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Inclusion properties for certain classes of meromorphic functions associated with the Choi–Saigo–Srivastava operator

Nak Eun Cho [a,*], K. Inayat Noor [b]

[a] *Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Puasn 608-737, South Korea*
[b] *Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan*

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Abstract

The purpose of the present paper is to introduce several new classes of meromorphic functions defined by using a meromorphic analogue of the Choi–Saigo–Srivastava operator for analytic functions and investigate various inclusion properties of these classes. Some interesting applications involving these and other classes of integral operators are also considered.

Keywords: Meromorphic functions; Differential subordination; Starlike functions; Convex functions; Close-to-convex functions; Choi–Saigo–Srivastava operator

1. Introduction

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

* Corresponding author.
E-mail addresses: necho@pknu.ac.kr (N.E. Cho), khalidanoor@hotmail.com (K.I. Noor).

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which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon\ 0 < |z| < 1\}$. If f and g are analytic in $\mathbb{U} = \mathbb{D} \cup \{0\}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$. For $0 \leqslant \eta, \beta < 1$, we denote by $\mathcal{MS}(\eta)$, $\mathcal{MK}(\eta)$ and $\mathcal{MC}(\eta, \beta)$ the subclasses of $\mathcal{M}$ consisting of all meromorphic functions which are, respectively, starlike of order η, convex of order η and close-to-convex of order β and type η in $\mathbb{U}$ [5,9].

Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0$ $(z \in \mathbb{U})$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}(\eta, \phi)$, $\mathcal{MK}(\eta, \phi)$ and $\mathcal{MC}(\eta, \beta; \phi, \psi)$ of the class $\mathcal{M}$ for $0 \leqslant \eta, \beta < 1$ and $\phi, \psi \in \mathcal{N}$, which are defined by

$$\mathcal{MS}(\eta; \phi) := \left\{ f \in \mathcal{M}\colon\ \frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$
$$\mathcal{MK}(\eta; \phi) := \left\{ f \in \mathcal{M}\colon\ \frac{1}{1-\eta}\left(-\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$

and

$$\mathcal{MC}(\eta, \beta; \phi, \psi) := \left\{ f \in \mathcal{M}\colon\ \exists g \in \mathcal{MS}(\eta; \phi) \text{ s.t. } \frac{1}{1-\beta}\left(-\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z) \text{ in } \mathbb{U} \right\}.$$

We note that the classes mentioned above is the familiar classes which have been used widely on the space of analytic and univalent functions in $\mathbb{U}$ [2,7] and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of $\mathcal{M}$. For examples, we have

$$\mathcal{MS}\left(\eta; \frac{1+z}{1-z}\right) = \mathcal{MS}(\eta), \qquad \mathcal{MK}\left(\eta; \frac{1+z}{1-z}\right) = \mathcal{MK}(\eta)$$

and

$$\mathcal{MC}\left(\eta, \beta; \frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = \mathcal{MC}(\eta, \beta).$$

Now we define the function $\lambda(a, b; z)$ by

$$\lambda(a, b; z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(a)_{k+1}}{(b)_{k+1}} z^k \quad (a > 0;\ b \neq 0, -1, -2, \ldots;\ z \in \mathbb{D}), \tag{1.1}$$

where $(x)_k$ is the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k := \begin{cases} 1 & \text{if } k = 0, \\ x(x+1)\ldots(x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

Let $f \in \mathcal{M}$. Denote by $L(a, b)\colon \mathcal{M} \to \mathcal{M}$ the operator defined by

$$L(a, b)f(z) = \lambda(a, b; z) * f(z) \quad (z \in \mathbb{D}),$$

where the symbol $(*)$ stands for the Hadamard product (or convolution). The operator $L(a,b)$ was introduced and studied by Liu and Srivastava [6]. Further, we remark in passing that this operator $L(a,b)$ is closely related to the Carlson–Shaffer operator defined on the space of analytic and univalent functions in $\mathbb{U}$ [11].

Corresponding to the function $\lambda(a,b;z)$, let $\lambda^{\dagger}(a,b;z)$ be defined such that

$$\lambda(a,b;z) * \lambda^{\dagger}(a,b;z) = \frac{1}{z(1-z)^{\mu}} \quad (\mu > 0).$$

Analogous to $L(a,b)$, we now introduce a linear operator $\mathcal{I}_{\mu}(a,b)$ on $\mathcal{M}$ as follows:

$$\mathcal{I}_{\mu}(a,b)f(z) = \lambda^{\dagger}(a,b;z) * f(z) \quad (\mu > 0;\ a > 0;\ b \neq 0,-1,-2,\ldots;\ z \in \mathbb{D}). \tag{1.1}$$

We note that $\mathcal{I}_2(2,1)f(z) = f(z)$ and $\mathcal{I}_2(1,1)f(z) = zf'(z) + 2f(z)$. It is easily verified from the definition of the operator $\mathcal{I}_{\mu}(a,b)$ that

$$z\big(\mathcal{I}_{\mu}(a+1,b)f(z)\big)' = a\mathcal{I}_{\mu}(a,b)f(z) - (a+1)\mathcal{I}_{\mu}(a+1,b)f(z) \tag{1.2}$$

and

$$z\big(\mathcal{I}_{\mu}(a,b)f(z)\big)' = \mu\mathcal{I}_{\mu+1}(a,b)f(z) - (\mu+1)\mathcal{I}_{\mu}(a,b)f(z). \tag{1.3}$$

We note that the operator $\mathcal{I}_{\mu}(a,b)$ is motivated essentially by the Choi–Saigo–Srivastava operator [2], which extends the integral operator studied by Noor and Noor [10].

Next, by using the operator $\mathcal{I}_{\mu}(a,b)$, we introduce the following classes of meromorphic functions for $\phi,\psi \in \mathcal{N}$, $\mu > 0$ and $0 \leqslant \eta,\beta < 1$:

$$\mathcal{MS}^{\mu}_{a,b}(\eta;\phi) := \big\{f \in \mathcal{M}\colon \mathcal{I}_{\mu}(a,b)f \in \mathcal{MS}(\eta;\phi)\big\},$$
$$\mathcal{MK}^{\mu}_{a,b}(\eta;\phi) := \big\{f \in \mathcal{M}\colon \mathcal{I}_{\mu}(a,b)f \in \mathcal{MK}(\eta;\phi)\big\},$$

and

$$\mathcal{MC}^{\mu}_{a,b}(\eta,\beta;\phi,\psi) := \big\{f \in \mathcal{M}\colon \mathcal{I}_{\mu}(a,b)f \in \mathcal{MC}(\eta,\beta;\phi,\psi)\big\}.$$

We also note that

$$f(z) \in \mathcal{MK}^{\mu}_{a,b}(\eta;\phi) \quad \Leftrightarrow \quad -zf'(z) \in \mathcal{MS}^{\mu}_{a,b}(\eta;\phi). \tag{1.4}$$

In particular, we set

$$\mathcal{MS}^{\mu}_{a,b}\left(\eta;\frac{1+Az}{1+Bz}\right) = \mathcal{MS}^{\mu}_{a,b}(\eta;A,B) \quad (-1 < B < A \leqslant 1)$$

and

$$\mathcal{MK}^{\mu}_{a,b}\left(\eta;\frac{1+Az}{1+Bz}\right) = \mathcal{MK}^{\mu}_{a,b}(\eta;A,B) \quad (-1 < B < A \leqslant 1).$$

In this paper, we investigate several inclusion properties of the classes $\mathcal{MS}^{\mu}_{a,b}(\eta;\phi)$, $\mathcal{MK}^{\mu}_{a,b}(\eta;\phi)$ and $\mathcal{MC}^{\mu}_{a,b}(\eta;\phi)$ associated with the operator $\mathcal{I}_{\mu}(a,b)$. Some applications involving integral operators are also considered.

2. Inclusion properties involving the operator $\mathcal{I}_\mu(a, b)$

The following results will be required in our investigation.

Lemma 2.1. [3] *Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and* $\operatorname{Re}\{\kappa\phi(z) + \nu\} > 0$ $(\kappa, \nu \in \mathbb{C})$. *If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \nu} \prec \phi(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

Lemma 2.2. [8] *Let ϕ be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* $\operatorname{Re}\{\omega(z)\} \geqslant 0$. *If p is analytic in $\mathbb{U}$ and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

At first, with the help of Lemma 2.1, we obtain the following theorem.

Theorem 2.1. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \operatorname{Re}\bigl\{\phi(z)\bigr\} < \min\bigl\{(\mu + 1 - \eta)/(1 - \eta), (a + 1 - \eta)/(1 - \eta)\bigr\}$$
$$(\mu, a > 0;\ 0 \leqslant \eta < 1).$$

Then

$$\mathcal{MS}_{a,b}^{\mu+1}(\eta; \phi) \subset \mathcal{MS}_{a,b}^{\mu}(\eta; \phi) \subset \mathcal{MS}_{a+1,b}^{\mu}(\eta; \phi).$$

Proof. To prove the first part of Theorem 2.1, let $f \in \mathcal{MS}_{a,b}^{\mu+1}(\eta; \phi)$ and set

$$p(z) = \frac{1}{1 - \eta}\left(-\frac{z(\mathcal{I}_\mu(a, b)f(z))'}{\mathcal{I}_\mu(a, b)f(z)} - \eta\right), \tag{2.1}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. By a simple calculation with (1.3) and (2.1), we obtain

$$\frac{1}{1 - \eta}\left(-\frac{z(\mathcal{I}_{\mu+1}(a, b)f(z))'}{\mathcal{I}_{\mu+1}(a, b)f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{-(1 - \eta)p(z) + \mu + 1 - \eta}$$
$$(z \in \mathbb{U}). \tag{2.2}$$

Applying Lemma 2.1 to (2.2), it follows that $f \in \mathcal{MS}_{a,b}^{\mu}(\eta; \phi)$. Moreover, by using the arguments similar to those detailed above with (1.2), we can prove the second part of Theorem 2.1. Therefore we complete the proof of Theorem 2.1. □

Theorem 2.2. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \mathrm{Re}\{\phi(z)\} < \min\{(\mu+1-\eta)/(1-\eta), (a+1-\eta)/(1-\eta)\}$$
$$(\mu, a > 0;\ 0 \leqslant \eta < 1).$$

Then

$$\mathcal{MK}_{a,b}^{\mu+1}(\eta;\phi) \subset \mathcal{MK}_{a,b}^{\mu}(\eta;\phi) \subset \mathcal{MK}_{a+1,b}^{\mu}(\eta;\phi).$$

Proof. Applying (1.4) and Theorem 2.1, we observe that

$$\begin{aligned} f(z) \in \mathcal{MK}_{a,b}^{\mu+1}(\eta;\phi) \quad &\Leftrightarrow \quad -zf'(z) \in \mathcal{MS}_{a,b}^{\mu+1}(\eta;\phi) \\ &\Rightarrow \quad -zf'(z) \in \mathcal{MS}_{a,b}^{\mu}(\eta;\phi) \\ &\Leftrightarrow \quad f(z) \in \mathcal{MK}_{a,b}^{\mu}(\eta;\phi), \end{aligned}$$

and

$$\begin{aligned} f(z) \in \mathcal{MK}_{a,b}^{\mu}(\eta;\phi) \quad &\Leftrightarrow \quad -zf'(z) \in \mathcal{MS}_{a,b}^{\mu}(\eta;\phi) \\ &\Rightarrow \quad -zf'(z) \in \mathcal{MS}_{a+1,b}^{\mu}(\eta;\phi) \\ &\Leftrightarrow \quad f(z) \in \mathcal{MK}_{a+1,b}^{\mu}(\eta;\phi), \end{aligned}$$

which evidently proves Theorem 2.2. □

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leqslant 1;\ z \in \mathbb{U})$$

in Theorems 2.1 and 2.2, we have

Corollary 2.1. *Let*

$$(1+A)/(1+B) < \min\{(\mu+1-\eta)/(1-\eta), (a+1-\eta)/(1-\eta)\}$$
$$(\mu, a > 0;\ 0 \leqslant \eta < 1;\ -1 < B < A \leqslant 1).$$

Then

$$\mathcal{MS}_{a,b}^{\mu+1}(\eta;A,B) \subset \mathcal{MS}_{a,b}^{\mu}(\eta;A,B) \subset \mathcal{MS}_{a+1,b}^{\mu}(\eta;A,B)$$

and

$$\mathcal{MK}_{a,b}^{\mu+1}(\eta;A,B) \subset \mathcal{MK}_{a,b}^{\mu}(\eta;A,B) \subset \mathcal{MK}_{a+1,b}^{\mu}(\eta;A,B).$$

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class $\mathcal{MC}_{a,b}^{\mu}(\eta,\beta;\phi,\psi)$.

Theorem 2.3. *Let $\phi, \psi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \mathrm{Re}\{\phi(z)\} < \min\{(\mu+1-\eta)/(1-\eta), (a+1-\eta)/(1-\eta)\}$$
$$(\mu, a > 0;\ 0 \leqslant \eta < 1).$$

Then

$$\mathcal{MC}^{\mu+1}_{a,b}(\eta,\beta;\phi,\psi)\subset\mathcal{MC}^{\mu}_{a,b}(\eta,\beta;\phi,\psi)\subset\mathcal{MC}^{\mu}_{a+1,b}(\eta,\beta;\phi,\psi).$$

Proof. To prove the first inclusion of Theorem 2.3, let $f\in\mathcal{MC}^{\mu+1}_{a,b}(\eta,\beta;\phi,\psi)$. Then, in view of the definition $\mathcal{MC}^{\mu+1}_{a,b}(\eta,\beta;\phi,\psi)$, there exists a function $g\in\mathcal{MS}^{\mu+1}_{a,b}(\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(\mathcal{I}_{\mu+1}(a,b)f(z))'}{\mathcal{I}_{\mu+1}(a,b)g(z)}-\beta\right)\prec\psi(z)\quad(z\in\mathbb{U}).\tag{2.3}$$

Now let

$$p(z)=\frac{1}{1-\beta}\left(-\frac{z(\mathcal{I}_{\mu}(a,b)f(z))'}{\mathcal{I}_{\mu}(a,b)g(z)}-\beta\right),\tag{2.4}$$

where p is analytic in $\mathbb{U}$ with $p(0)=1$. Using (1.3), we obtain

$$\begin{aligned}&\frac{1}{1-\beta}\left(-\frac{z(\mathcal{I}_{\mu+1}(a,b)f(z))'}{\mathcal{I}_{\mu+1}(a,b)g(z)}-\beta\right)\\&\quad=\frac{1}{1-\beta}\left(\frac{\frac{z(\mathcal{I}_{\mu}(a,b)(-zf'(z)))'}{\mathcal{I}_{\mu}(a,b)g(z)}+(\mu+1)\frac{\mathcal{I}_{\mu}(a,b)(-zf'(z))}{\mathcal{I}_{\mu}(a,b)g(z)}}{\frac{z(\mathcal{I}_{\mu}(a,b)g(z))'}{\mathcal{I}_{\mu}(a,b)g(z)}+\mu+1}-\beta\right).\end{aligned}\tag{2.5}$$

Since $g\in\mathcal{MS}^{\mu+1}_{a,b}(\eta;\phi)\subset\mathcal{MS}^{\mu}_{a,b}(\eta;\ \phi)$, by Theorem 2.1, we set

$$q(z)=\frac{1}{1-\eta}\left(-\frac{z(\mathcal{I}_{\mu}(a,b)g(z))'}{\mathcal{I}_{\mu}(a,b)g(z)}-\eta\right),\tag{2.6}$$

where $q\prec\phi$ in $\mathbb{U}$ with the assumption for $\phi\in\mathcal{N}$. Then, by virtue of (2.4), (2.5) and (2.6), we get

$$\frac{1}{1-\beta}\left(-\frac{z(\mathcal{I}_{\mu+1}(a,b)f(z))'}{\mathcal{I}_{\mu+1}(a,b)g(z)}-\beta\right)=p(z)+\frac{zp'(z)}{-(1-\eta)q(z)+\mu+1-\eta}\prec\psi(z)\quad(z\in\mathbb{U}).\tag{2.7}$$

Hence, by taking

$$\omega(z)=\frac{1}{-(1-\eta)q(z)+\mu+1-\eta},$$

in (2.7), and applying Lemma 2.2, we can show that $p\prec\psi$ in $\mathbb{U}$, so that

$$f\in\mathcal{MC}^{\mu}_{a,b}(\eta,\beta;\phi,\psi).$$

Moreover, we have the second inclusion by using arguments similar to those detailed above with (1.2). Therefore we complete the proof of Theorem 2.3. $\square$

3. Inclusion properties involving the integral operator F_c

In this section, we consider the integral operator F_c (see, e.g., [5]) defined by

$$F_c(f) := F_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\,dt \quad (f \in \mathcal{M};\ c > 0). \tag{3.1}$$

From the definition of F_c defined by (3.1), we observe that

$$z\big(\mathcal{I}_\mu(a,b)F_c(f)(z)\big)' = c\mathcal{I}_\mu(a,b)f(z) - (c+1)\mathcal{I}_\mu(a,b)F_c(f)(z). \tag{3.2}$$

We first state Theorem 3.1 below, the proof of which is much akin to that of Theorem 2.1.

Theorem 3.1. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \operatorname{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta) \quad (c > 0;\ 0 \leqslant \eta < 1).$$

If $f \in \mathcal{MS}^{\mu}_{a,b}(\eta;\phi)$, then $F_c(f) \in \mathcal{MS}^{\mu}_{a,b}(\eta;\phi)$.

Next, we derive an inclusion property involving F_c, which is obtained by applying (1.4) and Theorem 3.1.

Theorem 3.2. *Let $\phi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \operatorname{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta) \quad (c > 0;\ 0 \leqslant \eta < 1).$$

If $f \in \mathcal{MK}^{\mu}_{a,b}(\eta;\phi)$, then $F_c(f) \in \mathcal{MK}^{\mu}_{a,b}(\eta;\phi)$.

From Theorems 3.1 and 3.2, we have

Corollary 3.1. *Let*

$$(1+A)/(1+B) < (c+1-\eta)/(1-\eta) \quad (c > 0;\ -1 < B < A \leqslant 1;\ 0 \leqslant \eta < 1).$$

Then if $f \in \mathcal{MS}^{\mu}_{a,b}(\eta;\ A,B)$ and $\mathcal{MK}^{\mu}_{a,b}(\eta;\ A,B)$, then $F_c(f) \in \mathcal{MS}^{\mu}_{a,b}(\eta;A,B)$ and $\mathcal{MK}^{\mu}_{a,b}(\eta;\phi)$, respectively.

Finally, we obtain Theorem 3.3 below by using (3.2) and the same techniques as in the proof of Theorem 2.3.

Theorem 3.3. *Let $\phi, \psi \in \mathcal{N}$ with*

$$\max_{z\in\mathbb{U}} \operatorname{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta) \quad (c > 0;\ 0 \leqslant \eta < 1).$$

If $f \in \mathcal{MC}^{\mu}_{a,b}(\eta,\beta;\phi,\psi)$, then $F_c(f) \in \mathcal{MC}^{\mu}_{a,b}(\eta,\beta;\phi,\psi)$.

Remark. If we take $\mu = 2$, $a = 2$ and $b = 1$ in all theorems of this section, then we extend the results by Goel and Sohi [4], which reduce the results earlier obtained by Bajpai [1].

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References

[1] S.K. Bajpai, A note on a class of meromorphic univalent functions, Rev. Roumaine Math. Pures Appl. 22 (1977) 295–297.
[2] J.H. Choi, M. Saigo, H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (2002) 432–445.
[3] P. Eenigenberg, S.S. Miller, P.T. Mocanu, M.O. Reade, On a Briot–Bouquet differential subordination, General Inequal. 3 (1983) 339–348.
[4] R.M. Goel, N.S. Sohi, On a class of meromorphic functions, Glas. Mat. 17 (1982) 19–28.
[5] V. Kumar, S.L. Shukla, Certain integrals for classes of p-valent meromorphic functions, Bull. Austral. Math. Soc. 25 (1982) 85–97.
[6] J.-L. Liu, H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, J. Math. Anal. Appl. 259 (2001) 566–581.
[7] W.C. Ma, D. Minda, An internal geometric characterization of strongly starlike functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A 45 (1991) 89–97.
[8] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28 (1981) 157–171.
[9] S.S. Miller, P.T. Mocanu, Differential Subordination, Dekker, New York, 2000.
[10] K.I. Noor, M.A. Noor, On integral operators, J. Math. Anal. Appl. 238 (1999) 341–352.
[11] H.M. Srivastava, S. Owa, Some characterizations and distortions theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, Nagoya Math. J. 106 (1987) 1–28.