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A Method for Stepwise Refinement and Abstraction of Petri Nets*

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This paper is concerned with a method for expanding (or reducing) a Petri net representation to the desired level of detail using step-by-step refinement of transitions and places (or abstraction of subnets to transitions). In particular, we present conditions under which a subnet can be substituted for a single transition while preserving properties such as liveness and boundedness. The present method is general enough to include previously reported methods as special cases. The refinement technique can be used as a top-down approach for synthesizing Petri net models of concurrent systems, while the abstraction technique can be used as a "divide-and-conquer" approach to the analysis of Petri nets.

1. INTRODUCTION

Petri nets and related graph models have been proposed for a wide variety of applications [1, 3, 12]. These models are particularly suitable for representing concurrent hardware and software systems. They serve as intermediate tools between detailed circuit diagrams and block diagrams (or flowcharts), when the former are too complex to analyze or the latter too coarse to predict the behavior of systems. Yet, a difficulty in the use of Petri nets for large scale systems is that the net representation may still become too large to handle.

To cope with the above large scale problem, this paper presents a method for refining (or abstracting) a Petri net representation to the desired level of detail using step-by-step transformations of transitions into subnets (or vice versa). In particular, we present conditions under which a subnet can be substituted for a single transition while preserving properties such as liveness (absence of deadlooks) and boundedness (absence of overflows). Our method is closely related to and generalizes the method

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of Valette [14] in the following points: (1) Our method is applicable if a transition to be refined is not $(k+1)$-enabled for some integer $k \geqslant 1$, whereas the method of Valette is applicable only for $k=1$; and (2) the condition (well-formedness) that a subnet must satisfy in [14] is a sufficient condition for our counterpart (1-well-behavedness).

After defining the terminology used in this paper in Sec. 2, the main results on transformations of transitions are given in Sec. 3. Section 4 presents a method for refining places using the transition refinement techniques described in Sec. 3. Section 5 shows that $(k+1)$-enabledness and k-well-behavedness are decidable properties. Also, two additional properties used in the theory are shown to be decidable. Finally, in Sec. 6 our method is illustrated with two examples. The first example illustrates how marked graph transformations reported in [5, 11] can be interpreted by our transformation method for transitions, and the second one illustrates a divide-and-conquer approach to the liveness analysis of Petri nets.

2. Definitions

We denote by $\mathbb{N}$ and $\mathbb{N}^+$, the set of non-negative integers and the set of positive integers, respectively. For any set S, S^* is the set of all finite sequences of elements of S, including the empty sequence λ. $S^* - \{\lambda\}$ is written as S^+. $|\sigma|$ denotes the length of $\sigma \in S^*$. $\#(\sigma, s)$ is the number of occurrences of $s \in S$ in $\sigma \in S^*$.

A *Petri net* is a five-tuple $N = (P, T, \mathrm{IN}, \mathrm{OUT}, M_0)$ where P is a finite set of *places*, T is a finite set of *transitions* $(P \cap T = \phi)$, $\mathrm{IN}: T \times P \to \mathbb{N}$ and $\mathrm{OUT}: T \times P \to \mathbb{N}$ are functions. Any function $M: P \to \mathbb{N}$ is called a *marking*, and M_0 denotes an *initial marking*. A place $p \in P$ such that $\mathrm{IN}(t, p) \neq 0$ (or $\mathrm{OUT}(t, p) \neq 0$) is called an *input place* (or an *output place*) of $t \in T$. A transition $t \in T$ such that $\mathrm{OUT}(t, p) \neq 0$ (or $\mathrm{IN}(t, p) \neq 0$) is called an *input transition* (or an *output transition*) of $p \in P$. A transition $t \in T$ is said to be *enabled* at a marking M_1 iff (if and only if) $\mathrm{IN}(t, p) \leqslant M_1(p)$ for all $p \in P$. If t is enabled at M_1, the marking M_2 such that $M_2(p) = M_1(p) - \mathrm{IN}(t, p) + \mathrm{OUT}(t, p)$ for all $p \in P$ is *reachable* from M_1 by a *firing* of t, and we write $M_1(t\rangle M_2$. If there exist transitions $t_1, t_2, ..., t_n$ such that $M_i(t_i\rangle M_{i+1}$ $(i = 1, 2, ..., n)$ for markings $M_1, M_2, ..., M_{n+1}$, we say that M_{n+1} is *reachable* from M_1 by *firing sequence* $\sigma = t_1 t_2 \cdots t_n$ (starting from M_1), and we write $M_1(\sigma\rangle M_{n+1}$. We define $M(\lambda\rangle M$ for any marking M. $L(N)$ denotes the set of all firing sequences starting from the initial marking M_0, and $R(N)$ denotes the set of all markings reachable from M_0. A transition $t \in T$ is *live* iff for any $\sigma \in L(N)$ there exists a sequence $\sigma' \in T^*$ such that $\sigma\sigma' t \in L(N)$. N is *live* iff every $t \in T$ is live. A place $p \in P$ is *k-bounded* for $k \in \mathbb{N}^+$ iff $M(p) \leqslant k$ for all $M \in R(N)$. N is *k-bounded* iff every $p \in P$ is k-bounded. N is *safe* iff it is 1-bounded. A transition $t \in T$ is *k-enabled* for $k \in \mathbb{N}^+$ in N iff there exists a marking $M \in R(N)$ such that $k \cdot \mathrm{IN}(t, p) \leqslant M(p)$ for every $p \in P$.

A Petri net $N = (P, T, \mathrm{IN}, \mathrm{OUT}, M_0)$ is drawn as a graph in which a place is represented by a circle, and a transition by a box. Whenever $\mathrm{IN}(t, p) > 0$, there is an arc with weight $\mathrm{IN}(t, p)$ from the circle for p to the box for t; and whenever

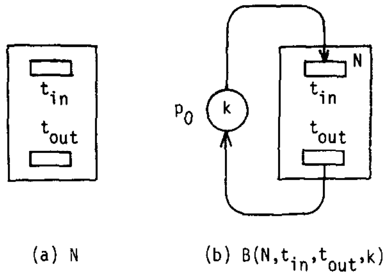


FIG. 1. (a) N, and (b) $B(N, t_{\text{in}}, t_{\text{out}}, k)$.

$\text{OUT}(t, p) > 0$, there is an arc with weight $\text{OUT}(t, p)$ from the box for t to the circle for p (the weight may not be indicated when it is 1). A marking M is represented by drawing $M(p)$ dots called *tokens* or writing "$M(p)$" in the circle for p.

For a Petri net $N = (P, T, \text{IN}, \text{OUT}, M_0)$, two distinct transitions t_{in}, $t_{\text{out}} \in T$ and $k \in \mathcal{N}^+$, the Petri net $B(N, t_{\text{in}}, t_{\text{out}}, k) = (P \cup \{p_0\}, T, \text{IN}_B, \text{OUT}_B, M_{B0})$ ($p_0 \notin P$ is a new place) is defined as follows (see Fig. 1):

$$\begin{aligned}
\text{IN}_B(t, p) &= 1, && t = t_{\text{in}} \text{ and } p = p_0, \\
&= 0, && t \neq t_{\text{in}} \text{ and } p = p_0, \\
&= \text{IN}(t, p), && p \in P, \\
\text{OUT}_B(t, p) &= 1, && t = t_{\text{out}} \text{ and } p = p_0, \\
&= 0, && t \neq t_{\text{out}} \text{ and } p = p_0, \\
&= \text{OUT}(t, p), && p \in P, \\
M_{B0}(p) &= k, && p = p_0, \\
&= M_0(p), && p \in P.
\end{aligned}$$

For $k \in \mathcal{N}^+$, a Petri net $N = (P, T, \text{IN}, \text{OUT}, M_0)$ is said to be *k-well-behaved* (*k-WB*) with respect to two distinct transitions t_{in}, $t_{\text{out}} \in T$ iff the following three conditions hold:

($WB1$) t_{in} is live in $B(N, t_{\text{in}}, t_{\text{out}}, k)$.

($WB2$) For each $\sigma_1 \in L(B(N, t_{\text{in}}, t_{\text{out}}, k))$ such that $\#(\sigma_1, t_{\text{in}}) > \#(\sigma_1, t_{\text{out}})$, there exists $\sigma_2 \in (T - \{t_{\text{in}}\})^+$ such that $\sigma_1\sigma_2 \in L(B(N, t_{\text{in}}, t_{\text{out}}, k))$ and $\#(\sigma_1, t_{\text{in}}) = \#(\sigma_1\sigma_2, t_{\text{out}})$.

($WB3$) $\#(\sigma, t_{\text{in}}) \geqslant \#(\sigma, t_{\text{out}})$ for any $\sigma \in L(B(N, t_{\text{in}}, t_{\text{out}}, k))$.

($WB1$) states that t_{in} never "gets blocked." ($WB2$) and ($WB3$) say that t_{in} can "get ahead" of t_{out} in firings, but t_{out} can always "catch up." We have:

Property 1. *For $n \in \mathbb{N}^+$, if Petri net N is $(n+1)$-WB with respect to t_{in} and t_{out}, then N is n-WB.*

Proof. Suppose that N is $(n+1)$-WB for $n \in \mathbb{N}^+$, and let $B(N, t_{\text{in}}, t_{\text{out}}, n+1)$ and $B(N, t_{\text{in}}, t_{\text{out}}, n)$ be denoted by N_{n+1} and N_n, respectively. $(WB2)$ and $(WB3)$ for $k = n+1$ imply $(WB2)$ and $(WB3)$ for $k = n$, respectively. Let $\sigma \in L(N_n)$ be an arbitrary firing sequence. By $(WB2)$ and $(WB3)$ for $k = n$, there exists $\sigma_1 \in (T - \{t_{\text{in}}\})^*$ such that $\sigma\sigma_1 \in L(N_n)$ and $\#(\sigma, t_{\text{in}}) = \#(\sigma\sigma_1, t_{\text{out}})$, where T is the set of transitions of N. Clearly $\sigma\sigma_1 \in L(N_{n+1})$, and by $(WB1)$ for $k = n+1$ we have $\sigma\sigma_1\sigma_2 t_{\text{in}} \in L(N_{n+1})$ for some $\sigma_2 \in (T - \{t_{\text{in}}\})^*$. Since the markings of N_{n+1} and N_n reached by $\sigma\sigma_1$ are identical except that p_0 has $n+1$ tokens in N_{n+1} and n tokens in N_n, $\sigma\sigma_1\sigma_2 t_{\text{in}}$ is also a firing sequence of N_n. So $(WB1)$ holds for $k = n$, and thus N is n-WB. ∎

Example 1. Transition t_0 of Petri net N shown in Fig. 2(a) is 1-, 2-, and 3-enabled, but not k-enabled for $k \geq 4$. Petri net N' shown in Fig. 2(b) is 1-, 2-, and 3-WB with respect to t_{in} and t_{out}, but not k-WB for $k \geq 4$. $B(N', t_{\text{in}}, t_{\text{out}}, 3)$ is illustrated in Fig. 2(c). ∎

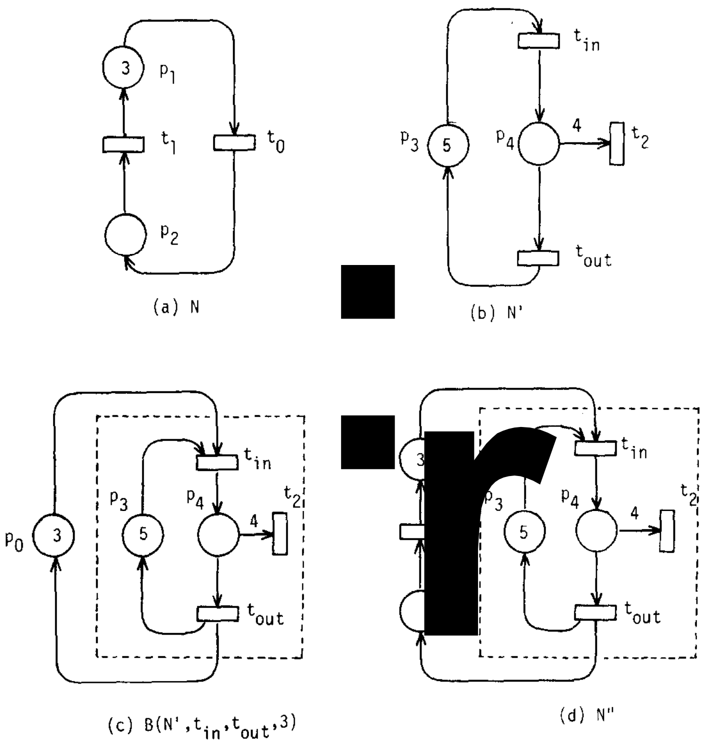


Fig. 2. Examples 1, 2, and 3: (a) Petri net N, (b) N', (c) $B(N', t_{\text{in}}, t_{\text{out}}, 3)$ and (d) $N'' = TR(N, N', t_0, t_{\text{in}}, t_{\text{out}})$.

If N is k-WB with respect to t_{in} and t_{out}, then the *synchronic distance* [3] $d(t_{\text{in}}, t_{\text{out}})$ between t_{in} and t_{out} in $B(N, t_{\text{in}}, t_{\text{ou}}, k)$ is no larger than k, where $d(t_{\text{in}}, t_{\text{out}}) = \text{Max}_{\sigma \in L(B(N, t_{\text{in}}, t_{\text{out}}, k))}(|\#(\sigma, t_{\text{in}}) - \#(\sigma, t_{\text{out}})|)$.

3. Refinement of Transitions

In this section we will investigate what properties are preserved, when a transition t_0 which is not $(k+1)$-enabled in a Petri net N is replaced by a k-WB Petri net N' to obtain a refined Petri net N''. Thus N is a reduced representation of N'', and N'' is a refined representation of N.

Let $N = (P, T, \text{IN}, \text{OUT}, M_0)$ and $N' = (P', T', \text{IN}', \text{OUT}', M_0')$ $(P \cap P' = \phi$, $T \cap T' = \phi)$ be Petri nets such that for some $k \in \mathcal{N}^+$, a transition $t_0 \in T$ is not $(k+1)$-enabled in N and N' is k-WB with respect to two distinct transitions t_{in}, $t_{\text{out}} \in T'$. Let $B(N', t_{\text{in}}, t_{\text{out}}, k)$ be denoted by $N_B' = (P' \cup \{p_0\}, T', \text{In}_B', \text{OUT}_B', M_{B0}')$. Let $N'' = TR(N, N', t_0, t_{\text{in}}, t_{\text{out}}) = (P'', T'', \text{IN}'', \text{OUT}'', M_0'')$ be the refined Petri net defined as follows (see Fig. 3):

$$
\begin{aligned}
P'' &= P \cup P', \\
T'' &= (T \cup T') - \{t_0\}, \\
\text{IN}''(t,p) &= 0, && (t \in T - \{t_0\} \text{ and } p \in P') \text{ or} \\
& && (t \in T', t \neq t_{\text{in}} \text{ and } p \in P), \\
&= \text{IN}(t,p), && t \in T - \{t_0\} \text{ and } p \in P, \\
&= \text{IN}'(t,p), && t \in T' \text{ and } p \in P', \\
&= \text{IN}(t_0,p), && t = t_{\text{in}} \text{ and } p \in P, \\
\text{OUT}''(t,p) &= 0, && (t \in T - \{t_0\} \text{ and } p \in P') \text{ or} \\
& && (t \in T', t \neq t_{\text{out}} \text{ and } p \in P), \\
&= \text{OUT}(t,p), && t \in T - \{t_0\} \text{ and } p \in P, \\
&= \text{OUT}'(t,p), && t \in T' \text{ and } p \in P', \\
&= \text{OUT}(t_0,p), && t = t_{\text{out}} \text{ and } p \in P, \\
M_0''(p) &= M_0(p), && p \in P, \\
&= M_0'(p), && p \in P'.
\end{aligned}
$$

Example 2. $N'' = TR(N, N', t_0, t_{\text{in}}, t_{\text{out}})$ for N and N' of Fig. 2(a); (b) is shown in Fig. 2(d). ▮

Let f: $(T'')^* \to T^*$ and f': $(T'')^* \to (T')^*$ be functions as defined below. As will be shown in the following lemmas, the function f converts a firing sequence of N'' into a

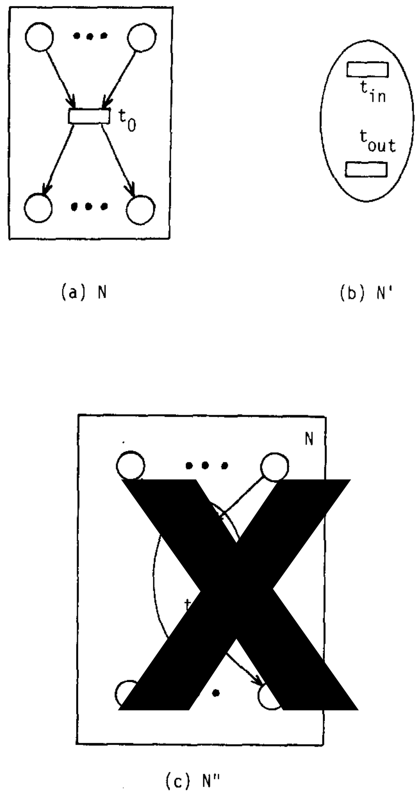


FIG. 3. Definition of N'': (a) N, (b) N' and $N'' = TR(N, N', t_0, t_{\text{in}}, t_{\text{out}})$.

firing sequence of N. Similarly, f' converts a firing sequence of N'' to a firing sequence of N'_B.

(a) $f(\lambda) = \lambda$.

(b) For $\sigma'' \in (T'')^*$,

$$\begin{aligned} f(\sigma'' t) &= f(\sigma''), && t \in T' - \{t_{\text{in}}\}, \\ &= f(\sigma'')\, t_0, && t = t_{\text{in}}, \\ &= f(\sigma'') t, && t \in T - \{t_0\}. \end{aligned}$$

(a′) $f'(\lambda) = \lambda$.

(b′) For $\sigma'' \in (T'')^*$,

$$\begin{aligned} f'(\sigma'' t) &= f'(\sigma'') t, && t \in T', \\ &= f'(\sigma''), && t \in T - \{t_0\}. \end{aligned}$$

Lemmas 2 and 4, in the following, state that if we refine a transition of N by a well-behaved Petri net, any firing sequence in the original net N can be simulated, using the correspondence given by f, by some firing sequence of the resulting net N'', and conversely, any possible firing sequence in N'' is a simulation of some firing sequence in N.

LEMMA 2. *For any $\sigma \in L(N)$, there exists $\sigma'' \in L(N'')$ such that $f(\sigma'') = \sigma$.*

Proof. By ($WB1$) and ($WB2$) there exists an infinite number of elements of $(T')^*$, $\sigma'_1, \sigma'_2, \ldots, \sigma'_{2i-1}, \sigma'_{2i}, \ldots$, such that $\#(\sigma'_{2i-1}, t_{\text{in}}) = 1$, $\#(\sigma'_{2i-1}, t_{\text{out}}) = 0$, $\#(\sigma'_{2i}, t_{\text{in}}) = 0$ and $\#(\sigma'_{2i}, t_{\text{out}}) = 1$ for all $i \in \mathbf{N}^+$, and $\sigma'_1 \sigma'_2 \cdots \sigma'_{2i-1} \sigma'_{2i} \cdots$ is a firing sequence of N'_B. So for any $\sigma \in L(N)$, $\sigma'' \in (T'')^*$ where σ'' is obtained from σ by replacing the ith occurrence of t_0 by $\sigma'_{2i-1}\sigma'_{2i}$ is a firing sequence of N''; i.e., $\sigma'' \in L(N'')$. Clearly $f(\sigma'') = \sigma$. ∎

LEMMA 3. *Given a firing sequence $\sigma'' \in L(N'')$, suppose that $f(\sigma'') \in L(N)$ and $f'(\sigma'') \in L(N'_B)$ (which always holds as proved in Lemma 4), and let $M''_0(\sigma''\rangle M''_1$, $M_0(f(\sigma''))\rangle M_1$ and $M'_{B0}(f'(\sigma''))\rangle M'_{B1}$. Then we have*

$$M_1(p) = M''_1(p) + \mathrm{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma'', t_{\text{in}}) - \#(\sigma'', t_{\text{out}})\} \tag{1}$$

for all $p \in P$, and

$$M'_{B1}(p) = M''_1(p) \tag{2}$$

for all $p \in P'$.

Proof. As (2) is trivial, we will only prove (1). The proof is by induction on the length of σ'':

(basis): (1) holds for $\sigma'' = \lambda$.

(induction): Suppose that (1) holds for all σ'' with $|\sigma''| \leqslant n$. For $\sigma'' = \sigma''_1 t \in L(N'')$ with $|\sigma''_1| = n$, let $M''_0(\sigma''_1\rangle M''$, $M''(t\rangle M''_1$, $M_0(f(\sigma''_1))\rangle M$, and $M(f(t))\rangle M_1$. By the inductive hypothesis we have

$$M(p) = M''(p) + \mathrm{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma''_1, t_{\text{in}}) - \#(\sigma''_1, t_{\text{out}})\} \tag{1'}$$

for all $p \in P$.

If $t \in T - \{t_0\}$, then $M_1''(p) = M''(p) - \mathrm{IN}(t, p) + \mathrm{OUT}(t, p)$ and $M_1(p) = M(p) - \mathrm{IN}(t, p) + \mathrm{OUT}(t, p)$ for all $p \in P$, and thus (1) is derived from (1′).

If $t \in T' - \{t_{\text{in}}, t_{\text{out}}\}$, then $M''_1(p) = M''(p)$ and $M_1(p) = M(p)$ for all $p \in P$ by definition, and again (1′) yields (1).

Now suppose that $t = t_{\text{in}}$. Since $f(t_{\text{in}}) = t_0$, for all $p \in P$ we have

$$\begin{aligned} M_1(p) &= M(p) - \text{IN}(t_0, p) + \text{OUT}(t_0, p), \\ &= M''(p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma_1'', t_{\text{in}}) - \#(\sigma_1'', t_{\text{out}})\} - \text{IN}(t_0, p) + \text{OUT}(t_0, p) \\ &\qquad \text{(by (1'))}, \\ &= M''(p) - \text{IN}''(t_{\text{in}}, p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma'', t_{\text{in}}) - \#(\sigma'', t_{\text{out}})\} \\ &\qquad \text{(since IN}(t_0, p) = \text{IN}''(t_{\text{in}}, p) \text{ and OUT}(t_0, p) = \text{OUT}''(t_{\text{out}}, p)), \\ &= M_1''(p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma'', t_{\text{in}}) - \#(\sigma'', t_{\text{out}})\} \\ &\qquad \text{(since OUT}''(t_{\text{in}}, p) = 0), \end{aligned}$$

and thus (1) is derived.

Finally suppose that $t = t_{\text{out}}$. Since $f(t_{\text{out}}) = \lambda$, for all $p \in P$ we have

$$\begin{aligned} M_1(p) &= M(p) \\ &= M''(p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma_1'', t_{\text{in}}) - \#(\sigma_1'', t_{\text{out}})\}, \\ &\qquad \text{(by (1'))} \\ &= M''(p) + \text{OUT}''(t_{\text{out}}, p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma'', t_{\text{in}}) - \#(\sigma'', t_{\text{out}})\}, \\ &= M_1''(p) + \text{OUT}''(t_{\text{out}}, p) \cdot \{\#(\sigma'', t_{\text{in}}) - \#(\sigma'', t_{\text{out}})\} \\ &\qquad \text{(since IN}''(t_{\text{out}}, p) = 0), \end{aligned}$$

and (1) is derived. This completes the proof of (1). ∎

LEMMA 4. *For any $\sigma'' \in L(N'')$, the following hold*:

$$f(\sigma'') \in L(N), \tag{3}$$

$$\#(\sigma'', t_{\text{in}}) \geqslant \#(\sigma'', t_{\text{out}}), \tag{4}$$

$$f'(\sigma'') \in L(N_B'). \tag{5}$$

Proof. The proof is by induction on the length of σ'':

(basis): For $\sigma'' = \lambda$, (3), (4), and (5) hold trivially.

(induction): Suppose (3), (4), and (5) hold for any $\sigma'' \in L(N'')$ with $|\sigma''| \leqslant n$.

(induction for (3)): Let $\sigma'' = \sigma_1'' t \in L(N'')$ be an arbitrary firing sequence where $|\sigma_1''| = n$.

If $t \in T' - \{t_{\text{in}}\}$, $f(\sigma'') = f(\sigma_1'') \in L(N)$ by the inductive hypothesis for (3).

Suppose that $t \in (T - \{t_0\}) \cup \{t_{\text{in}}\}$. By (1), (4), and the inductive hypothesis for (3), we have $M_1''(p) \leqslant M_1(p)$ for $p \in P$ where $M_0''(\sigma_1''\rangle M_1''$ and $M_0(f(\sigma_1''))\rangle M_1$. Therefore, since t is enabled at M_1'' in N'', t (or t_0 if $t = t_{\text{in}}$) is enabled also at M_1 in N; i.e., $f(\sigma'') = f(\sigma_1'')t$ (or $f(\sigma_1'')\, t_0$ if $t = t_{\text{in}}) \in L(N)$.

(induction for (4)): Suppose that $\#(\sigma'', t_{\text{in}}) < \#(\sigma'', t_{\text{out}})$ for some $\sigma'' \in L(N'')$ with $|\sigma''| = n + 1$. By the inductive hypothesis for (4), this is possible only when $\sigma'' = \sigma_1'' t_{\text{out}}$, where $\#(\sigma_1'', t_{\text{in}}) = \#(\sigma_1'', t_{\text{out}})$. By (2) and the inductive hypothesis for (5), $M_1''(p) = M_{B1}'(p)$ for all $p \in P'$, where $M_0''(\sigma_1''\rangle M_1$ and $M_{B0}'(f'(\sigma_1''))\rangle M_{B1}'$. So we have $f'(\sigma_1'')\, t_{\text{out}} \in L(N_B')$ and $\#(f'(\sigma_1'')\, t_{\text{out}}, t_{\text{in}}) < \#(f'(\sigma_1'')\, t_{\text{out}}, t_{\text{ou}})$, a contradiction to $(WB3)$. Therefore (4) is true for all $\sigma'' \in L(N'')$ of length $n + 1$.

(induction for (5)): Let $\sigma'' = \sigma_1'' t \in L(N'')$, where $|\sigma_1''| = n$ and $t \in T''$.
If $t \notin T', f'(\sigma_1'' t) = f'(\sigma_1'') \in L(N'')$ by the inductive hypothesis for (5).
Suppose that $t \in T'$. Since $M_1''(p) = M_{B1}'(p)$ for all $p \in P'$ where $M_0''(\sigma_1''\rangle M_1''$ and $M_{B0}'(f'(\sigma_1''))\rangle M_{B1}'$ by (2) and the inductive hypothesis for (5), we have $f'(\sigma_1'' t) = f'(\sigma_1'')t \in L(N_B')$ for $t \in T' - \{t_{\text{in}}\}$. For $t = t_{\text{in}}$, we have to show that $M_{B1}'(p_0) \geqslant 1$. If $M_{B1}'(p_0) = 0$, then $\#(\sigma'', t_{\text{in}}) = \#(\sigma'', t_{\text{out}}) + (k + 1)$. Since the last $(k + 1)$ firings of t_{in} in σ'' have no "corresponding" firings of t_{out} in σ'', we see, from the construction of N'', that in the firing sequence $f(\sigma'')$ (note that $f(\sigma'') \in L(N)$ by (induction for (5))) the last $(k + 1)$ firings of t_0 need not "produce" tokens for completing $f(\sigma'')$. So we can "postpone" the last $(k + 1)$ firings of t_0, and obtain another firing sequence $\sigma(t_0)^{k+1} \in L(N)$, which is a permutation of $f(\sigma'')$ such that $(k + 1) \cdot \text{IN}(t_0, p) \leqslant M_1(p)$ for all $p \in P$ where $M_0(\sigma\rangle M_1$. That is, t_0 is $(k + 1)$-enabled in N (contradiction). Therefore $M'_{B1}(p_0) \geqslant 1$, and so $f'(\sigma'') = f'(\sigma_1'')\, t_{\text{in}} \in L(N'')$. ∎

Lemma 5. *For any $\sigma'' \in L(N'')$, there exists $\sigma_1'' \in (T' - \{t_{\text{in}}\})^*$ such that $\sigma''\sigma_1'' \in L(N'')$ and $\#(\sigma'', t_{\text{in}}) = \#(\sigma''\sigma_1'', t_{\text{out}})$.*

Proof. Let $\sigma'' \in L(N'')$ be an arbitrary firing sequence. Since $f'(\sigma'') \in L(N_B')$ by (5), by $(WB2)$ there exists $\sigma_1'' \in (T' - \{t_{\text{in}}\})^+$ such that $f'(\sigma'')\,\sigma_1'' \in L(N_B')$ and $\#(f'(\sigma''), t_{\text{in}}) = \#(f'(\sigma'')\,\sigma_1'', t_{\text{out}})$. So $\sigma''\sigma_1'' \in L(N'')$ by (2) and $\#(\sigma'', t_{\text{in}}) = \#(f'(\sigma''), t_{\text{in}}) = \#(f'(\sigma'')\,\sigma_1'', t_{\text{out}}) = \#(\sigma''\sigma_1'', t_{\text{out}})$. ∎

Theorem 6. *If N'' is m''-bounded, then N is m''-bounded.*

Proof. By Lemmas 2 and 5, for every $\sigma \in L(N)$ there exists $\sigma'' \in L(N'')$ such that $f(\sigma'') = \sigma$ and $\#(\sigma'', t_{\text{in}}) = \#(\sigma'', t_{\text{out}})$. Let $M_0(\sigma\rangle M_1$ and $M_0''(\sigma''\rangle M_1''$. Then we have $M_1(p) = M_1''(p) \leqslant m''$ for all $p \in P$ by (1) and (4). Thus M is m''-bounded. ∎

A special case of Theorem 6 is stated as a corollary.

Corollary 7. *If N'' is safe, then N is safe.*

Proof. Set m'' to 1 in Theorem 6. ∎

Theorem 8. *If N is m-bounded and every place $p \in P'$ of N_B' is m'-bounded, then N'' is m''-bounded where $m'' = \text{Max}(m, m')$.* ∎

Proof. Let $\sigma'' \in L(N'')$ and $M_0''(\sigma''\rangle M_1''$. By (1), (3), and (4) we let $M_0(f(\sigma''))\rangle M_1$ and we have $M_1''(p) \leqslant M_1(p) \leqslant m$ for all $p \in P$. Further, $M_1''(p) =$

$M'_{B1}(p) \leqslant m'$ for all $p \in P'$, where $M'_{B0}(f'(\sigma'')\rangle M'_{B1}$ by (2) and (5). Therefore $M''_1(p) \leqslant \mathrm{Max}(m, m')$ for all $p \in P''$; i.e., N'' is m''-bounded where $m'' = \mathrm{Max}(m, m')$. ∎

A special case of Theorem 8 is stated as a corollary.

COROLLARY 9. *If N and $B(N', t_{\text{in}}, t_{\text{out}}, 1)$ are safe, then N'' is safe.*

Proof. Set m and m' to 1 in Theorem 8. ∎

EXAMPLE 3. In Fig. 2, N is 3-bounded, and p_3 and p_4 of $B(N', t_{\text{in}}, t_{\text{out}}, 3)$ are 5-bounded. So N'' is 5-bounded by Theorem 8. ∎

THEOREM 10. *If N'' is live, then N is live.*

Proof. Let $\sigma \in L(N)$ be an arbitrary firing sequence. By Lemma 2 there exists $\sigma'' \in L(N'')$ such that $f(\sigma'') = \sigma$. Since N'' is live, for any $t \in T$ there exists $\sigma''_1 \in (T'')^*$ such that $\sigma''\sigma''_1 t \in L(N'')$. Then by (3) and the definition of f, we have $f(\sigma''\sigma''_1) = \sigma f(\sigma''_1) t \in L(N)$. Therefore N is live. ∎

Condition A, stated next, is used to obtain a sufficient condition for N'' to be live, when a transition t_0, which is not $(k+1)$-enabled in N, is replaced by a k-WB subnet N'.

CONDITION A. *For any reachable marking $M \in R(N)$ of $N = (P, T, \mathrm{IN}, \mathrm{OUT}, M_0)$, t_0 is k-enabled in the Petri net $(P, T, \mathrm{IN}, \mathrm{OUT}, M)$; or equivalently, from any marking $M \in R(N)$ we can again reach a marking M_1 such that $k \cdot \mathrm{IN}(t_0, p) \leqslant M_1(p)$ for all $p \in P$.*

THEOREM 11. *Suppose that N satisfies Condition A. If N and N'_B are live, then N'' is live.*

Proof. Let $\sigma'' \in L(N'')$ be an arbitrary firing sequence. By Lemma 5 there exists $\sigma_1'' \in (T' - \{t_{\text{in}}\})^*$ such that $\sigma''\sigma''_1 \in L(N'')$ and $\#(\sigma'', t_{\text{in}}) = \#(\sigma''\sigma''_1, t_{\text{out}})$.

(i) By (3) and the liveness of N, for any $t \in T$ we have $f(\sigma''\sigma''_1)\,\sigma_2 t \in L(N)$ for some $\sigma_2 \in T^*$. By (1), (4), and an argument similar to the one used in the proof of Lemma 2, $\sigma_2 t$ can be "simulated" by some firing sequence of N''. So there exists $\sigma''_2 \in (T'')^*$ such that $\sigma''\sigma''_1\sigma''_2 t \in L(N'')$ (here $f(\sigma''\sigma''_1\sigma''_2 t) = f(\sigma''\sigma''_1)\,\sigma_2 t$). Thus every $t \in T - \{t_0\}$ is live in N''.

(ii) By (3) we have $f(\sigma''\sigma''_1) \in L(N)$. Since N satisfies Condition A there exists some $\sigma_2 \in T^*$ such that $f(\sigma''\sigma''_1)\,\sigma_2 \in L(N)$ and $k \cdot \mathrm{IN}(t_0, p) \leqslant M_1(p)$ for every $p \in P$ where $M_0(f(\sigma''\sigma''_1)\,\sigma_2\rangle M_1$. By (1), (4), Lemma 5, and an argument similar to the one used in the proof of Lemma 2, σ_2 can be "simulated" by a firing sequence $\sigma''_2 \in (T'')^*$ of N'', and we obtain $\sigma''\sigma''_1\sigma''_2 \in L(N'')$, $f(\sigma''\sigma''_1\sigma''_2) = f(\sigma''\sigma''_1)\,\sigma_2$, and $\#(\sigma''\sigma''_1\sigma''_2, t_{\text{in}}) = \#(\sigma''\sigma''_1\sigma''_2, t_{\text{out}})$. Note that by (1) and (4) at marking M''_1 where $M''_0(\sigma''\sigma''_1\sigma''_2\rangle M''_1$, we have $k \cdot \mathrm{IN}(t_{\text{in}}, p) \leqslant M''_1(p)$ for all $p \in P$. Consider

$\sigma' = f(\sigma''\sigma_1''\sigma_2'') \in (T')^*$. σ' is a firing sequence of N_B' by (5). Since N_B' is live, for any $t' \in T'$ there exists some $\sigma_1' \in (T')^*$ such that $\sigma'\sigma_1't' \in L(N_B')$. Now we show that $\sigma_1't'$ can be "simulated" by a firing sequence of N'' by the correspondence of transition firings described below. A firing of any transition $t \in T' - \{t_{\text{in}}\}$ in σ_1' can be simulated by a firing of the same transition in N''. The 1st, 2nd,..., and kth firings of t_{in} in σ_1' can also be simulated by the firings of t_{in} in N'', since we have $k \cdot \text{IN}(t_{\text{in}}, p) \leqslant M_1''(p)$ for all $p \in P$. The $k+i$th firing of t_{in} in σ_1', $i \geqslant 1$, can be simulated in N'' by some $\sigma_i''t_{\text{in}}$, where $\sigma_i'' \in (T - \{t_0\})^*$, since by assumption N is live and thus t_{in} can be enabled by some firing sequence σ_i''. Let $\sigma_3''t' \in (T'')^*$ be the firing sequence obtained when N'' simulates $\sigma_1't'$ of N_B' by the correspondence given above. We have shown that for any $\sigma'' \in L(N'')$ and $t' \in T'$ there are some σ_1'', σ_2'', and σ_3'' such that $\sigma''\sigma_1''\sigma_2''\sigma_3''t' \in L(N'')$. Thus all $t' \in T'$ is live in N''.

So N'' is live by (i) and (ii). ∎

An example to illustrate Theorem 11 will be given later in Sec. 6.

EXAMPLE 4. In order to see why Condition A is necessary in Theorem 11, consider the nets N and N' shown in Fig. 4. We observe that:

(1) Transition t_0 in N is not 5-enabled (i.e., $k = 4$).

(2) N' is 4-*WB* with respect to t_{in} and t_{out}.

(3) N is live.

(4) $B(N', t_{\text{in}}, t_{\text{out}}, 4)$ is live.

However, the refined net N'' obtained from N by substituting N' for t_0 is not live, since t_1 can never be enabled. This means that some condition in Theorem 11 does not hold. Indeed, it can be seen that Condition A with $k = 4$ in N does not hold (although it holds for $k = 3$). ∎

For the Petri net N, let $W: P \to \mathbf{N}$ be a function that assigns a "weight" to each place. The weight may be interpreted as the amount of resources necessary to accom-

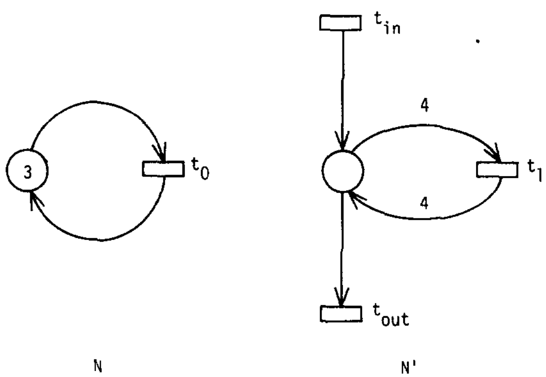


FIG. 4. Illustrations for Example 4.

modate a data item in the place. Then $\sum_{p\in P} M(p)\, W(p)$ represents the total resources required for any marking M. Similarly we let $W': P' \to \mathbb{N}$ be a weight function for N'. We define the weight function $W'': P'' \to \mathbb{N}$ for N'', where $P'' = P \cup P'$, as

$$\begin{aligned} W''(p) &= W(p), \qquad p \in P, \\ &= W'(p), \qquad p \in P'. \end{aligned}$$

We define the followings:

$$M_{\max} = \max_{M \in R(N)} \Big(\sum_{p \in P} M(p)\, W(p) \Big),$$

$$M'_{\max} = \max_{M'_B \in R(N'_B)} \Big(\sum_{p \in P'} M'_B(p)\, W'(p) \Big),$$

$$M''_{\max} = \max_{M'' \in R(N'')} \Big(\sum_{p \in P''} M''(p)\, W''(p) \Big).$$

We have the following theorem.

THEOREM 12. $M''_{\max} \leqslant M_{\max} + M'_{\max}$.

Proof. Let $\sigma'' \in L(N'')$ be an arbitrary firing sequence. By (1), (2), (3), (4), and (5), $M''_1(p) \leqslant M_1(p)$ for all $p \in P$ and $M''_1(p) = M'_{B1}(p)$ for all $p \in P'$ where $M''_0(\sigma''\rangle M''_1$, $M_0(f(\sigma''))\rangle M_1$ and $M'_{B0}(f'(\sigma''))\rangle M'_{B1}$. Thus we have

$$\sum_{p \in P''} M''_1(p)\, W''(p) \leqslant \sum_{p \in P} M_1(p)\, W(p) + \sum_{p \in P'} M'_{B1}(p)\, W'(p).$$

Hence the theorem. ∎

The above theorem is useful for estimating the maximum resource requirement at each stage of refinement.

4. REFINEMENT OF PLACES

In this section we consider a refinement method for places. We analyze this method using the results obtained in the previous section, as these two methods are closely related.

First we define the S-transformation for Petri nets ("S" for splitting). For a Petri net $N = (P, T, \mathrm{IN}, \mathrm{OUT}, M_0)$ and a place $p_0 \in P$, the Petri net $N^s = S(N, p_0, p_{01}, t_0, p_{02}) = ((P - \{p_0\}) \cup \{p_{01}, p_{02}\},\ T \cup \{t_0\},\ \mathrm{IN}^s,\ \mathrm{OUT}^s, M_0^s) = (P^s, T^s, \mathrm{IN}^s, \mathrm{OUT}^s, M_0^s)$ (see Fig. 5(a), (b)) is a copy of N, except that p_0 is split into two new places p_{01} and $p_{02} (\notin P)$. New transition $t_0 (\notin T)$ is connected only to p_{01} and p_{02}. Transition t_0 is the only output transition of p_{01} and the only input transition of

p_{02}. The input transitions of p_{01} are exactly those of p_0, and the output transitions of p_{02} are exactly those of p_0. $M_0^s(p) = M_0(p)$ for $p \in P - \{p_0\}$, $M_0^s(p_{01}) = M_0(p_0)$ and $M_0^s(p_{02}) = 0$. We have the following two lemmas.

LEMMA 13. (a) *For any $\sigma \in L(N)$, there exists $\sigma^s \in L(N^s)$ such that σ is a copy of σ^s in which all occurrences of t_0 are deleted.*

(b) *For any $\sigma^s \in L(N^s)$, $\sigma \in L(N)$ when σ is obtained from σ^s by deleting all occurrences of t_0.*

Proof. (a) Let $\sigma \in L(N)$. Let $\sigma^s \in (T^s)^*$ be the sequence obtained from σ by replacing every occurrence of each $t \in T$ by $t_0^{\mathrm{IN}(t,p_0)}t$. Then $\sigma^s \in L(N^s)$ and satisfies the condition of the lemma.

(b) Let $\sigma \in L(N)$ and $\sigma^s \in L(N^s)$ be firing sequences such that $\#(\sigma, t) = \#(\sigma^s, t)$ for all $t \in T$. Let $M_0(\sigma\rangle M_1$ and $M_0^s(\sigma^s\rangle M_1^s$. Then we have $M_1(p_0) \geqslant M_1^s(p_{02})$ and $M_1(p) = M_1^s(p)$ for all $p \in P - \{p_0\}$. Therefore, for any $\sigma^s \in L(N^s)$, the sequence σ which is obtained from σ^s by deleting all occurrences of t_0 is a firing sequence of N. ∎

LEMMA 14. (a) *N is k-bounded iff N^s is k-bounded.*

(b) *p_0 is k-bounded in N iff t_0 is not $(k+1)$-enabled in N^s.*

(c) *If N^s is live, then N is live.*

(d) *If p_0 has at least one input transition and N is live, then N^s is live.*

Proof.

(a) and (b): (if): Let $\sigma \in L(N)$ be an arbitrary firing sequence and $M_0(\sigma\rangle M_1$. Let $\sigma^s \in L(N^s)$ be the firing sequence obtained from σ by replacing every occurrence of each $t \in T$ by $t_0^{\mathrm{IN}(t,p_0)}t$, and $M_0^s(\sigma^s\rangle M_1^s$. Then we have $M_1(p_0) = M_1^s(p_{01})$ and $M_1(p) = M_1^s(p)$ for all $p \in P - \{p_0\}$. Thus if N is not k-bounded, N^s is not k-bounded. If p_0 is not k-bounded in N, then t_0 is $(k+1)$-enabled in N^s.

(only if): Let $\sigma^s \in L(N^s)$ be an arbitrary firing sequence and $M_0^s(\sigma^s\rangle M_1^s$. Let $\sigma \in L(N)$ be the firing sequence obtained from σ^s by deleting all occurrences of t_0, and $M_0(\sigma\rangle M_1$. Then we have $M_1(p_0) \geqslant M_1^s(p_{01})$, $M_1(p_0) \geqslant M_1^s(p_{02})$, and $M_1(p) = M_1^s(p)$ for all $p \in P - \{p_0\}$. So if N^s is not k-bounded, N is not k-bounded. If t_0 is $(k+1)$-enabled in N^s, then p_0 is not k-bounded in N.

(c): For arbitrary $\sigma \in L(N)$, let $\sigma^s \in L(N^s)$ be the firing sequence obtained from σ by replacing every occurrence of each $t \in T$ by $t_0^{IN(t,p_0)}t$. Since N^s is live, for any $t \in T$ there exists $\sigma_1^s \in (T^s)^*$ such that $\sigma^s \sigma_1^s t \in L(N^s)$. Then we have $\sigma\sigma_1 \in L(N)$, where σ_1 is the sequence obtained from σ_1^s by deleting all occurrences of t_0. Thus N is live.

(d): For arbitrary $\sigma^s \in L(N^s)$, let $\sigma \in L(N)$ be the sequence obtained from σ^s by deleting all occurrences of t_0. Since N is live, for any $t \in T$ there exists $\sigma_1 \in T^*$ such that $\sigma\sigma_1 t \in L(N)$. Let $\sigma_1^s \in (T^s)^*$ be the sequence obtained from σ_1 by replacing every occurrence of each $t' \in T$ by $t't_0^{\mathrm{OUT}(t',p_0)}$. Let x be a non-negative integer such that $\sigma^s t_0^x \in L(N^s)$ and $\sigma^s t_0^{(x+1)} \notin L(N^s)$. Then $\sigma^s t_0^x \sigma_1^s t \in L(N^s)$. So each $t \in T$ is live in N^s. Since $\mathrm{OUT}(t, p_0) \geqslant 1$ for some $t \in T$, t_0 is also live in N^s by a similar argument. Thus N^s is live. ∎

The hypothesis in (d) is necessary to avoid the case shown in Fig. 6. Here $N^s = S(N, p_0, p_{01}, t_0, p_{02})$ is not live, whereas N is live.

Let $N = (P, T, \mathrm{IN}, \mathrm{OUT}, M_0)$ and $N' = (P', T', \mathrm{IN}', \mathrm{OUT}', M_0')$ $(P \cap P' = \phi$, $T \cap T' = \phi)$ be Petri nets such that for some $k \in \mathcal{N}^+$, place $p_0 \in P$ is k-bounded in N, and N' is k-*WB* with respect to two distinct transitions t_{in}, $t_{\mathrm{out}} \in T'$. Let $N'' = (P'', T'', \mathrm{IN}'', \mathrm{OUT}'', M_0'') = TR(S(N, p_0, p_{01}, t_0, p_{02}),\ N', t_0, t_{\mathrm{in}}, t_{\mathrm{out}})$, where $p_{01}, p_{02} \notin P$, $p_{01}, p_{02} \notin P'$, $t_0 \notin T$, and $t_0 \notin T'$ (see Fig. 5). That is, N'' is the Petri net

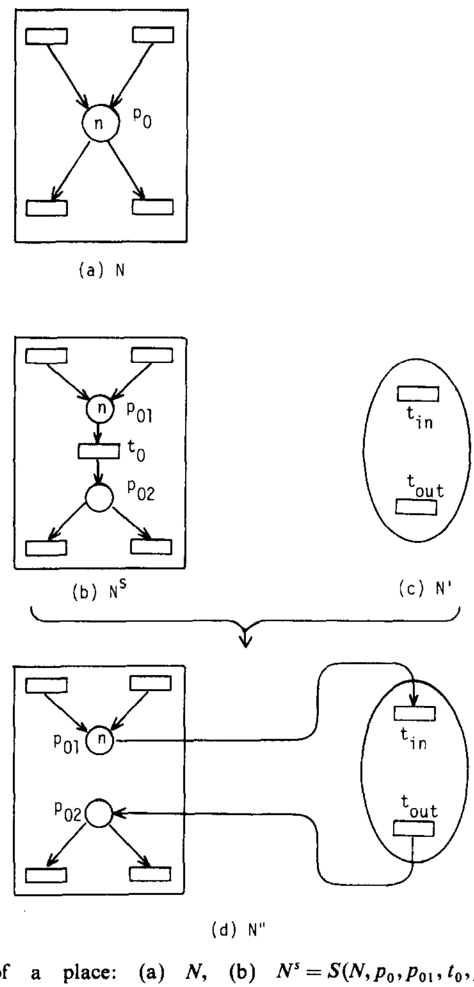


FIG. 5. Refinement of a place: (a) N, (b) $N^s = S(N, p_0, p_{01}, t_0, p_{02})$, (c) N' and (d) $N'' = TR(N^s, N', t_0, t_{\mathrm{in}}, t_{\mathrm{out}})$.

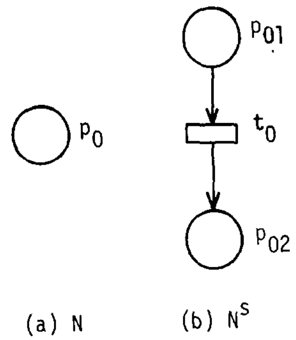


FIG. 6. A place without input transitions: (a) live N and (b) non-live N^s.

obtained from N by refining place p_0 by N'. Again, $B(N', t_{\text{in}}, t_{\text{out}}, k)$ is denoted by N'_B.

Some properties of the N'' constructed above are stated below. The proofs of the following results follow from Theorems 6, 8, 10, 12, and Lemmas 13 and 14.

THEOREM 15. *If N'' is m''-bounded, then N is m''-bounded.* ∎

THEOREM 16. *If N is m-bounded and every place $p \in P'$ of N'_B is m'-bounded, then N'' is m''-bounded where $m'' = \text{Max}(m, m')$.* ∎

THEOREM 17. *If N'' is live, then N is live.* ∎

Theorems 15, 16, and 17 are analogs of Theorems 6, 8, and 10, respectively. In order to state an analog of Theorem 11, we need the following condition.

CONDITION B. *For any marking $M \in R(N)$ there exists $M_1 \in R(N)$ such that $M(\sigma\rangle M_1$ for some σ and $M_1(p_0) = k$;* i.e., *from any reachable marking M we can again reach a marking M_1 such that $M_1(p_0) = k$.*

THEOREM 18. *Suppose that p_0 has at least one input transition and N satisfies Condition B. If N and N'_B are live, then N'' is live.* ∎

Let $W: P \to \mathbf{N}$ and $W': P' \to \mathbf{N}$ be weight functions for Petri nets N and N', respectively. Let $W'': P''$ $(=(P - \{p_0\}) \cup \{p_{01}, p_{02}\} \cup P') \to \mathbf{N}$ be the weight function for N'' defined as follows:

$$
\begin{aligned}
W''(p) &= W(p), && p \in P - \{p_0\},\\
&= W(p_0), && p = p_{01},\\
&= 0, && p = p_{02},\\
&= W'(p), && p \in P'.
\end{aligned}
$$

We define the following:

$$M_{\max} = \underset{M \in R(N)}{\mathrm{Max}} \left(\sum_{p \in P} M(p)\, W(p) \right),$$

$$M'_{\max} = \underset{M'_B \in R(N'_B)}{\mathrm{Max}} \left(\sum_{p \in P'} M'_B(p)\, W'(p) \right),$$

$$M''_{\max} = \underset{M'' \in R(N'')}{\mathrm{Max}} \left(\sum_{p \in P''} M''(p)\, W''(p) \right).$$

We have the following theorem.

THEOREM 19. $M''_{\max} \leqslant M_{\max} + M'_{\max}$. ∎

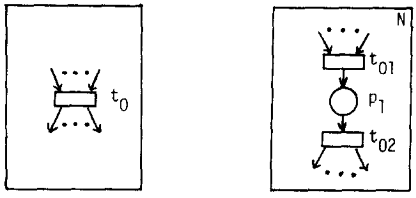
5. RELATED DECISION PROBLEMS

In this section we investigate the problem of deciding whether a given transition in a Petri net is $(k+1)$-enabled for given $k \in \mathbf{N}$, and the problem of deciding whether a given Petri net is k-WB for given $k \in \mathbf{N}^+$. Also discussed is the decidability of Conditions A and B.

We have the following theorems.

THEOREM 20. *It is decidable whether a given transition of a Petri net is $(k+1)$-enabled for given $k \in \mathbf{N}$.*

Proof. Let N be a Petri net for which we wish to test whether a transition t_0 is $(k+1)$-enabled. We construct a new Petri net $\hat{N}$ as shown in Fig. 7. $\hat{N}$ is a copy of N except that t_0 is split into new transitions t_{01}, t_{02}, and a new place p_1. The set of input places (or output places) of t_{01} (or t_{02}) is the same as that of t_0. Place p_1 is the only output place (or input place) of t_{01} (or t_{02}). $\hat{N}$ has the same initial marking as N, with p_1 holding no token. It should be clear from the construction in Fig. 7 that t_0 is $(k+1)$-enabled in N iff p_1 is not k-bounded in $\hat{N}$. Now, since whether a given place



(a) N (b) $\hat{N}$

FIG. 7. N and $\hat{N}$ in Theorem 20.

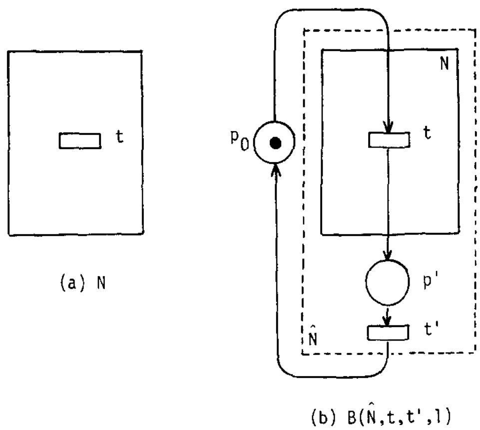


FIG. 8. N and $B(\hat{N}, t, t', 1)$ in the proof (i) of Theorem 21.

of a Petri net is k-bounded or not is decidable using the reachability tree [4], it is also decidable whether t_0 is $(k+1)$-enabled in N. ∎

THEOREM 21. *The decision problem for k-well-behavedness and the liveness problem for Petri nets are recursively equivalent.*

Proof. (i) Let N be a Petri net for which we wish to test whether a given transition t is live. As shown in Fig. 8, we construct a new Petri net $\hat{N}$ by adding the following to copy of N:

— a new place p' as an output place of t (initially p' has no token).

— a new transition t' which has only one input place p' and no output place.

Consider $B(\hat{N}, t, t', 1)$ shown in Fig. 8(b). If t is live in N, it is easy to see that the conditions for well-behavedness (WB 1), (WB 2), and (WB 3) hold for $k = 1$ with respect to t and t'. Conversely, if $\hat{N}$ is 1-WB with respect to t and t', then t is live in N. Thus we conclude that t is live in N iff $\hat{N}$ is 1-WB with respect to t and t'.

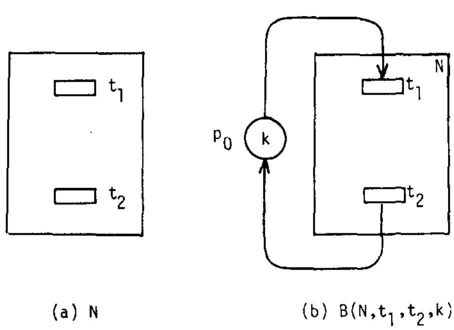


FIG. 9. N and $B(N, t_1, t_2, k)$ in the proof (ii) of Theorem 21.

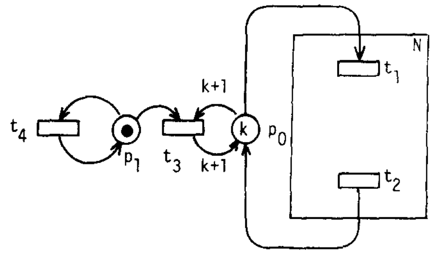


FIG. 10. Test(0) in the proof of Theorem 21.

(ii) Let N be a Petri net which we wish to test for k-WB, given $k \in \mathbb{N}^+$, with respect to two distinct transitions t_1 and t_2 (see Fig. 9(a)). That is, we wish to test whether $B(N, t_1, t_2, k)$ shown in Fig. 9(b) satisfies (WB 1), (WB 2), and (WB 3). We construct $k+1$ new Petri nets: Test(0), Test(1),..., and Test(k) as follows.

Test(0): Test(0) is a copy of $B(N, t_1, t_2, k)$ with two new transitions t_3, t_4, and a new place p_1 as shown in Fig. 10. Initially p_1 has one token. Place p_0 must hold more than k tokens for transition t_3 to fire.

Test(i) ($1 \leqslant i \leqslant k$): Test(i), $1 \leqslant i \leqslant k$, is shown in Fig. 11 (the arc (*) from p_3 to t_5 does not exist in Test(k)). The operation of Test(i) is as follows. Before t_5 fires, Test(i) simulates the firings of N by the copy of N and t_8. Transition t_5 can fire only

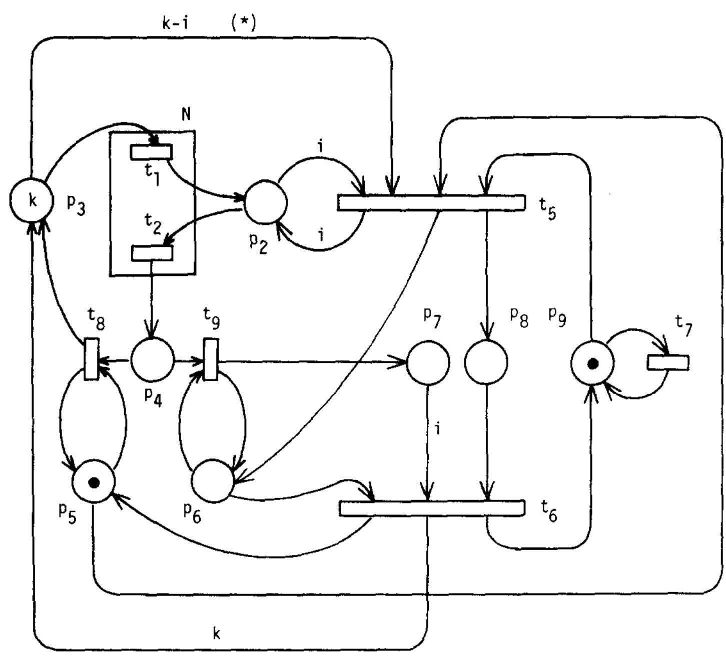


FIG. 11. Test(i), $1 \leqslant i \leqslant k$, in the proof of Theorem 21.

when t_1 has fired exactly i times more than t_2. The firing of t_5 moves the token in p_5 to p_6, disabling t_8. When p_7 has i tokens by the firings of t_2 and t_9, t_6 fires and returns k tokens to p_3. Transition t_7 can fire iff t_6 has fired exactly as many times as t_5.

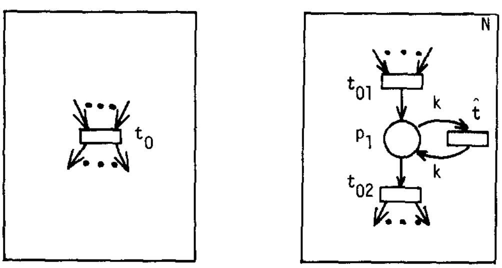
(ii-1): Suppose that N is k-WB with respect to t_1 and t_2. By (WB 1), t_1 is live in Test(0). By (WB 3), the number of tokens in p_0 is always less than or equal to k. Thus t_3 never fires, and t_4 is live in Test(0). Suppose that t_7 is not live in Test(i) for some i, $1 \leqslant i \leqslant k$. This means that there is a firing sequence $\sigma \in L(\text{Test(i)})$ such that $\sigma t_5 \in L(\text{Test(i)})$ and t_6 cannot fire after σt_5. That is, there is a firing sequence $\sigma_1 \in L(B(N, t_1, t_2, k))$ such that $\#(\sigma_1, t_1) = \#(\sigma_1, t_2) + i$, and there is no $\sigma_2 \in (T - \{t_1\})^+$ (T is the set of transitions of N) such that $\sigma_1\sigma_2 \in L(B(N, t_1, t_2, k))$ and $\#(\sigma_1, t_1) = \#(\sigma_1\sigma_2, t_2)$; i.e., ($WB$ 2) is not satisfied (contradiction). Therefore t_7 is live in Test(i) for each i, $1 \leqslant i \leqslant k$.

(ii-2): If t_1 is live in Test(0), t_1 is live in $B(N, t_1 t_2, k)$ ((WB 1)). If t_4 is live in Test(0), t_3 never fires, so the number of tokens in p_0 is always less than or equal to k ((WB 3)). Now suppose that (WB 3) holds and (WB 2) does not hold. Then for some i, $1 \leqslant i \leqslant k$, there exists $\sigma_1 \in L(B(N, t_1, t_2, k))$ such that $\#(\sigma_1, t_1) = \#(\sigma_1, t_2) + i$ and there is no $\sigma_2 \in (T - \{t_1\})^+$ with $\sigma_1\sigma_2 \in L(B(N, t_1 t_2, k))$ and $\#(\sigma_1, t_1) = \#(\sigma_1\sigma_2, t_2)$. Then in Test(i), after simulating σ_1 by the copy of N and t_8 (since (WB 3) holds by assumption, Test(i) can always simulate N), t_5 can fire, and t_6 cannot fire after that, since t_2 cannot fire i times without t_1 firing. So t_7 is not live in Test(i). Therefore if (WB 3) holds and t_7 is live in Test(i) for each i, $1 \leqslant i \leqslant k$, then (WB 2) holds.

From (ii-1) and (ii-2), we see that N is k-WB with respect to t_1 and t_2 iff t_1 and t_4 are live in Test(0) and t_7 is live in Test(i) for all i, $1 \leqslant i \leqslant k$.

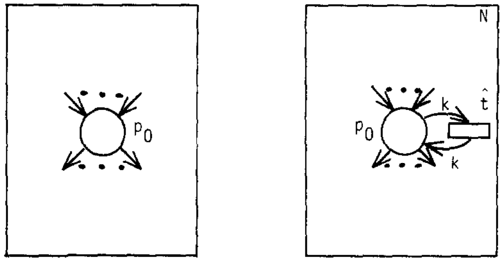
From (i) and (ii), we conclude that the decision problem for k-well-behavedness and the liveness problem are recursively equivalent. ∎

So k-well-behavedness is a decidable property as shown in the next corollary, since the reachability problem for Petri nets, which is equivalent [4] to the liveness problem, has recently been shown to be solvable [7, 8]. However, Theorem 21 tells us that the decision problem is computationally intractable in general.



(a) N (b) $\hat{N}$

FIG. 12. N and $\hat{N}$ in the proof of Theorem 23.



(a) N (b) $\hat{N}$

FIG. 13. N and $\hat{N}$ in the proof of Theorem 24.

COROLLARY 22. *The decision problem for k-well-behavedness is solvable.* ▮

THEOREM 23. *It is decidable whether a given Petri net satisfies Condition A.*

Proof. To test whether Petri net N satisfies Condition A with respect to transition t_0, construct another net $\hat{N}$ which is a copy of N except that t_0 is replaced by new transitions t_{01}, t_{02}, $\hat{t}$ and a place p_1 with no token initially, as shown in Fig. 12. Transition $\hat{t}$ can fire only when p_1 has at least k tokens. By a similar argument to the one in the proof of Theorem 20, we see that N satisfies Condition A with respect to t_0 iff $\hat{t}$ is live in $\hat{N}$. Then the theorem follows from the decidability of liveness. ▮

THEOREM 24. *It is decidable whether a given Petri net satisfies Condition B.*

Proof. To test whether Petri net N satisfies Condition B with respect to place p_0, construct another Petri net $\hat{N}$ as shown in Fig. 13. $\hat{N}$ is obtained by adding to N a new transition $\hat{t}$ that self-loops on p_0 with arcs weighted k. Clearly N satisfies Condition B with respect to p_0 iff $\hat{t}$ is live in $\hat{N}$. Then the theorem follows by the decidability of liveness. ▮

6. ILLUSTRATIVE EXAMPLES

In this section, two examples are given to illustrate our method of Petri net transformations. The first example shows that the marked graph transformations reported earlier in [5, 11] can be interpreted by our transformations of transitions. The second example illustrates the use of Theorems 10 and 11 for the liveness analysis of Petri nets.

6.1. EXAMPLE 5. Recently it has been shown [5, 11] that $\rho(G)$, the number of equivalence classes of live and safe markings where the equivalence is defined by the mutual reachability of two markings, is either invariant, or computed by a formula when certain transformations are applied to a marked graph G. So far, six types of

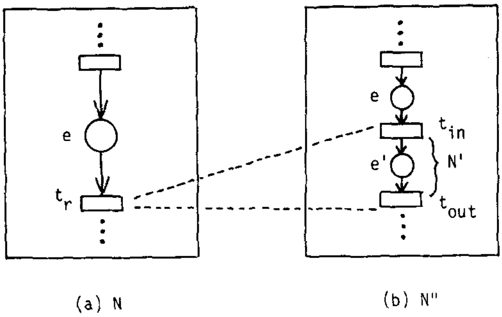


FIG. 14. Series expansion.

such transformations are known, and they are referred to as series, parallel, unique circuit, Y-V, separable graph, and unique path transformations. These transformations are useful for both analysis and synthesis of marked graphs. They can be used to synthesize decision-free concurrent systems, with the following prescribed properties: $\rho(G)$, liveness, safeness, performance, and resource requirements [6, 9]. The marked graph transformations can deal with more porperties than the Petri net transformations, but the former turn out to be special case of the latter as far as liveness and safeness are concerned. In the following, the first four of the six marked graph transformations are interpreted in terms of our transformations of transitions (interpretation of the last two transformations are found in [13]). Thus, it provides another way of proving that the marked graph transformations preserve liveness and safeness. The term "transformation", as used here, means both expansion (refinement) and reduction (abstraction), where the latter is the reverse operation of the former.

Note that since the marked graph transformations considered here are transformations among live and safe marked graphs, each transition t_r to be refined is not 2-enabled and Condition A is satisfied with $k = 1$.

(i) *Series Transformation.* Series expansion adds a place e' in series with a place e as is shown in Fig. 14(a) and (b). This transformation can be regarded as the refinement of transition t_r by subnet N' indicated in Fig. 14(b). Since N' is 1-well-behaved, N'' is live and safe by Corollary 9 and Theorem 11. Conversely, the series

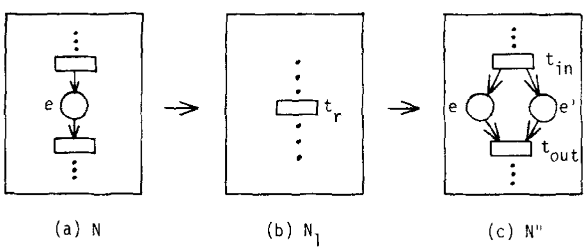


FIG. 15. Parallel expansion.

reduction from N'' to N preserves liveness and safeness by Corollary 7 and Theorem 10.

(ii) *Parallel Transformation.* Parellel expansion adds a place e' in parallel with a place e as is shown in Fig. 15(a) and (c). This expansion is regarded as a combination of two Petri net transformations from Fig. 15(a) to (b) and then from (b) to (c), where the former is an abstraction of a subnet to a transition t_r, and latter is a refinement of t_r.

(iii) *Y-V Transformation.* Figure 16 illustrates *Y-V* expansion. This is the same type of refinement of a transition t_r as the series expansion. Thus the same argument as in (i) applies.

(iv) *Unique Circuit Transformation.* When there exists a unique path p_{12} from a transition t_1 to another transition t_2, addition of a place having a token together with

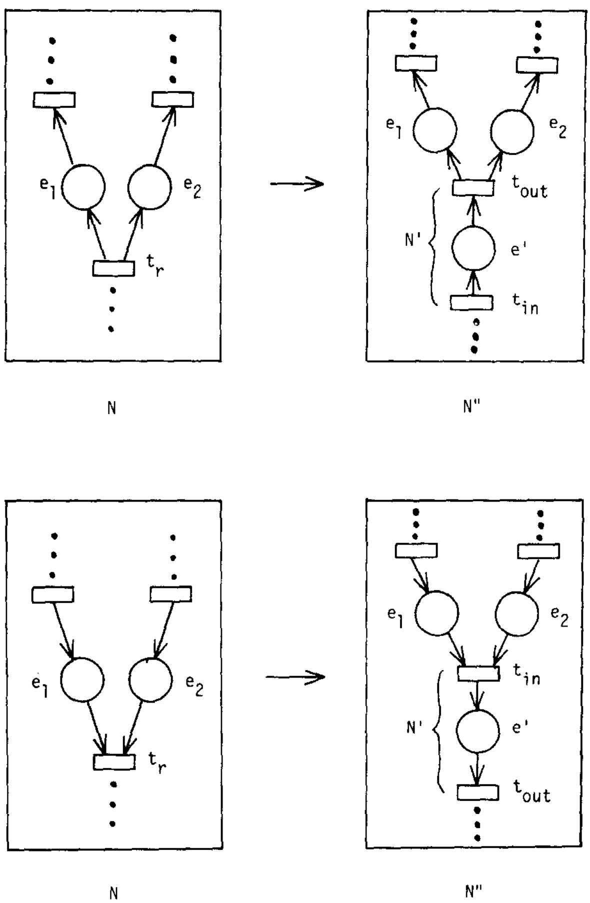


FIG. 16. *Y-V* expansion.

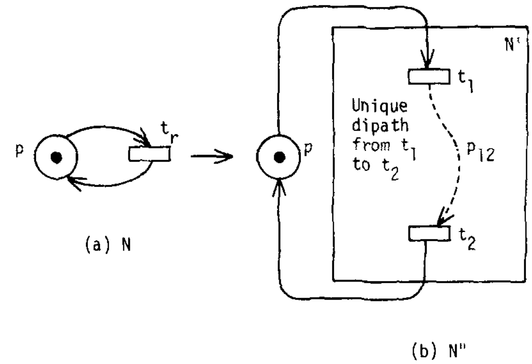


FIG. 17. Unique circuit expansion.

an arc from t_2 to p and another arc from p to t_1 creates a unique directed circuit containing p. This transformation is referred to as the unique circuit expansion. Before adding the place p with one token, each place on p_{12} must be made token-free (see Fig. 17(b)). In this case the given Petri net itself is 1-well-behaved with respect to t_1 and t_2, and thus this expansion can be viewed as the refinement of transition t_r in the live and safe Petri net N shown in Fig. 17(a) by the given Petri net.

Note that the subnet used in each of the Petri net transformations considered above is k-well-behaved, not only for $k=1$ but also for any integer $k \geqslant 2$. However, it is sufficient to use $k=1$ for live and safe Petri nets. In this respect, the stepwise refinement method of Valette [14] can be used to interpret the above marked graph transformations.

6.2. EXAMPLE 6. The Petri net N shown in Fig. 18 is a non-free choice and non-simple net (in the sense of Hack [4]). The net can be interpreted as a representation of a system consisting of one producer (subnet N_A) and two consumers (subnets N_B and N_C). The producer puts items in the buffer represented by place f. The two consumers can remove items from the buffer in a mutually exclusive manner. (Note that consumer N_B removes two items at a time.) The number of tokens in place e represents the size of the empty space in the buffer. The initial marking shown in Fig. 18 shows that the buffer is of size $n \geqslant 2$ and initially empty. Using the above net N, we illustrate how to apply Theorems 10 and 11 to divide the liveness analysis of a large Petri net into the analysis of smaller nets. First we reduce N to the net N_1 shown in Fig. 19 by substituting transitions t_A, t_B, and t_C for the subnets, N_A, N_B, and N_C shown in Fig. 18, respectively. It is easy to see that t_A is not $(n+1)$-enabled in N_1, and that t_B and t_C are not 2-enabled in N_1. Then by Theorem 11, the liveness analysis of N can be "divided" into the following analyses of the smaller nets N_1, N_A, N_B, and N_C. That is, N is live if the following statements (1) to (7) are true:

(1) N_1 is live.

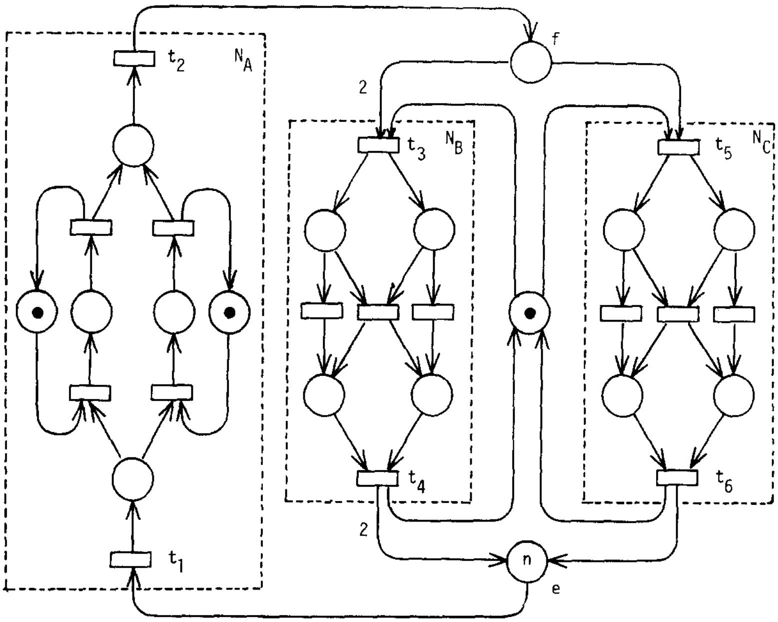


FIG. 18. N of Example 6.

(2) N_1 satisfies Condition A with respect to t_A and $k = n$.

(3) N_1 satisfies Condition A with respect to t_B and $k = 1$.

(4) N_1 satisfies Condition A with respect to t_C and $k = 1$.

(5) N_A is n-*WB* with respect to t_1 and t_2, and the net $B(N_A, t_1, t_2. n)$ is live.

(6) N_B is 1-*WB* with respect to t_3 and t_4, and the net $B(N_B, t_3, t_4, 1)$ is live.

(7) N_C is 1-*WB* with respect to t_5 and t_6, and the net $B(N_C, t_5, t_6, 1)$ is live.

For this particular example, the subnets are so small that the above statements (1) to (7) can be verified by inspection, and it is easy to see that N is live.

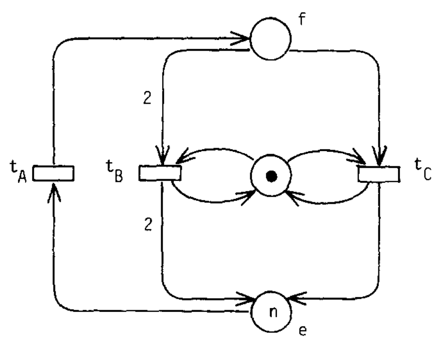


FIG. 19. N_1 of Example 6.

Now suppose that $n = 1$ in N_1 shown in Fig. 19, i.e., the size of the buffer is one. It is easy to see that t_B is dead, and that N_1 is not live. By Theorem 10, the net N shown in Fig. 18, which is a refinement of N_1, is not live.

7. Conclusion

A technique for the stepwise refinement and abstraction of Petri nets has been presented. The presented method is more general than those reported earlier in [5, 11, 14]. As was illustrated elsewhere [2, 6, 9–11, 14], the application of these methods is two-fold. First, the abstraction (reduction) technique can be used as a "divide-and-conquer" approach for the analysis of liveness, boundedness, resource requirements, etc., for large scale Petri nets. Second, the refinement (expansion) technique can be used as a top-down approach for growing (synthesizing) a Petri net model of a system from an abstract level to a desired level of detail. During this process of growth, it is possible to prescribe liveness, boundedness, resource requirements, etc. In this respect, the technique can serve a paradigm for writing "good" (deadlock-free and overflow-free) programs or design plans of concurrent systems in a top-down manner. However, the present transformation techniques can apply only to a limited topology (a subnet having a pair of t_{in} and t_{out}). Further study is suggested on transformations applicable to more general topologies.

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