Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 51 (2006) 999–1010
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

A Family of Root-Finding Methods with Accelerated Convergence

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Dedicated to Professor Jürgen Herzberger on the occasion of his 65$^{\text{th}}$ birthday.

(Received November 2004; revised and accepted October 2005)

Abstract—A parametric family of iterative methods for the simultaneous determination of simple complex zeros of a polynomial is considered. The convergence of the basic method of the fourth order is accelerated using Newton's and Halley's corrections thus generating total-step methods of orders five and six. Further improvements are obtained by applying the Gauss-Seidel approach. Accelerated convergence of all proposed methods is attained at the cost of a negligible number of additional operations. Detailed convergence analysis and two numerical examples are given.

Keywords—Determination of polynomial zeros, Simultaneous iterative methods, Convergence analysis, Accelerated convergence, R-order of convergence.

1. INTRODUCTION

The problem of determining polynomial zeros has a great importance in theory and practice (for instance, in the theory of control systems, digital signal processing, stability of systems, analysis of transfer functions, various mathematical models, differential and difference equations, eigenvalue problems). Iterative methods for the simultaneous finding all polynomial zeros belong to the most efficient approaches and became practically applicable with the rapid development of digital computers, see, e.g., [1–9] and references cited therein. Since the corresponding iterative formulas run in identical versions, simultaneous methods are very suitable for the implementation on parallel computers, which additionally increases their importance, see [10–13]. Quantitative (initial) conditions for predicting the immediate appearance of a safe and fast convergence of the simultaneous methods, in the spirit of Smale's point estimation theory [14], were studied in details in [8,15].

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The authors would like to thank the anonymous referees who made valuable comments and suggestions that improved the presentation.

 Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2005.10.013

In this paper, we use a fixed-point relation involving a complex parameter to construct a new family of simultaneous methods of the fourth order for solving the polynomial equation $P(z) = 0$ in ordinary complex arithmetic. Interval versions of this family for the inclusion of simple and multiple zeros were studied in [16,17]. The motivation and reasons for developing higher-order methods were discussed in [13]. In order to decrease the computational cost of the basic method, we state another modification of this family which possesses very fast convergence (Section 2). The proposed methods have a high computational efficiency since the acceleration of convergence is attained with only few additional computations. Actually, the increase of the convergence rate is attained by means of Newton's and Halley's corrections which use the already calculated values of P, P', P'' at the points $z_1, \dots, z_n$—the current approximations to the wanted zeros. The main convergence theorems for the total-step as well as single-step methods are established in Section 3. The results of numerical experiments are presented in Section 4.

2. SIMULTANEOUS METHODS FOR FINDING SIMPLE ZEROS

Let P be a monic polynomial with simple zeros $\zeta_1, \dots, \zeta_n$ and let $z_1, \dots, z_n$ be their mutually distinct approximations. For the point

$$z = z_i \qquad (i \in I_n := \{1, \dots, n\}),$$

and a complex parameter $\alpha \neq -1$, let us introduce the notations,

$$\begin{gathered}
\Sigma_{\lambda,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(z_i - \zeta_j)^{\lambda}}, \quad S_{\lambda,i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(z_i - z_j)^{\lambda}} \qquad (\lambda = 1, 2), \quad \varepsilon_i = z_i - \zeta_i, \\
\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \qquad \delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}, \\
F_i = (\alpha+1)\Sigma_{2,i} - \alpha(\alpha+1)\Sigma_{1,i}^2, \qquad f_i = (\alpha+1)S_{2,i} - \alpha(\alpha+1)S_{1,i}^2,
\end{gathered} \tag{2.1}$$

LEMMA 2.1. *For $i \in I_n$ the following identity is valid,*

$$(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - F_i = \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2. \tag{2.2}$$

PROOF. Starting from the identities,

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} = \frac{1}{\varepsilon_i} + \Sigma_{1,i}$$

and

$$\delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2} = -\left(\frac{P'(z_i)}{P(z_i)}\right)' = \sum_{j=1}^{n} \frac{1}{(z_i - \zeta_j)^2} = \frac{1}{\varepsilon_i^2} + \Sigma_{2,i},$$

we obtain

$$\begin{aligned}
(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^* &= (\alpha+1)\left(\frac{1}{\varepsilon_i^2} + \Sigma_{2,i}\right) - \alpha\left(\frac{1}{\varepsilon_i} + \Sigma_{1,i}\right)^2 - (\alpha+1)\Sigma_{2,i} + \alpha(\alpha+1)\Sigma_{1,i}^2 \\
&= \frac{1}{\varepsilon_i^2} - \frac{2\alpha}{\varepsilon_i}\Sigma_{1,i} + \alpha^2\Sigma_{1,i}^2 - \frac{2\alpha}{\varepsilon_i}\Sigma_{1,i} \\
&= \frac{1}{\varepsilon_i^2} - \frac{2\delta_{1,i}}{\varepsilon_i}\left(\delta_{1,i} - \frac{1}{\varepsilon_i}\right) + \alpha^2\left(\delta_{1,i} - \frac{1}{\varepsilon_i}\right)^2 \\
&= \left(\frac{\alpha+1}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2.
\end{aligned}$$

∎

From identity (2.2), we obtain the following fixed-point relation,

$$\zeta_i = z_i - \frac{\alpha + 1}{\alpha\delta_{1,i} + \left[(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - F_i\right]_*^{1/2}} \qquad (i \in I_n), \tag{2.3}$$

assuming that two values of the square root have to be taken in (2.3). The symbol $*$ points to the choice of the proper value of the square root appearing in (2.3) as well as later iterative formulas and some expressions used in the convergence analysis.

Let us introduce some additional notation.

1° The approximations $z_1^{(m)}, \dots, z_n^{(m)}$ of the zeros at the m^{th} iterative step are denoted by $z_1, \dots, z_n$, and the new approximations $z_1^{(m+1)}, \dots, z_n^{(m+1)}$, obtained by some simultaneous iterative method, by $\hat{z}_1, \dots, \hat{z}_n$, respectively.

2°

$$N_i = N(z_i) = \frac{1}{\delta_{1,i}} = \frac{P(z_i)}{P'(z_i)} \qquad \text{(Newton's correction)}, \tag{2.4}$$

$$H_i = H(z_i) = \left[\frac{P'(z_i)}{P(z_i)} - \frac{P''(z_i)}{2P'(z_i)}\right]^{-1} = \frac{2\delta_{1,i}}{\delta_{1,i}^2 + \delta_{2,i}} \qquad \text{(Halley's correction)}. \tag{2.5}$$

3°

$$S_{k,i}(\boldsymbol{a}, \boldsymbol{b}) = \sum_{j=1}^{i-1} \frac{1}{(z_i - a_j)^k} + \sum_{j=i+1}^{n} \frac{1}{(z_i - b_j)^k},$$

$$f_i(\boldsymbol{a}, \boldsymbol{b}) = (\alpha + 1) S_{2,i}(\boldsymbol{a}, \boldsymbol{b}) - \alpha(\alpha + 1) S_{1,i}^2(\boldsymbol{a}, \boldsymbol{b}),$$

where $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ are vectors of distinct complex numbers. If $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{z} = (z_1, \dots, z_n)$, then we will write $S_{k,i}(\boldsymbol{z}, \boldsymbol{z}) = S_{k,i}$ and $f_i(\boldsymbol{z}, \boldsymbol{z}) = f_i$.

4°

$$\boldsymbol{z} = (z_1, \dots, z_n) \qquad \text{(the current approximation)},$$

$$\hat{\boldsymbol{z}} = (\hat{z}_1, \dots, \hat{z}_n) \qquad \text{(the new approximation)},$$

$$\boldsymbol{z}_N = (z_{N,1}, \dots, z_{N,n}), \quad z_{N,i} = z_i - N(z_i) \qquad \text{(the Newton approximation)},$$

$$\boldsymbol{z}_H = (z_{H,1}, \dots, z_{H,n}), \quad z_{H,i} = z_i - H(z_i) \qquad \text{(the Halley approximation)}.$$

We recall that the correction terms (2.4) and (2.5) appear in the iterative formulas,

$$\hat{z} = z - N(z) \qquad \text{(Newton's method)}$$

and

$$\hat{z} = z - H(z) \qquad \text{(Halley's method)},$$

which have quadratic and cubic convergence, respectively.

Putting $\zeta_i := \hat{z}_i$ in (2.3), where $\hat{z}_i$ is a new approximation to the zero ζ_i, and taking certain approximations z_j of ζ_j in the sums involved in F_i (see (2.1)) on the right-hand side of the fixed-point relation (2.3), we obtain approximations f_i of F_i. Then, from (2.3), we construct some families of iterative methods for the simultaneous determination of all simple zeros of a polynomial.

For the total-step methods ("Jacobi" or parallel mode) and single-step methods (serial or "Gauss-Seidel" mode) the abbreviations TS and SS will be used. Besides, we denote the corresponding vectors of approximations as follows,

$$\boldsymbol{z}^{(1)} = \boldsymbol{z} = (z_1, \dots, z_n),$$

$$\boldsymbol{z}^{(2)} = \boldsymbol{z}_N = (z_{N,1}, \dots, z_{N,n}),$$

$$\boldsymbol{z}^{(3)} = \boldsymbol{z}_H = (z_{H,1}, \dots, z_{H,n}).$$

Now, we are able to present three families of total-step methods, denoted with (TS(k)) ($k = 1, 2, 3$), in the unique form,

$$\hat{z}_i = z_i - \frac{\alpha + 1}{\alpha\delta_{1,i} + \left[(\alpha+1)\,\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i\left(\boldsymbol{z}^{(k)}, \boldsymbol{z}^{(k)}\right)\right]_*^{1/2}} \qquad (i \in I_n;\ k=1,2,3;\ \alpha \neq -1). \quad (\mathrm{TS}(k))$$

For some specific values of the parameter α, from the families of methods listed above we obtain special cases of these families as *Ostrowski-like method* ($\alpha = 0$, studied in [18,19], *Laguerre-like method* ($\alpha = 1/(n-1)$, considered in [20]), *Euler-like method* ($\alpha = 1$) and *Halley-like method* ($\alpha = -1$), see [21,22]. The names come from the similarity with the quoted classical methods. Indeed, omitting the sums $S_{1,i}$ and $S_{2,i}$ (involved in f_i) in the above formulas, we obtain the corresponding well-known classical methods.

In our consideration, we will always assume that $\alpha \neq -1$. However, this particular case reduces the proposed family (by applying a limiting process) to the already known method of Halley's type studied in [21,22].

The convergence rate of each of the three total-step methods presented above can be accelerated using any new approximation as soon as it is found. In this way, we construct the following families of single-step methods,

$$\hat{z}_i = z_i - \frac{\alpha + 1}{\alpha\delta_{1,i} + \left[(\alpha+1)\,\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i\left(\hat{\boldsymbol{z}}, \boldsymbol{z}^{(k)}\right)\right]_*^{1/2}} \qquad (i \in I_n;\ k=1,2,3;\ \alpha \neq -1)\,. \quad (\mathrm{SS}(k))$$

REMARK 1. Considering the iterative formulas (TS(k)), (SS(k)), and fixed-point relation (2.3), we observe that a "proper" sign in front of the square root should be chosen (indicated by $*$). We take the sign so that a smaller step $|\hat{z}_i - z_i|$ is obtained.

In a particular case $\alpha = 1/(n-1)$, from the iterative formulas (TS(k)) and (SS(k)) we obtain the Laguerre-like methods considered in [20]. Computationally verifiable initial conditions which provide the guaranteed convergence of the basic total-step method ($k = 1$) of Laguerre's type were established in [23].

3. CONVERGENCE ANALYSIS

In this section, we state the main convergence theorems concerning the total-step methods (TS(k)) and the single-step methods (SS(k)). For simplicity, we will often omit the iteration index m and denote quantities in the latter $(m+1)^{\text{th}}$ iteration by $\widehat{\ }$.

Let us introduce the notation,

$$d = \min_{\substack{i,j \\ i \neq j}} |\zeta_i - \zeta_j|\,, \qquad q = \frac{4n}{d},$$

and suppose that the conditions,

$$|\varepsilon_i| < \frac{d}{4n} = \frac{1}{q} \qquad (i = 1, \dots, n), \tag{3.1}$$

are satisfied. In the sequel, we will assume that $n \geq 3$. Also, in our convergence analysis, we will deal with the parameter α lying in the disk $\{z : |z| < 2.4\}$ centered at the origin.

LEMMA 3.1. *Let $z_1 \dots, z_n$ be distinct approximations to the zeros $\zeta_1, \dots, \zeta_n$, and let $\varepsilon_i = z_i - \zeta_i$, $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$, where $\hat{z}_1, \dots, \hat{z}_n$ are approximations produced by the iterative methods* (TS(k)) *If (3.1) holds and $|\alpha| < 2.4 \wedge \alpha \neq -1$, then*

(i)

$$|\hat{\varepsilon}_i| \leq \frac{q^{k+2}}{n-1} |\varepsilon_i|^3 \sum_{j \neq i} |\varepsilon_j|^k \qquad (i \in I_n;\ k = 1, 2, 3)\,;$$

(ii)

$$|\hat{\varepsilon}_i| < \frac{d}{4n} = \frac{1}{q} \qquad (i = 1, \ldots, n).$$

The proof of Assertions (i) and (ii) is laborious and extensive but elementary, and can be derived applying a technique similar to that used in [20]. For these reasons, we omit the proof.

Let $z_1^{(0)}, \ldots, z_n^{(0)}$ be initial approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of the polynomial P, and let

$$\varepsilon_i^{(m)} = z_i^{(m)} - \zeta_i,$$
$$\varepsilon^{(m)} = \max_{1 \le i \le n} \left|\varepsilon_i^{(m)}\right|,$$

where $z_1^{(m)}, \ldots, z_n^{(m)}$ are approximations obtained in the m^{th} iterative step.

THEOREM 3.1. *Let $|\alpha| < 2.4 \wedge \alpha \neq -1$ and let the inequalities,*

$$\left|\varepsilon_i^{(0)}\right| < \frac{d}{4n} = \frac{1}{q} \qquad (i = 1, \ldots, n), \tag{3.2}$$

hold. Then, the total-step methods (TS(k)) are convergent with the order $k+3$ ($k = 1, 2, 3$).

PROOF. Starting from condition (3.2) (which coincides with (3.1)) and using Assertion (i) of Lemma 3.1, we came to the following inequalities,

$$\left|\varepsilon_i^{(1)}\right| \le \frac{q^{k+2}}{n-1} \left|\varepsilon_i^{(0)}\right|^3 \sum_{j \ne i} \left|\varepsilon_j^{(0)}\right|^k < \frac{1}{q} \qquad (i \in I_n;\ k = 1, 2, 3),$$

which means that the implication,

$$\left|\varepsilon_i^{(0)}\right| < \frac{d}{4n} = \frac{1}{q} \Rightarrow \left|\varepsilon_i^{(1)}\right| < \frac{d}{4n} = \frac{1}{q},$$

is valid (see, also, Assertion (ii) of Lemma 3.1). We can prove by mathematical induction that condition (3.2) implies

$$\left|\varepsilon_i^{(m+1)}\right| \le \frac{q^{k+2}}{n-1} \left|\varepsilon_i^{(m)}\right|^3 \sum_{j \ne i} \left|\varepsilon_j^{(m)}\right|^k < \frac{1}{q} \qquad (i \in I_n;\ k = 1, 2, 3), \tag{3.3}$$

for each $m = 0, 1, \ldots$ and $i \in I_n$. Replacing $|\varepsilon_i^{(m)}| = t_i^{(m)}/q$ in (3.3), we get

$$t_i^{(m+1)} \le \frac{\left(t_i^{(m)}\right)^3}{n-1} \sum_{j \ne i} \left(t_j^{(m)}\right)^k \qquad (i \in I_n;\ k = 1, 2, 3). \tag{3.4}$$

Let $t^{(m)} = \max_{1 \le i \le n} t_i^{(m)}$. First, from (3.2), it follows

$$q|\varepsilon_i^{(0)}| = t_i^{(0)} \le t^{(0)} < 1 \qquad (i \in I_n).$$

Successive application of the inequalities of this type to (3.4) gives $t_i^{(m)} < 1$ for all $i \in I_n$ and $m = 1, 2, \ldots$. According to this, we get from (3.4),

$$t_i^{(m+1)} \le \left(t_i^{(m)}\right)^3 \left(t^{(m)}\right)^k \le \left(t^{(m)}\right)^{k+3} \qquad (k = 1, 2, 3). \tag{3.5}$$

From (3.5), we infer that the sequences $\{t_i^{(m)}\}$ $(i \in I_n)$ converge to 0, which means that the sequences $\{|\varepsilon_i^{(m)}|\}$ are also convergent, that is $z_i^{(m)} \to \zeta_i$ $(i \in I_n)$. Finally, from (3.5), we conclude that the total-step methods (TS(k)) have the convergence order $k+3$, that is, the basic total-step methods (TS(1)), the total-step methods with Newtons's corrections (TS(2)) and the total-step methods with Halley's corrections (TS(3)) have the order of convergence *four*, *five*, and *six*, respectively. ∎

Let us consider now the convergence rate of the single-step method (SS(k)). Applying the same technique and argumentations presented in [20] and starting from the initial conditions (3.2), we can prove that the inequalities,

$$\left|\varepsilon_i^{(m+1)}\right| \le \frac{q^{k+2}}{n-1}\left|\varepsilon_i^{(m)}\right|^3 \left(\sum_{j=1}^{i-1}\left|\varepsilon_j^{(m+1)}\right| + q^{k-1}\sum_{j=i+1}^{n}\left|\varepsilon_j^{(m)}\right|^k\right) < \frac{1}{q} \qquad (i \in I_n;\ k = 1,2,3), \tag{3.6}$$

hold for each $m = 0,1,\dots$ and $i \in I_n$, supposing that for $i = 1$, the first sum in (3.6) is omitted.

Substituting $|\varepsilon_i^{(m)}| = t_i^{(m)}/q$ in (3.6), we obtain

$$t_i^{(m+1)} \le \frac{\left(t_i^{(m)}\right)^3}{n-1}\left(\sum_{j=1}^{i-1} t_j^{(m+1)} + \sum_{j=i+1}^{n}\left(t_j^{(m)}\right)^k\right) \qquad (i \in I_n;\ k = 1,2,3). \tag{3.7}$$

The convergence analysis of the single-step methods (SS(k)), similar to that presented by Alefeld and Herzberger [5], uses the notion of the R-order of convergence introduced by Ortega and Rheinboldt [24]. The R-order of an iterative process IP with the limit point ζ will be denoted by $O_R((\mathrm{IP}),\ \zeta)$.

THEOREM 3.2. *Assume that initial conditions (3.2) and inequalities (3.6) are valid for the single-step method (SS(k)). Then, the R-order of convergence of (SS(k)) is given by*

$$O_R((\mathrm{SS(k)}), \zeta) \ge 3 + \tau_n(k), \tag{3.8}$$

where $\tau_n(k) > k$ is the unique positive root of the equation,

$$\tau^n - k^{n-1}\tau - 3k^{n-1} = 0. \tag{3.9}$$

PROOF. As in the proof of Theorem 3.1, we first note that condition (3.2) implies

$$\left|\varepsilon_i^{(0)}\right| q = t_i^{(0)} \le t = \max_{1\le i\le n} t_i^{(0)} < 1. \tag{3.10}$$

According to this and (3.7), we conclude that the sequences $\{t_i^{(m)}\}$ $(i \in I_n)$ converge to 0. Hence, the sequences $\{|\varepsilon_i^{(m)}|\}$ are also convergent which means that $z_i^{(m)} \to \zeta_i$ $(i \in I_n)$. Following Alefeld and Herzberger [5], the following system of inequalities can be derived from relations (3.7) and (3.10),

$$t_i^{(m+1)} \le t^{r_i^{(m)}} \qquad (i = 1,\dots,n;\ m = 0,1,\dots). \tag{3.11}$$

The column vectors $\boldsymbol{r}^{(m)} = [r_1^{(m)} \cdots r_n^{(m)}]^\top$ are successively computed by

$$\boldsymbol{r}^{(m+1)} = A_n(k)\,\boldsymbol{r}^{(m)} \tag{3.12}$$

starting with $\boldsymbol{r}^{(0)} = [1 \cdots 1]^\top$. The $n \times n$ matrix $A_n(k)$ in (3.12) is given by

$$A_n(k) = \begin{bmatrix} 3 & k & & & & & \\ & 3 & k & & \mathbf{O} & & \\ & & 3 & k & & & \\ & & & \ddots & \ddots & & \\ & & \mathbf{O} & & & & \\ & & & & & 3 & k \\ 3 & k & 0 & 0 & \cdots & 0 & 3 \end{bmatrix} \qquad (k = 1,2,3),$$

(see [7, Sec. 2.3] for more general case). The characteristic polynomial of the matrix $A_n(k)$ is

$$g_n(\lambda; k) = (\lambda - 3)^n - (\lambda - 3) k^{n-1} - 3k^{n-1}.$$

Setting $\tau = \lambda - 3$, we get

$$g_n(\tau + 3; k) = \tau^n - k^{n-1}\tau - 3k^{n-1}.$$

It is easy to show that the equation,

$$\tau^n - k^{n-1}\tau - 3k^{n-1} = 0,$$

has the unique positive root $\tau_n(k) > k$. The corresponding (positive) eigenvalue of the matrix $A_n(k)$ is $3 + \tau_n(k)$. Using some elements of the matrix analysis we find that the matrix $A_n(k)$ is irreducible and primitive so that it has the unique positive eigenvalue equal to its spectral radius $\rho(A_n(k))$. According to the analysis presented in [5], it can be shown that the spectral radius $\rho(A_n(k))$ gives the lower bound of the R-order of iterative method (SS(k)), for which the inequalities (3.8) are valid. Therefore, we have

$$O_R(\ (\mathrm{SS}(k)),\ \zeta) \geq \rho(A_n(k)) = 3 + \tau_n(k),$$

where $\tau_n(k) > k$ is the unique positive root of equation (3.9). ∎

The lower bounds of $O_R((\mathrm{SS}(1)),\ \zeta)$, $O_R((\mathrm{SS}(2)),\ \zeta)$, and $O_R((\mathrm{SS}(3)),\ \zeta)$ are displayed in Table 1.

Table 1. The lower bound of the R-order of convergence.

Methods \ n	3	4	5	6	7	8	9	10	Very Large n
(SS(1)):	4.672	4.453	4.341	4.274	4.229	4.196	4.172	4.153	$\to 4$
(SS(2)):	5.862	5.586	5.443	5.357	5.299	5.257	5.225	5.201	$\to 5$
(SS(3)):	6.974	6.662	6.503	6.404	6.339	6.291	6.255	6.228	$\to 6$

4. NUMERICAL RESULTS

To illustrate the convergence rate of the considered iterative methods, we tested a great number of polynomials. Total-step as well as single-step methods with the Newton and Halley corrections, presented in this paper, use the already calculated values P, P', P'' at the points $z_1, \dots, z_n$ so that the convergence rate of these iterative methods is accelerated at the price of a negligible number of additional operations. Therefore, the employed approach provides high computational efficiency of the proposed methods with corrections.

For the selection between the two values of the square root, we adopt the criterion proposed by Henrici [25, p. 532].

The argument of the square root is to be chosen to differ by less than $\pi/2$ from the argument of $P'(z_i)$.

This criterion provides the greater denominator in magnitude (between two values), giving a smaller step $|\hat{z}_i - z_i|$. The tested numerical examples showed that the above criterion gives quite satisfactory results even for crude initial approximations.

The performed numerical experiments demonstrated very fast convergence of the modified methods, which is illustrated in the first of the two presented numerical examples.

EXAMPLE 1. The methods (TS(k)) and (SS(k)) were applied to simultaneously approximate the zeros of the polynomial,

$$P(z) = z^{20} - (5 - 4i)z^{19} + (22 - 20i)z^{18} - (72 - 68i)z^{17} + (52 - 188i)z^{16} - 132iz^{15}$$

$$
\begin{aligned}
&- (1586 - 940i)z^{14} + (5026 - 5684i)z^{13} - (13337 - 15404i)z^{12} \\
&+ (31425 - 24928i)z^{11} - (44644 - 48680i)z^{10} + (24414 - 53936i)z^{9} \\
&+ (10964 - 145744i)z^{8} - (360780 - 313536i)z^{7} + (793808 - 714400i)z^{6} \\
&- (1103368 - 1607552i)z^{5} + (2420320 - 841472i)z^{4} - (1944640 - 1643520i)z^{3} \\
&+ (3782400 - 3571200i)z^{2} - (4464000 - 6912000i)z + 8640000 \\
= &\, (z-4)(z-i)(z+5i)(z+3)(z^2+9)(z^2+2z+5)(z^2+4)(z^2-4) \\
&\times (z^2+2z+2)(z^2-2z+2)(z^2-4z+5)(z^2-2z+10).
\end{aligned}
$$

All tested methods started with the same initial approximations.

As a measure of accuracy of the produced approximations, we have calculated Euclid's norm,

$$
e^{(m)} := \left\| z^{(m)} - \zeta \right\|_2 = \left(\sum_{i=1}^{n} \left| z_i^{(m)} - \zeta_i \right|^2 \right)^{1/2}.
$$

The entries $e^{(m)}$ $(m = 1, 2, 3)$ are given in Table 2 where $A(-q)$ means $A \times 10^{-q}$. In the presented example for the initial approximations, we have $e^{(0)} = 0.80$.

From Table 2 and a number of other tested polynomials we conclude that the results obtained by the proposed methods coincide with the theoretical results given in Theorems 3.1 and 3.2. Also, note that two iterative steps are usually sufficient for solving most of the practical problems when initial approximations are reasonably good and polynomials are well-conditioned. The third iteration is given to confirm fast convergence of our new family of root-finding methods and good matching the convergence orders with those given in theoretical analysis.

The studied methods from the families (TS(k)) and (SS(k)), as the majority of iterative methods with similar structure, work very well when the sought zeros are simple and their measure of separation (given by $\min_{i \neq j} |\zeta_i - \zeta_j|$ or $\min_{i \neq j} |z_i - z_j|$) is not very small entry. A precise quantitative measure of separation was given in [26]. The case of multiple zeros was considered in [17]. As well known, a direct application of numerical iterative methods, in general, is not efficient for the calculation of clusters of zeros without the help of some other procedures. As discussed in [17,27–30], the problem of finding clusters of zeros requires a multistage composite

Table 2. Euclid's norm of errors. The three first iterations.

Methods		TS(1)	TS(2)	TS(3)	SS(1)	SS(2)	SS(3)
$\alpha = 0$	$e^{(1)}$	8.11 (−3)	3.02 (−3)	1.19 (−3)	7.55 (−3)	2.06 (−3)	8.84 (−4)
	$e^{(2)}$	1.36 (−9)	1.75 (−14)	3.54 (−19)	1.02 (−9)	5.45 (−15)	4.37 (−20)
	$e^{(3)}$	3.29 (−38)	8.34 (−70)	5.12 (−113)	3.58 (−39)	2.30 (−74)	3.05 (−124)
$\alpha = \frac{1}{n-1}$	$e^{(1)}$	8.40 (−3)	3.13 (−3)	1.20 (−3)	7.68 (−3)	2.13 (−3)	8.59 (−4)
	$e^{(2)}$	1.36 (−9)	1.91 (−14)	3.67 (−19)	9.99 (−10)	6.98 (−15)	5.75 (−20)
	$e^{(3)}$	1.78 (−37)	1.30 (−69)	4.79 (−113)	4.43 (−39)	9.01 (−74)	2.18 (−123)
$\alpha = 1$	$e^{(1)}$	1.78 (−2)	5.85 (−3)	1.94 (−3)	1.22 (−2)	4.47 (−3)	1.44 (−3)
	$e^{(2)}$	3.30 (−7)	1.74 (−11)	2.23 (−16)	7.09 (−8)	5.40 (−12)	8.93 (−18)
	$e^{(3)}$	1.42 (−27)	1.22 (−54)	3.01 (−95)	2.58 (−30)	8.32 (−59)	6.07 (−104)
$\alpha = -1$	$e^{(1)}$	1.78 (−2)	5.46 (−3)	2.04 (−3)	1.25 (−2)	4.62 (−3)	1.55 (−3)
	$e^{(2)}$	2.29 (−8)	4.68 (−13)	2.68 (−18)	4.50 (−9)	1.14 (−13)	6.85 (−19)
	$e^{(3)}$	1.20 (−31)	8.74 (−63)	1.57 (−106)	1.81 (−34)	3.44 (−66)	7.40 (−110)

algorithm that includes some auxiliary methods such as a detection, localization, enclosure, splitting, and, in final step, a refinement. These additional procedures were omitted in this paper and we did not consider the problem of clusters. We hope to address the problem concerning clusters in future works.

In the next example, we present a globally convergent properties of the considered methods in a restrictive sense. For comparison purpose, we have also tested two another methods.

Weierstrass' (or Durand-Kerner) quadratically convergent method, one of the most efficient methods for the simultaneous determination of polynomial zeros, that possesses a global convergence for almost all arbitrary distinct initial approximations (the conjecture that has not been proved yet),

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod\limits_{j\neq i}(z_i - z_j)} \qquad (i \in I_n). \tag{W}$$

Ehrlich-Aberth's method with Newton's corrections (EAN) of the fourth order that has very high computational efficiency,

$$\hat{z}_i = z_i - \frac{N_i}{1 - N(z_i)\sum\limits_{j\neq i}(1/(z_i - z_j + N_j))} \qquad (i \in I_n). \tag{EAN}$$

EXAMPLE 2. We applied the proposed methods (TS(k)), (EAN), and (W) for the simultaneous approximation to the zeros of the polynomial,

$$P(z) = z^{50} + z^{49} + 1,$$

often used in the literature as a test example. We find that all zeros of the above polynomial lie in the annulus

$$\{z \in \mathbb{C} \,:\, r = 0.5 < |z| < 2 = R\},$$

where r and R are calculated by

$$r = \frac{1}{2} \min_{1\le k\le n} \left|\frac{a_n}{a_{n-k}}\right|^{1/k},$$

$$R = 2 \max_{1\le k\le n} \left|\frac{a_k}{a_0}\right|^{1/k},$$

(see [25, Theorem 6.4b, Corollary 6.4k]. All tested methods started with Aberth's initial approximations [31],

$$z_\nu^{(0)} = -\frac{a_1}{n} + r_0 \exp(i\theta_\nu), \qquad i = \sqrt{-1}, \quad \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right) \quad (\nu = 1, \dots, n),$$

equidistantly distributed along the circle $|z + a_1/n| = r_0$. In this concrete case, we have $n = 50$, $a_1 = 1$, and choose $r_0 = 2$. The stopping criterion was given by

$$E^{(m)} = \max_{1\le i\le 50} |f(z_i^{(m)})| < \tau = 10^{-12}. \tag{4.1}$$

Table 3 gives the number of iterative steps for the considered iterative procedures.

Table 3. The number of iterations for Aberth's initial approximations and $\tau = 10^{-12}$.

Methods	$\alpha = 0$	$\alpha = \frac{1}{(n-1)}$	$\alpha = -1$	$\alpha = 1$	EAN	W
TS(1)	13	13	17	> 50	17	40
TS(2)	12	12	15	> 50	-	-
TS(3)	11	11	13	> 50	-	-

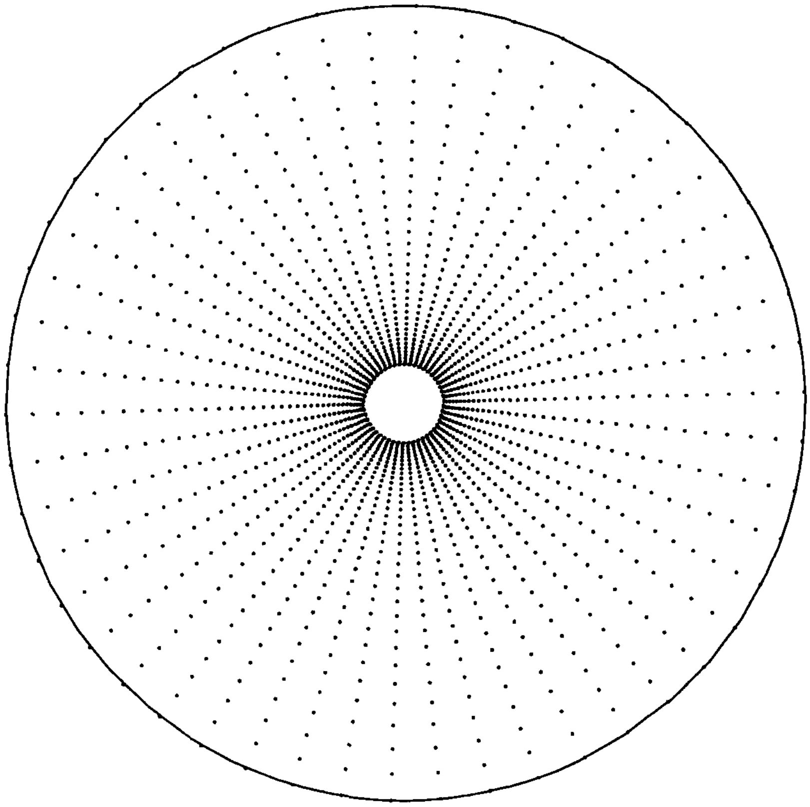

Figure 1. The approach of approximations to the zeros.

Aberth's initial approximations should be regarded as very crude ones, especially when r_0 is considerably large (see Figure 1). As a consequence, first iterations give very rough approximations until they reach the zone of local convergence. Then, the applied iterative methods start to converge very fast and only 3 to 4 iterations are needed to fulfill the required stopping criterion.

In other words, the great deal of procedure happens in the range of slow convergence. In this concrete example, the best results were obtained for small $|\alpha|$, while Euler-like method ($\alpha = 1$) showed the worst convergence. It is evident that the methods with corrections cannot be accelerated during the slow-convergent iterating since Newton's and Halley's corrections are calculated at the points which are very crude approximations to the zeros. For these reasons, the number of iterations needed to satisfy the stopping criterion is slightly less for the methods with corrections compared to the basic method (TS(1)). The initial approximations are closer to the zeros (as in Example 1), the total efficiency of the methods with corrections is greater. Let us note that the implementation of single step methods is pointless in the case of Aberth's approximations; indeed, the use of the already calculated approximations is aimless since they are very crude and cannot cause any improvement.

To display visually the process of convergence and clarify the above discussion, we have tested the considered methods to find the zeros of the polynomial $P(z) = z^{50} + z^{49} + 1$, but with very crude initial approximations taking $r_0 = 10$. For illustration, we have chosen Laguerre-like method ($\alpha = 1/(n-1) = 1/49$). The stopping criterion (4.1) was satisfied after 36 iterations. The approaching course of current approximations is shown in Figure 1. One can observe that the approximations converge radially straightforward to the exact zeros, distributed along the unit circle $|z| = 1$. Indeed, from the list of exact zeros, we found that all zeros lie in the annulus $\{z \in \mathbb{C} : 0.986 < |z| < 1.055\}$, which means that the distance between neighboring zeros is very small - the average distance is about 5×10^{-2}. In a certain sense, we could regard that very close zeros make groups of clusters. We have also observed that $\log E^{(m)}$ is positive monotonically decreasing arithmetical sequence until the approximations are far from the zone of local convergence. In the domain of convergence, $\log E^{(m)}$ behaves as negative geometrical sequence with the quotient equal to the order of convergence of the applied method.

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