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A NEW PICK-TYPE THEOREM ON THE HEXAGONAL LATTICE

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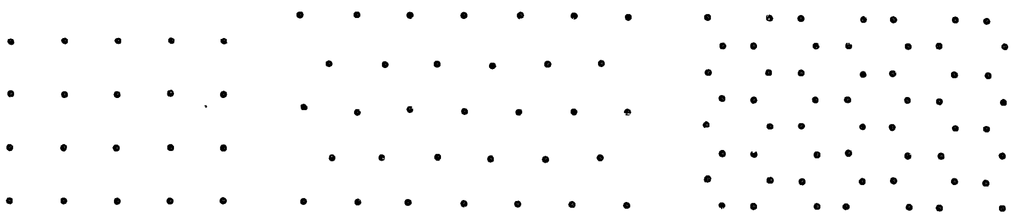
Let L be the lattice of all vertex points in a tiling of the plane by regular hexagons of unit area. Suppose the vertices of a planar polygon P are points of the lattice L, and points of L occur frequently along the edges of P. Then the area of P is $A(P)=\frac{1}{4}b+\frac{1}{2}i+\frac{1}{12}c-1$, where b is the number of lattice points on the boundary of P, i is the number of lattice points in the interior of P, and c is the boundary characteristic of P.

1. Introduction and notation

By a "Pick-type theorem" we mean a result which gives the exact area of any planar polygon P when
 (1) the vertices of P lie in a given lattice, and
 (2) information about P is known only at the lattice points.
Figure 1 shows the patterns of the lattice points generated by the face-to-face tilings of the plane using, respectively, regular squares, triangles, and hexagons of unit area. Given a *lattice polygon* P (i.e., a planar polygon whose vertices are lattice points), let b denote the number of lattice points on the boundary of P, let i denote the number of lattice points in the interior of P, and let $A(P)$ be the area of P.

The classical theorem of Pick [2] asserts that if P is a lattice polygon using the square lattice, then $A(P)=\frac{1}{2}b+i-1$. It is easy to show (see Ding and Reay [1]) that if P uses the triangular lattice, then $A(P)=b+2i-2$. For the rest of this paper we assume P uses the hexagonal lattice L. Since the area of P is not uniquely determined by the parameters b and i, an additional parameter c, the boundary characteristic, was introduced in [1]. Each lattice point x on the boundary ∂P of a hexagonal lattice polygon P is incident to exactly 3 edges of the hexagonal tiling. Each of these edges either
 (1) lies in the boundary of P, or
 (2) extends locally into the exterior of P near x, or
 (3) extends locally into the interior of P near x.



The square lattice — The triangular lattice — The hexagonal lattice L

Fig. 1. Lattices from planar tilings by regular polygons of unit area.

We will denote these three disjoint sets of edges by $B(x)$, $F(x)$, and $G(x)$, respectively. We will define the *boundary characteristic* $c(x, P)$ *at the boundary point* x as $c(x, P) = |F(x)| - |G(x)|$, and the *boundary characteristic* $c(P)$ of P is defined by

$$c(P) = \sum_{x \in \partial P} c(x, P).$$

(Throughout the paper, $|S|$ denotes the cardinality of the set S.) Intuitively, to compute the boundary characteristic $c = c(P)$ of the lattice polygon P, travel once around the boundary of P, and add 1 each time we find a lattice edge that starts on the boundary and sticks out locally into the exterior of P, and add -1 each time we find such an edge that locally pokes into the interior of P.

The following results ([1, Theorem 5]) will be used and extended in the next section.

Theorem A. *If the vertices of the planar polygon P lie in the hexagonal lattice L, and if the boundary of P is the union of edges and/or minor diagonals and/or major diagonals of the hexagonal tiling, then*

$$A(P) = \tfrac{1}{4}b + \tfrac{1}{2}i + \tfrac{1}{12}c - 1. \tag{1}$$

Figure 2 shows examples of the boundary characteristic for three simple lattice polygons of area $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}$ and also shows that the parameter c may vary even though the parameters $b = 3$ and $i = 0$ are the same for each of these examples. Theorem A applies in each case.

Theorem A also applies to the parallelogram $XYZW$ in Fig. 3, where $b = 8$,

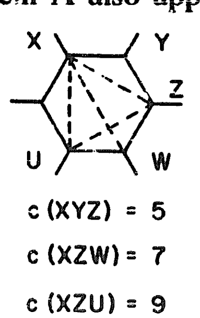


Fig. 2.

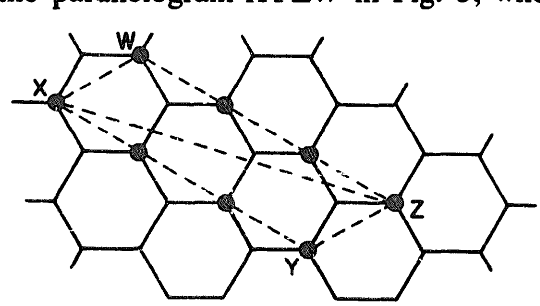


Fig. 3.

$i = 3$, $c = 6$, and thus the area is 3. However Theorem A does not apply to triangle XYZ of Fig. 3. Its area is clearly 1.5 (half the area of the parallelogram), and $b = 5$, $i = 1$, $c = 7$. Thus for triangle XYZ, $(\frac{1}{4}b + \frac{1}{2}i + \frac{1}{12}c - 1) = \frac{4}{3}$ and formula (1) is not valid. Hence formula (1) is not valid for all lattice polygons using L.

2. Main result

In this section we extend Theorem A to get a much broader class of polygons satisfying (1). We will define a *lattice segment* to be a line segment $S = [a, b]$ in a tiling of the plane by regular hexagons of unit area, with $[a, b] \cap L = \{a, b\}$. There is a discrete increasing sequence $\{a_1, a_2, \ldots\}$ of all possible real values that may be realized as the length of a lattice segment. The numbers $a_1 = 0.620\ldots$, $a_2 = 1.074\ldots$, and $a_3 = 1.240\ldots$ are respectively the lengths of an edge, a minor diagonal, and a major diagonal of a hexagon in the tiling. Lattice segments of lengths a_4 to a_{11} are shown with solid lines in Fig. 4. The diagonal XZ in Fig. 3 also has length a_{11}.

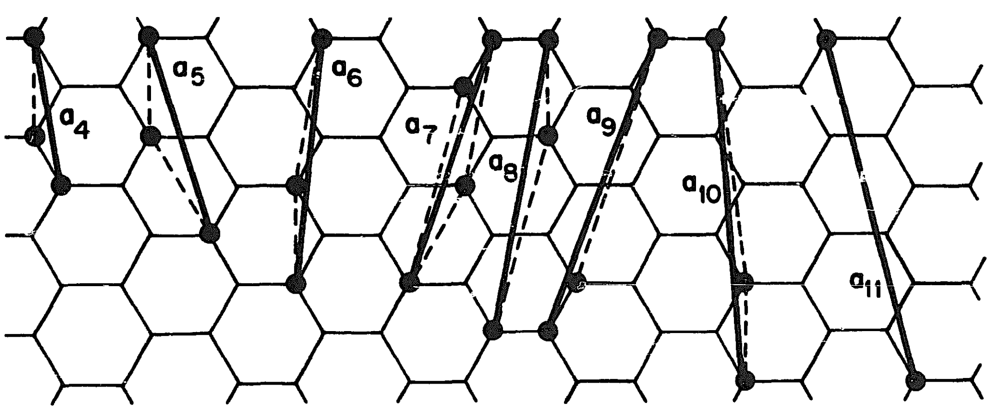


Fig. 4. The first few lattice segments in a hexagonal tiling.

Theorem. *If the lattice polygon P has vertices in the hexagonal lattice L, and if each lattice segment in the boundary of P has length at most $a_{10} = 3.77376\ldots$, then $A(P) = \frac{1}{4}b + \frac{1}{2}i + \frac{1}{12}c - 1$.*

Overview of the Proof. Let d be the length of the longest lattice segment in the boundary of P. If $d \leq a_3$, then Theorem A applies and the proof is completed. The proof now proceeds by induction on the lengths $d = a_4$ through length $d = a_{10}$. Note that the counterexample of Fig. 3 shows that the theorem fails, in general if $d \geq a_{11}$.

If $d = a_i$ is the maximal length of a lattice segment in ∂P, let $m = m(P)$ stand for the number of lattice segments in ∂P with length a_i, and proceed with an induction on m. Choose a lattice segment $[v_1 v_2]$ of length a_i in ∂P. We will add a lattice-triangle $T = \operatorname{conv}\{v_1 v_2 v_3\}$, as shown in Fig. 4, to obtain a larger (possibly non-simple) polygon $P' = P \cup T$ whose area we can determine by the induction hypotheses. Vertex v_3 is always chosen to be the lattice point on the external side of the lattice segment $[v_1 v_2]$ which is closest to the segment. From the area of P' we may determine the area of P. □

Lemma 1. *Vertex v_3 is not in the interior of P and hence $A(P') = A(P) + A(T)$.*

Proof. (Note that Lemma 1 is true in general only if $d \leq a_{10}$.) If v_3 is in the interior of P, then there must exist a lattice segment in ∂P which is longer than $d = a_i$. □

The following results are clear using Fig. 4.

Lemma 2. *The sides $[v_1 v_3]$ and $[v_2 v_3]$ of triangle T have length less than $d = a_i$. Hence $m(P') = m(P) - 1$.*

Lemma 3. *The area of triangle T is $\frac{1}{6}$ for a_4, a_6, a_9, and a_{10}. The area of triangle T is $\frac{1}{3}$ for a_5 and a_8. For a_7 the area of T is either $\frac{1}{3}$ or $\frac{1}{6}$, depending on which side of $[v_1 v_2]$ is the external side.*

Lemma 4. *Suppose formula (1) is true for P', and $\partial P'$ is a simple closed curve. Then formula (1) is true for P if and only if*

$$c(P') = c(P) + 12A(T) - 3. \tag{2}$$

Proof. We must consider two cases. First, if $v_3 \notin P$, then it is obvious that $b(P') = b(P) + 1$ and $i(P') = i(P)$. (See Fig. 5.) Secondly, if $v_3 \in P$, then, by Lemma 1, $v_3 \in \partial P$. In this case it is clear that $b(P') = b(P) - 1$ and $i(P') = i(P) + 1$. (See Fig. 6.) Also $A(P) = A(P') - A(T)$ by Lemma 1. In either case, formula (1) is true for P iff

$$\tfrac{1}{4}b(P) + \tfrac{1}{2}i(P) + \tfrac{1}{12}c(P) - 1 = \tfrac{1}{4}b(P') + \tfrac{1}{2}i(P') + \tfrac{1}{12}c(P') - 1 - A(T)$$

iff $\frac{1}{12}c(P) = \frac{1}{4} + \frac{1}{12}c(P') - A(T)$ iff formula (2) is valid. □

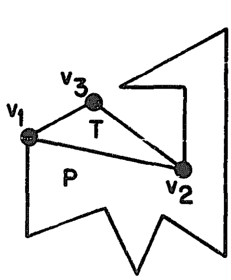


Fig. 5.

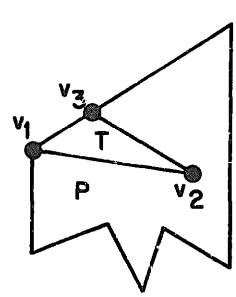


Fig. 6.

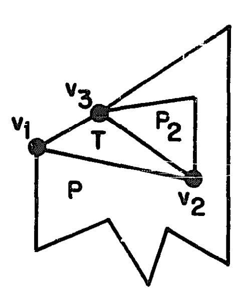


Fig. 7.

When $v_3 \in \partial P$ and $\partial P'$ is not a simple curve then the complement of P' consists of a lattice polygon P_2, and an unbounded region. (See Fig. 7.) Define $P_1 = P \cup T \cup P_2 = P' \cup P_2$.

Lemma 5. *Suppose $\partial P'$ is a non-simple curve and formula* (1) *is valid for P_1 and P_2. Then formula* (1) *is true for P if and only if*

$$c(P_1) - c(P_2) = c(P) + 12A(T) - 9. \tag{3}$$

Proof. It is obvious that $b(P) = b(P_1) + b(P_2) - 1$ and $i(P_1) = i(P) + i(P_2) + b(P_2) - 1$. (See Fig. 7.) Substituting these results into formula (1) for P_1 and P_2, and using the fact that $A(P) = A(P_1) - A(P_2) - A(T)$, we obtain the desired result. □

For convenient notation in the following proof, we always assume that the lattice segment $[v_2v_3]$ is longer than the lattice segment $[v_1v_3]$, and we define $c(x, P)$ to be zero whenever $x \notin \partial P$. With the above lemmas, the proof of the theorem is now reduced to verifying Eq. (2) or (3) for each case.

Lemma 6. *Suppose $d = a_i \leq a_{10}$, and $\partial P'$, is a simple closed curve. If $A(T) = \frac{1}{6}$ (so that $i \in \{4, 6, 7, 9, 10\}$), then $c(P') - c(P) = -1$. If $A(T) = \frac{1}{3}$ (so that $i \in \{5, 7, 8\}$), then $c(P') - c(P) = +1$. In each case formula* (2) *holds.*

Proof. We consider only the case when $A(T) = \frac{1}{6}$. The proof when $A(T) = \frac{1}{3}$ is similar.

Case 1. Suppose $v_3 \notin P$. Then

$$\begin{aligned} c(P') - c(P) &= c(v_1, P') - c(v_1, P) \\ &\quad + c(v_2, P') - c(v_2, P) + c(v_3, P') - c(v_3, P). \end{aligned}$$

In each case (see Fig. 4), $c(v_3, P) = 0$ and $c(v_2, P') - c(v_2, P) = 0$. Of the three edges of the hexagonal tiling which meet at v_3, exactly one extends into the exterior of P' so $c(v_3, P')$ is either 0 or -1. When $c(v_3, P') = 0$, then $c(v_1, P') = c(v_1, P) - 1$. When $c(v_3, P') = -1$, then $c(v_1, P') = c(v_1, P)$. In either case it is easy to show that $c(v_1, P') - c(v_1, P) + c(v_3, P') = -1$ and formula (2) holds.

Case 2. Suppose $v_3 \in P$, so either v_1 or v_2 is in the interior of P'. In either case the result is easily checked, as in Case 1. □

Lemma 7. *Suppose $d = a_i \leq a_{10}$, and $\partial P'$ is not a simple closed curve. If $A(T) = \frac{1}{6}$, then $c(P_1) - c(P_2) - c(P) = -7$. If $A(T) = \frac{1}{3}$, then $c(P_1) - c(P_2) - c(P) = -5$. In each case formula* (3) *holds.*

Proof. We will prove the lemma only for the case when $A(T)=\frac{1}{6}$ and thus $i\in\{4,6,7,9,10\}$. The reasoning for the case when $A(T)=\frac{1}{3}$, so $i\in\{5,7,8\}$, is similar and is left to the reader.

Case 1. Suppose $i\in\{4,7,9\}$. In these cases $[v_1v_3]$ is an edge of a hexagon of the tiling and we denote this edge as e_1. Also, exactly one hexagonal edge at v_3, denoted as e_2, enters the interior of T from v_3. The third hexagonal edge at v_3, denoted by e_3, meets T only at v_3. (See Fig. 8.) We must show that $c(P_1)-c(P_2)-c(P)=-7$. Now $c(x,P_1)=c(x,P_2)+c(x,P)$ at all lattice points except the vertices of T, and it is easy to show that $c(v_1,P_1)-c(v_1,P_2)-c(v_1,P)=-1$ and $c(v_2,P_1)-c(v_2,P_2)-c(v_2,P)=0$. Thus it is sufficient to prove that

$$c(v_3,P_1)-c(v_3,P_2)-c(v_3,P)=-6 \tag{4}$$

Let us define (e_i,P_j) to be the contribution of the edge e_i to $c(v_3,P_j)$ as an aid in the computations. Thus to show (4) it is sufficient to establish and sum the following:

$$(e_1,P_1)-(e_1,P_2)-(e_1,P)=-2, \tag{4.1}$$

$$(e_2,P_1)-(e_2,P_2)-(e_2,P)=-3, \tag{4.2}$$

$$(e_3,P_1)-(e_3,P_2)-(e_3,P)=-1. \tag{4.3}$$

In all cases $(e_1,P)=1$, and $(e_1,P_1)=0$ iff $(e_1,P_2)=1$, and $(e_1,P_2)=0$ iff $(e_1,P_1)=-1$, so (4.1) is established. In all cases $(e_2,P_2)=(e_2,P)=1$ and $(e_2,P_1)=-1$, so (4.2) is established.

In all cases, if $(e_3,P_1)=1$, then $(e_3,P)=(e_3,P_2)=1$.

If $(e_3,P_1)=-1$, then either $e_3\subset(\partial P_2\cap\partial P)$ so $(e_3,P_2)=(e_3,P)=0$, or else $e_3\not\subset(\partial P_2\cap\partial P)$ so $(e_3,P_2)+(e_3,P)=0$.

If $(e_3,P_1)=0$, then e_3 is in the exterior of P_2 and in ∂P, so $(e_3,P)=0$ and $(e_3,P_2)=1$. Thus (4.3) is established.

Case 2. Suppose $i\in\{6,10\}$. In these cases two hexagonal edges at v_3, denoted by e_1 and e_2, extend into the interior of T. The third edge e_3 meets T only at vertex v_3. (See Fig. 9.)

It is easy to show that

$$c(v_1,P_1)-c(v_1,P_2)-c(v_1,P)=0$$

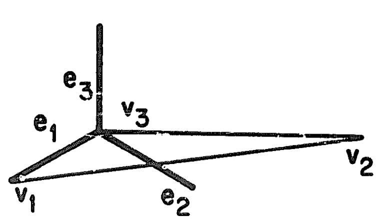


Fig. 8. Case 1.

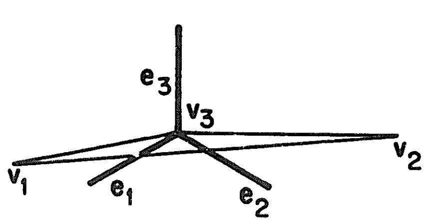


Fig. 9. Case 2.

and

$$c(v_2, P_1) - c(v_2, P_2) - c(v_2, P) = 0$$

so it is sufficient to prove

$$c(v_3, P_1) - c(v_3, P_2) - c(v_3, P) = -7. \tag{5}$$

To show (5) it is sufficient to establish and sum the following.

$$(e_1, P_1) - (e_1, P_2) - (e_1, P) = -3, \tag{5.1}$$

$$(e_2, P_1) - (e_2, P_2) - (e_2, P) = -3, \tag{5.2}$$

$$(e_3, P_1) - (e_3, P_2) - (e_3, P) = -1. \tag{5.3}$$

These are established in the same way, but with less work, as in Case 1 above. This completes the proof of Lemma 7 when $A(T) = \frac{1}{6}$. □

The above lemmas complete the proof of the main theorem. Note that the proof of the lemmas depends on the fact that no lattice segment in ∂P ever enters the interior of the triangle T from the vertex v_1 or v_2. This result is no longer valid in general for a lattice segment $[v_1 v_2]$ of length a_{11} no matter how the vertex v_3 is chosen, and shows why the proof of the theorem fails in the cases where $d \geqslant a_{11}$. It would be interesting to give a complete characterization of the planar lattice polygons which use the hexagonal tiling of the plane for which formula (1) is valid.

See Varberg [3] or Ding and Reay [1] for references of papers related to Pick's theorem. Also see [1] for Pick type theorems using lattices that come from any of the 11 Archimedean tilings of the plane.

References

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