (50) by p and sum the result with the second one. In the result obtained we take

$$\hat{\eta}(z,t) = \begin{cases} \eta(z,t), & \text{if } t \in (0, t_1], \\ 0, & \text{if } t \notin (0, t_1] \end{cases}$$

and

$$\hat{\zeta}(z,t) = \begin{cases} \zeta(z,t), & \text{if } t \in (0, t_1], \\ 0, & \text{if } t \notin (0, t_1], \end{cases}$$

where $\eta = v_k$, $\zeta = w_{kt}$, $t_1 \in (0, T-k]$, and represent the corresponding terms in the form

$$\int\limits_{Q_{T-k}} v_{kt} v_k \,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int\limits_{\Omega} v_k^2 \,\mathrm{d}z\Big|_{t=0}^{t=t_1},$$

$$\int\limits_{Q_{T-k}} w_{ktt} w_{kt} \,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int\limits_{\Omega} w_{kt}^2 \,\mathrm{d}z\Big|_{t=0}^{t=t_1}, \qquad \int\limits_{Q_{T-k}} w_{kt} w_k \,\mathrm{d}z\,\mathrm{d}t = \frac{1}{2}\int\limits_{\Omega} w_k^2 \,\mathrm{d}z\Big|_{t=0}^{t=t_1}.$$

Passing to the limit $k \to 0$ and using the initial data (49) gives us

$$\frac{p}{2}\big\|v(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|w_t(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|\nu w_z(\cdot,t_1)\big\|_{2,\Omega}^2 + \int\limits_{Q_{t_1}} pr v_z^2 \,\mathrm{d}z\,\mathrm{d}t$$
$$= \int\limits_{Q_{t_1}} p(u_{1t} v_z v - h_{1z} v w_t)\,\mathrm{d}z\,\mathrm{d}t. \tag{51}$$

The integral in the right-hand side is estimated using (44)

$$\left|\int\limits_{Q_{t_1}} p(u_{1t} v_z v + h_{1z} v w_t)\,\mathrm{d}z\,\mathrm{d}t\right| \leqslant pC_5\big(\|v_z\|_{2,Q_{t_1}} + \|w_t\|_{2,Q_{t_1}}\big)\cdot \max_{Q_{t_1}}|v|.$$

Thus, for almost all $t_1 \in [0,T]$ the following inequality holds

$$\frac{p}{2}\big\|v(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|w_t(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|\nu w_z(\cdot,t_1)\big\|_{2,\Omega}^2 + \int\limits_{Q_{t_1}} pr v_z^2 \,\mathrm{d}z\,\mathrm{d}t$$
$$\leqslant pC_5\big(\|v_z\|_{2,Q_{t_1}} + \|w_t\|_{2,Q_T}\big)\cdot \max_{Q_T}|v|. \tag{52}$$

Consider the first boundary-value problem for the inhomogeneous parabolic equation (46) with zero boundary and initial conditions and the free term $(h_2 w_t)_z$. If $\|h_2 w_t\|_{2,Q_T} = 0$, then $v \equiv 0$ and, consequently, $w \equiv 0$, see (47)–(49). Consider case $\|h_2 w_t\|_{2,Q_T} = q > 0$. From Theorem 7.1 of [12, p. 181] we deduce that

$$\max_{Q_T}\left|\frac{v}{q}\right| \leqslant C_6(T_0), \quad \forall T \in (0,T_0],$$

where $C_6(T_0)$ depends on T_0 only, whenever $T \in (0,T_0]$. This is proved in the same way as the constant C_5 was estimated previously. Then

$$\max_{Q_T}|v| \leqslant C_5(T_0)C_6(T_0)\|w_t\|_{2,Q_T}, \quad \forall T \in (0,T_0].$$

This gives

$$\frac{p}{2}\big\|v(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|w_t(\cdot,t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|\nu w_z(\cdot,t_1)\big\|_{2,\Omega}^2 + \int\limits_{Q_{t_1}} pr v_z^2 \,\mathrm{d}z\,\mathrm{d}t$$
$$\leqslant pC_5^2(T_0)C_6(T_0)\big(\|v_z\|_{2,Q_{t_1}}\cdot\|w_t\|_{2,Q_T} + \|w_t\|_{2,Q_T}^2\big).$$

Applying Cauchy's inequality with $\epsilon = 2r_0 C_5^{-2}(T_0) C_6^{-1}(T_0)$ to $\|v_z\|_{2,Q_{t_1}} \cdot \|w_t\|_{2,Q_T}$ with r_0 being the positive constant in assumption (a), we obtain the following inequality, valid for almost all $t_1 \in [0, T]$, $T \in (0, T_0]$,

$$\frac{p}{2}\big\|v(\cdot, t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|w_t(\cdot, t_1)\big\|_{2,\Omega}^2 + \frac{1}{2}\big\|\nu w_z(\cdot, t_1)\big\|_{2,\Omega}^2 \leqslant pC_7(T_0)\|w_t\|_{2,Q_T}^2,$$

where

$$C_7(T_0) = C_5^2(T_0)C_6(T_0)\left(1 + \frac{C_5^2(T_0)C_6(T_0)}{4r_0}\right).$$

Integrating the latter with respect to t_1 from 0 to T, with $T \in (0, T_0]$, we have

$$\frac{p}{2}\|v\|_{2,Q_T}^2 + \frac{1}{2}\|w_t\|_{2,Q_T}^2 + \frac{\nu_0^2}{2}\|w_z\|_{2,Q_T}^2 \leqslant TpC_7(T_0)\|w_t\|_{2,Q_T}^2.$$

But it is impossible for $TpC_7(T_0) < 1/2$ in the case when w_t, w_z, v are nonzero functions. Therefore $v = w \equiv 0$ in Q_{T_1} with, e.g., $T_1 = \min\{(4pC_7(T_0))^{-1}, T_0\}$, which proves Theorem 4.2. Applying this result recursively, after a finite number of steps, we conclude that $v = w \equiv 0$ in Q_T. □

4.3. Stability of a weak solution of the problem

Let us show that a weak solution of problem (12)–(19) is stable with respect to variations of the coefficients (except ν) and free terms of the equations, and also the initial conditions. This result will be established in the case when ν is supposed to be a smooth enough function. We have shown that problem (12)–(19) has the unique weak solution $h \in \mathring{V}_2(Q_T)$, $u \in \mathring{W}_2^{1,1}(Q_T)$ having additional properties

$$\max_{Q_T}|h| + \|h_z\|_{2,Q_T} + \big\|u_t(\cdot, t)\big\|_{2,\Omega} + \big\|u_z(\cdot, t)\big\|_{2,\Omega} \leqslant C_8, \tag{53}$$

where the positive constant C_8 does not depend on h, u. Along with problem (12)–(19) consider the family of problems

$$h_t^m = \big(r^m h_z^m\big)_z - \big(h^m u_t^m\big)_z - \big(r^m j^m\big)_z, \quad (z,t) \in Q_T, \tag{54}$$

$$u_{tt}^m = \big(\nu^2 u_z^m\big)_z - ph^m h_z^m - f^m, \quad (z,t) \in Q_T, \tag{55}$$

$$h^m(z,0) = h_0^m(z), \quad z \in \Omega, \tag{56}$$

$$u^m(z,0) = u_0^m(z), \qquad u_t^m(z,0) = u_1^m(z), \quad z \in \Omega, \tag{57}$$

$$u^m(\pm l, t) = h^m(\pm l, t) = 0, \quad t \in (0, T), \tag{58}$$

where $m \in \mathbb{N}$. Suppose that $r^m(z)$, $j^m(z,t)$, $h_0^m(z)$, $u_0^m(z)$, $u_1^m(z)$, $f^m(z,t)$ are smooth functions satisfying the conditions of the uniqueness and existence theorems. In this case transmission conditions (18)–(19) can be dropped owing to the smoothness of the solution. Problems (54)–(58) have the unique weak solutions $h^m, u^m, m \in \mathbb{N}$.

Theorem 4.3. *Suppose the sequence $\{r^m\}$ is uniformly bounded and converges a.e. to r, while the sequences $\{j^m\}$, $\{f^m\}$, $\{h_0^m\}$, $\{u_0^m\}$, $\{u_1^m\}$ converge to j, f, h_0, u_0, u_1 in the norms of the spaces to which they belong according to the conditions of Theorem* 4.1. *Then the weak solutions $h^m \in \mathring{V}_2^{1,0}(Q_T)$, $u^m \in \mathring{W}_2^{1,1}(Q_T)$ converge in such spaces to the weak solution h, u of the limit problem* (12)–(19).

For a proof of this proposition we need the following lemma.

Lemma 4.2. *Let h, u and h^m, u^m be weak solutions of problems* (12)–(19) *and* (54)–(58), *and $v^m(z,t) = h^m(z,t) - h(z,t)$, $w^m(z,t) = u^m(z,t) - u(z,t)$. Then there exists a positive number δ, independent of h, u, h^m, u^m, t_1, t_2, such that for any $t_1, t_2 \in [0,T]$, $0 \leqslant t_2 - t_1 < \delta$, the following inequality holds true*

$$\begin{aligned}
&\operatorname{vrai}\max_{t\in[t_1,t_2]}\bigl\{p\|v^m\|_{2,\Omega}^2 + \|w_t^m\|_{2,\Omega}^2 + 2\|\nu w_z^m\|_{2,\Omega}^2\bigr\} + 2pr_0\|v_z^m\|_{2,Q_{t_1t_2}}^2 \\
&\quad\leqslant pC_9\bigl\{\|(r^m - r)h_z\|_{2,Q_{t_1t_2}}^2 + \|r^m j^m - rj\|_{2,Q_{t_1t_2}}^2\bigr\} + \|f^m - f\|_{2,Q_{t_1t_2}}^2 \\
&\qquad + 2p\|v^m(\cdot,t_1)\|_{2,\Omega}^2 + 2\|w_t^m(\cdot,t_1)\|_{2,\Omega}^2 + 2\|\nu w_z^m(\cdot,t_1)\|_{2,\Omega}^2,
\end{aligned} \tag{59}$$

where $Q_{t_1t_2} = \Omega \times (t_1,t_2)$, and C_9 is a positive constant.

Proof. For a proof of this proposition we subtract from the integral identities (20)–(21) for h^m, u^m the corresponding ones for h, u and write the result of this subtraction in the form of integral identities for the functions v^m, w^m in the following manner

$$\begin{aligned}
&-\int_{Q_T} v^m\eta_t\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T} r^m v_z^m\eta_z\,\mathrm{d}z\,\mathrm{d}t - \int_{Q_T} v^m u_t^m\eta_z\,\mathrm{d}z\,\mathrm{d}t - \int_{Q_T} hw_t^m\eta_z\,\mathrm{d}z\,\mathrm{d}t \\
&\quad= \int_{Q_T}(r^m j^m - rj)\eta_z\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T}(r^m - r)h_z\eta_z\,\mathrm{d}z\,\mathrm{d}t + \int_{\Omega}(h_0^m - h_0)\eta(z,0)\,\mathrm{d}z,
\end{aligned} \tag{60}$$

$$\begin{aligned}
&-\int_{Q_T} w_t^m\zeta_t\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T}\nu^2 w_z^m\zeta_z\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T} ph_z^m v^m\zeta\,\mathrm{d}z\,\mathrm{d}t + \int_{Q_T} phv_z^m\zeta\,\mathrm{d}z\,\mathrm{d}t \\
&\quad= \int_{Q_T}(f^m - f)\zeta\,\mathrm{d}z\,\mathrm{d}t + \int_{\Omega}(u_1^m - u_1)\zeta(z,0)\,\mathrm{d}z, \quad w^m(z,0) = u_0^m(z) - u_0(z).
\end{aligned} \tag{61}$$

We take as η in (60) the function

$$\hat{\eta}_{\bar{k}}(z,t) = \frac{1}{k}\int_{t-k}^{t}\hat{\eta}(z,\tau)\,\mathrm{d}\tau,$$

where $\hat{\eta}(z,t)$ is an arbitrary element of $\mathring{W}_2^{1,1}(Q_{k,T})$ that is equal to zero for $t \geqslant T-k$ and $t \leqslant 0$. In a manner analogous to that of Lemma 4.1 we obtain (the latter integral in formula (60) is equal to zero by the choice of the test function)

$$\int\limits_{Q_{T-k}} v_{kt}^m \hat{\eta}\, \mathrm{d}z\, \mathrm{d}t + \int\limits_{Q_{T-k}} \left(r^m v_z^m - v^m u_t^m - h w_t^m\right)_k \hat{\eta}_z\, \mathrm{d}z\, \mathrm{d}t$$
$$= \int\limits_{Q_{T-k}} \left(r^m j^m - rj + \left(r^m - r\right)h_z\right)_k \hat{\eta}_z\, \mathrm{d}z\, \mathrm{d}t. \tag{62}$$

This equality is actually valid for a class of functions $\hat{\eta}(z,t)$ that is more extensive than the class just considered; namely, it is valid for any function $\hat{\eta}$ that is equal to zero for $t \geqslant \tau$, $\tau \leqslant T-k$ and is equal to some function $\eta(z,t)$ from $\mathring{V}_2^{1,0}(Q_\tau)$ for $t \in [0,\tau]$. Indeed, the set $\mathring{W}_2^{1,1}(Q_{k,T})$ is dense in $\mathring{V}_2^{1,0}(Q_{k,T})$. Thus for any η from $\mathring{V}_2^{1,0}(Q_{k,T})$ there is a sequence of functions η_n from $\mathring{W}_2^{1,1}(Q_{k,T})$ that is strongly convergent to it for $n \to \infty$ in $\mathring{V}_2^{1,0}(Q_{k,T})$. We denote by $\chi_l(t)$ the continuous piecewise-linear functions

$$\chi_l(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ lt, & \text{if } t \in [0, \frac{1}{l}], \\ 1, & \text{if } t \in [\frac{1}{l}, \tau - \frac{1}{l}], \\ l(\tau - t), & \text{if } t \in [\tau - \frac{1}{l}, \tau], \\ 0, & \text{if } t \geqslant \tau \end{cases}$$

with $\tau > 1/l$. Identity (62) is established for $\hat{\eta}_{nl}(z,t) = \eta_n(z,t)\chi_l(t)$ when $\tau \leqslant T-k$. It is easy to see that one can pass to the limit in it as n and $l \to \infty$ and thereby prove the identity

$$\int\limits_{Q_\tau} v_{kt}^m \eta\, \mathrm{d}z\, \mathrm{d}t + \int\limits_{Q_\tau} \left(r^m v_z^m - v^m u_t^m - h w_t^m\right)_k \eta_z\, \mathrm{d}z\, \mathrm{d}t$$
$$= \int\limits_{Q_\tau} \left(r^m j^m - rj + \left(r^m - r\right)h_z\right)_k \eta_z\, \mathrm{d}z\, \mathrm{d}t \tag{63}$$

for any function $\eta \in \mathring{V}_2^{1,0}(Q_\tau)$ when $\tau \leqslant T-k$. In (63) we take $\eta = v_k^m$ and represent the first term in the form

$$\int\limits_{Q_\tau} v_k^m \left(v_k^m\right)_t \mathrm{d}z\, \mathrm{d}t = \frac{1}{2} \int\limits_{\Omega} \left(v_k^m\right)^2 \mathrm{d}z \Big|_{t=0}^{t=\tau},$$

after which we let k tend to zero. This gives

$$\frac{1}{2} \left\| v^m(\cdot,t) \right\|_{2,\Omega}^2 \Big|_{t=0}^{t=\tau} + \int\limits_{Q_\tau} \left(r^m v_z^m - v^m u_t^m - h w_t^m\right) v_z^m\, \mathrm{d}z\, \mathrm{d}t$$
$$= \int\limits_{Q_\tau} \left(r^m j^m - rj + \left(r^m - r\right)h_z\right) v_z^m\, \mathrm{d}z\, \mathrm{d}t. \tag{64}$$

Analogously the following equality can be obtained from (61)

$$\frac{1}{2}\left(\left\|w_t^m(\cdot,t)\right\|_{2,\Omega}^2+\left\|\nu w_z^m(\cdot,t)\right\|_{2,\Omega}^2\right)\Big|_{t=0}^{t=\tau}+\int\limits_{Q_\tau} p v^m h_z^m w_t^m \,\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_\tau} p v_z^m h w_t^m \,\mathrm{d}z\,\mathrm{d}t$$
$$=\int\limits_{Q_\tau}\left(f^m-f\right)w_t^m\,\mathrm{d}z\,\mathrm{d}t. \tag{65}$$

Multiply (64) by p and sum the result with (65). This gives

$$\frac{1}{2}\left(p\left\|v^m(\cdot,t)\right\|_{2,\Omega}^2+\left\|w_t^m(\cdot,t)\right\|_{2,\Omega}^2+\left\|\nu w_z^m(\cdot,t)\right\|_{2,\Omega}^2\right)\Big|_{t=0}^{t=\tau}$$
$$+\int\limits_{Q_\tau} p\left(r^m\left(v_z^m\right)^2+v^m h_z^m w_t^m-v^m v_z^m u_t^m\right)\mathrm{d}z\,\mathrm{d}t$$
$$=\int\limits_{Q_\tau} p\left(v_z^m\left(r^m j^m-rj\right)+\left(r^m-r\right)h_z w_z^m\right)\mathrm{d}z\,\mathrm{d}t+\int\limits_{Q_\tau}\left(f^m-f\right)w_t^m\,\mathrm{d}z\,\mathrm{d}t. \tag{66}$$

Subtract from (66) for $\tau=t$ the same equality for $\tau=t_1$, $t_1<t<t_2$, and consider $\operatorname{vrai}\max_{t\in[t_1,t_2]}$ of the result. We obtain

$$\frac{1}{2}\operatorname{vrai}\max_{t\in[t_1,t_2]}\left\{p\left\|v^m\right\|_{2,\Omega}^2+\left\|w_t^m\right\|_{2,\Omega}^2+\left\|\nu w_z^m\right\|_{2,\Omega}^2\right\}+p\int\limits_{Q_{t_1t_2}} r^m\left(v_z^m\right)^2\mathrm{d}z\,\mathrm{d}t$$
$$\leqslant p\int\limits_{Q_{t_1t_2}}\left\{\left|\left(r^m j^m-rj\right)v_z^m\right|+\left|\left(r^m-r\right)h_z v_z^m\right|+\left|v^m h_z^m w_t^m\right|+\left|v^m u_t^m v_z^m\right|\right\}\mathrm{d}z\,\mathrm{d}t$$
$$+\int\limits_{Q_{t_1t_2}}\left|\left(f^m-f\right)w_t^m\right|\mathrm{d}z\,\mathrm{d}t+\frac{1}{2}p\left\|v^m(\cdot,t_1)\right\|_{2,\Omega}^2+\frac{1}{2}\left\|w_t^m(\cdot,t_1)\right\|_{2,\Omega}^2$$
$$+\frac{1}{2}\left\|\nu w_z^m(\cdot,t_1)\right\|_{2,\Omega}^2. \tag{67}$$

Using (33), (36) (with $q=\infty$, $\gamma=4$) and (53) gives us

$$\int\limits_{Q_{t_1t_2}}\left|v^m u_t^m v_z^m\right|\mathrm{d}z\,\mathrm{d}t\leqslant\left\|v_z^m\right\|_{2,Q_{t_1t_2}}\cdot\left\|u_t^m\right\|_{2,4,Q_{t_1t_2}}\cdot\left\|v^m\right\|_{\infty,4,Q_{t_1t_2}}$$
$$\leqslant\frac{C_8\delta^{1/4}}{\sqrt{2}}\left\|v_z^m\right\|_{2,Q_{t_1t_2}}\cdot\left(\left\|v_z^m\right\|_{2,Q_{t_1t_2}}+\operatorname{vrai}\max_{t\in[t_1,t_2]}\left\|v^m\right\|_{2,\Omega}\right),$$

with $t_2-t_1\leqslant\delta$. To prove the latter inequalities we used (53) and

$$\left\|u_t^m\right\|_{2,4,Q_{t_1t_2}}=\left\{\int\limits_{t_1}^{t_2}\left(\int\limits_{\Omega}\left|u_t^m\right|^2\mathrm{d}z\right)^2\mathrm{d}t\right\}^{1/4}\leqslant C_8\left\{\int\limits_{t_1}^{t_2}\mathrm{d}t\right\}^{1/4}\leqslant C_8\delta^{1/4}.$$

The other terms are estimated in the similar way

$$\int\limits_{Q_{t_1t_2}} \left|v^m h_z^m w_t^m\right| \mathrm{d}z\,\mathrm{d}t \leqslant \left\|h_z^m\right\|_{2,Q_{t_1t_2}} \cdot \left\|w_t^m\right\|_{2,4,Q_{t_1t_2}} \cdot \left\|v^m\right\|_{\infty,4,Q_{t_1t_2}}$$

$$\leqslant \frac{C_8\delta^{1/4}}{\sqrt{2}}\left(\left\|v_z^m\right\|_{2,Q_{t_1t_2}} + \operatorname{vrai}\max_{t\in[t_1,t_2]}\left\|v^m\right\|_{2,\Omega}\right) \cdot \operatorname{vrai}\max_{t\in[t_1,t_2]}\left\|w_t^m\right\|_{2,\Omega},$$

$$\int\limits_{Q_{t_1t_2}} \left|\left(r^m j^m - rj\right)v_z^m\right| \mathrm{d}z\,\mathrm{d}t \leqslant \frac{\epsilon}{2}\left\|v_z^m\right\|^2_{2,Q_{t_1t_2}} + \frac{1}{2\epsilon}\left\|r^m j^m - rj\right\|^2_{2,Q_{t_1t_2}},$$

$$\int\limits_{Q_{t_1t_2}} \left|\left(r^m - r\right)h_z v_z^m\right| \mathrm{d}z\,\mathrm{d}t \leqslant \frac{\epsilon}{2}\left\|v_z^m\right\|^2_{2,Q_{t_1t_2}} + \frac{1}{2\epsilon}\left\|\left(r^m - r\right)h_z\right\|^2_{2,Q_{t_1t_2}},$$

$$\int\limits_{Q_{t_1t_2}} \left|\left(f^m - f\right)w_t^m\right| \mathrm{d}z\,\mathrm{d}t \leqslant \left\|w_t^m\right\|^2_{2,Q_{t_1t_2}} + \frac{1}{4}\left\|f^m - f\right\|^2_{2,Q_{t_1t_2}}$$

$$\leqslant \delta\operatorname{vrai}\max_{t\in[t_1,t_2]}\left\|w_t^m\right\|^2_{2,\Omega} + \frac{1}{4}\left\|f^m - f\right\|^2_{2,Q_{t_1t_2}}.$$

We can choose ϵ, δ such that the following inequalities are valid

$$5C_8\delta^{1/4} \leqslant \sqrt{2}, \qquad p\sqrt{2}C_8\delta^{1/4} + 4\delta \leqslant 1, \qquad 2\epsilon\sqrt{2} + 6C_8\delta^{1/4} \leqslant r_0\sqrt{2}.$$

From (67) this gives

$$\operatorname{vrai}\max_{t\in[t_1,t_2]}\left\{p\left\|v^m\right\|^2_{2,\Omega} + \left\|w_t^m\right\|^2_{2,\Omega} + 2\left\|\nu w_z^m\right\|^2_{2,\Omega}\right\} + 2pr_0\left\|v_z^m\right\|^2_{2,Q_{t_1t_2}}$$
$$\leqslant C_9 p\left\{\left\|\left(r^m - r\right)h_z\right\|^2_{2,Q_{t_1t_2}} + \left\|r^m j^m - rj\right\|^2_{2,Q_{t_1t_2}}\right\} + \left\|f^m - f\right\|^2_{2,Q_{t_1t_2}}$$
$$+ 2p\left\|v^m(\cdot,t_1)\right\|^2_{2,\Omega} + 2\left\|w_t^m(\cdot,t_1)\right\|^2_{2,\Omega} + 2\left\|\nu w_z^m(\cdot,t_1)\right\|^2_{2,\Omega}$$

with some positive constant C_9. Lemma 4.2 is proved. □

Corollary 4.3. *Under the fulfillment of the conditions of Lemma* 4.2 *there is valid the following estimate*

$$p\left\|v^m(\cdot,t_2)\right\|^2_{2,\Omega} + \left\|w_t^m(\cdot,t_2)\right\|^2_{2,\Omega} + 2\left\|\nu w_z^m(\cdot,t_2)\right\|^2_{2,\Omega} + 2pr_0\left\|v_z^m\right\|^2_{2,Q_{t_1t_2}}$$
$$\leqslant C_9 p\left\{\left\|\left(r^m - r\right)h_z\right\|^2_{2,Q_{t_1t_2}} + \left\|r^m j^m - rj\right\|^2_{2,Q_{t_1t_2}}\right\} + \left\|f^m - f\right\|^2_{2,Q_{t_1t_2}}$$
$$+ 2p\left\|v^m(\cdot,t_1)\right\|^2_{2,\Omega} + 2\left\|w_t^m(\cdot,t_1)\right\|^2_{2,\Omega} + 2\left\|\nu w_z^m(\cdot,t_1)\right\|^2_{2,\Omega}.$$

Proof of Theorem 4.3. Let us partition the interval $[0,T]$ into a finite number of intervals of length less than δ by the points $0=t_0<t_1<t_2<\cdots<t_s=T$ with t_i, δ satisfied to the conditions of Lemma 4.2. The s is a finite number and does not depend on h^m, u^m, h, u. An inequality of type (59) is valid for each cylinder $Q_{t_{i-1}t_i}=\Omega\times(t_{i-1},t_i)$, $i=1,2,\ldots,s$. For this reason we obtain in s steps

$$\begin{aligned}
&\operatorname{vrai}\max_{t\in[0,T]}\bigl\{p\bigl\|v^m\bigr\|^2_{2,\Omega}+\bigl\|w^m_t\bigr\|^2_{2,\Omega}+2\nu_0^2\bigl\|w^m_z\bigr\|^2_{2,\Omega}\bigr\}+2pr_0\bigl\|v^m_z\bigr\|^2_{2,Q_T}\\
&\quad\leqslant C_{10}\bigl\{p\bigl\|\bigl(r^m-r\bigr)h_z\bigr\|^2_{2,Q_T}+p\bigl\|r^m j^m-rj\bigr\|^2_{2,Q_T}+\bigl\|f^m-f\bigr\|^2_{2,Q_T}\\
&\qquad+\bigl\|u^m_0-u_0\bigr\|^2_{2,\Omega}+\bigl\|u^m_1-u_1\bigr\|^2_{W^1_2(\Omega)}+p\bigl\|h^m_0-h_0\bigr\|^2_{2,\Omega}\bigr\},
\end{aligned}$$

where C_{10} is a positive constant. The sequence $\{r^m\}$ is bounded and tends to r a.e. in Ω. Then in view of $h_z\in L_2(Q_T)$ and the Dominated Convergence Theorem, see Theorem 5 in [8, p. 648],

$$\bigl\|\bigl(r^m-r\bigr)h_z\bigr\|^2_{2,Q_T}\to 0\quad\text{as } m\to\infty.$$

Analogously we can show that $\|r^m j^m-rj\|^2_{2,Q_T}\to 0$ as $m\to\infty$ too. This proves Theorem 4.3. □

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