Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Functional Analysis 238 (2006) 58–98
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Quantized Gromov–Hausdorff distance

Wei Wu

Department of Mathematics, East China Normal University, Shanghai 200062, PR China

Received 20 February 2005; accepted 20 February 2005
Available online 14 June 2006
Communicated by Alain Connes

Abstract

A quantized metric space is a matrix order unit space equipped with an operator space version of Rieffel's Lip-norm. We develop for quantized metric spaces an operator space version of quantum Gromov–Hausdorff distance. We show that two quantized metric spaces are completely isometric if and only if their quantized Gromov–Hausdorff distance is zero. We establish a completeness theorem. As applications, we show that a quantized metric space with 1-exact underlying matrix order unit space is a limit of matrix algebras with respect to quantized Gromov–Hausdorff distance, and that matrix algebras converge naturally to the sphere for quantized Gromov–Hausdorff distance.

Keywords: Quantized metric space; Matrix Lipschitz seminorm; Matrix seminorm; Matrix state space; Quantized Gromov–Hausdorff distance

1. Introduction

Following up the compact metric spaces given by Connes in connection with his theory of quantum Riemannian geometry defined by Dirac operators [3], Rieffel defined the notion of a compact quantum metric space (A, L_A) in [17] as an order unit space A equipped with a Lip-norm L_A, which is a generalization of the usual Lipschitz seminorm on functions which one associates to an ordinary metric. Many interesting examples of compact quantum metric space have been constructed [10,11,14,16]. Motivated by the type of convergence of spaces that has recently begun to play a central role in string theory, Rieffel introduces the quantum Gromov–Hausdorff distance for the compact quantum metric spaces as a quantum analogue of

E-mail address: wwu@math.ecnu.edu.cn.

0022-1236/$ – see front matter © 2006 Published by Elsevier Inc.
doi:10.1016/j.jfa.2005.02.017

Gromov–Hausdorff distance, and shows that the basic theorems of the classical theory have natural quantum analogues.

In [20,21], we formulated matrix Lipschitz seminorms on matrix order unit spaces. This operator space version of Lipschitz seminorm has many nice properties which are similar to those for ordinary metric spaces. These data may then be thought of as some "noncommutative metric spaces." So it is natural to ask, as does Rieffel in [17], if it is possible to develop a corresponding operator space version of quantum Gromov–Hausdorff distance. This is the aim of the present article.

In contrast to the matricial quantum Gromov–Hausdorff distance in [8] and operator Gromov–Hausdorff distance in [9], our quantized Gromov–Hausdorff distance operates entirely at the "matrix" level. Not only the matrix state spaces but also the matrix Lipschitz seminorms and the complete isometries are brought into our picture. This should be important in the background of operator systems.

The paper has eight sections. Section 2 contains preliminaries, mainly to fix some terminology and notation. In Section 3 we define quantized metric space and develop an operator "quotient." Section 4 defines our quantized Gromov–Hausdorff distance, and we prove that it satisfies the triangle inequality. Section 5 deals with the operator Gromov–Hausdorff distance zero. We establish that it implies a complete isometry. Section 6 treats the completeness theorem of the complete isometry classes of quantized metric spaces. In Section 7 we show that a quantized metric space with 1-exact underlying matrix order unit space is a limit of matrix algebras with respect to quantized Gromov–Hausdorff distance. It is established in Section 8 that matrix algebras converge naturally to the sphere for quantized Gromov–Hausdorff distance.

2. Preliminaries

All vector spaces are assumed to be complex throughout this paper. Given a vector space V, we let $M_{m,n}(V)$ denote the matrix space of all m by n matrices $v = [v_{ij}]$ with $v_{ij} \in V$, and we set $M_n(V) = M_{n,n}(V)$. If $V = \mathbb{C}$, we write $M_{m,n} = M_{m,n}(\mathbb{C})$ and $M_n = M_{n,n}(\mathbb{C})$, which means that we may identify $M_{m,n}(V)$ with the tensor product $M_{m,n} \otimes V$. We identify $M_{m,n}$ with the normed space $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$. We use the standard matrix multiplication and *-operation for compatible scalar matrices, and 1_n for the identity matrix in M_n.

There are two natural operations on the matrix spaces. For $v \in M_{m,n}(V)$ and $w \in M_{p,q}(V)$, the direct sum $v \oplus w \in M_{m+p,n+q}(V)$ is defined by letting

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix},$$

and if we are given $\alpha \in M_{m,p}$, $v \in M_{p,q}(V)$ and $\beta \in M_{q,n}$, the matrix product $\alpha v \beta \in M_{m,n}(V)$ is defined by

$$\alpha v \beta = \left[\sum_{k,l} \alpha_{ik} v_{kl} \beta_{lj} \right].$$

A **-vector space* V is a complex vector space together with a conjugate linear mapping $v \mapsto v^*$ such that $v^{**} = v$. A complex vector space V is said to be *matrix ordered* if:

(1) V is a *-vector space;
(2) each $M_n(V)$, $n \in \mathbb{N}$, is partially ordered;
(3) $\gamma^* M_n(V)^+ \gamma \subseteq M_m(V)^+$ if $\gamma = [\gamma_{ij}]$ is any $n \times m$ matrix of complex numbers.

A *matrix order unit space* $(\mathcal{V}, 1)$ is a matrix ordered space $\mathcal{V}$ together with a distinguished order unit 1 satisfying the following conditions:

(1) $\mathcal{V}^+$ is a proper cone with the order unit 1;
(2) each of the cones $M_n(\mathcal{V})^+$ is Archimedean.

Each matrix order unit space $(\mathcal{V}, 1)$ may be provided with the norm

$$\|v\| = \inf\left\{t \in \mathbb{R}\colon \begin{bmatrix} t1 & v \\ v^* & t1 \end{bmatrix} \geqslant 0\right\}.$$

As in [17], we will not assume that $\mathcal{V}$ is complete for the norm.

If V and W are *-vector spaces and $\varphi : V \mapsto W$ is a linear mapping, we have a linear mapping $\varphi^* : V \mapsto W$ defined by $\varphi^*(v) = \varphi(v^*)^*$.

Given vector spaces V and W and a linear mapping $\varphi : V \mapsto W$ and $n \in \mathbb{N}$, we have a corresponding $\varphi_n : M_n(V) \mapsto M_n(W)$ defined by

$$\varphi_n\big([v_{ij}]\big) = \big[\varphi(v_{ij})\big].$$

If V and W are vector spaces in duality, then they determine the matrix pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : M_n(V) \times M_m(W) \mapsto M_{nm},$$

where

$$\big\langle\!\big\langle [v_{ij}], [w_{kl}] \big\rangle\!\big\rangle = \big[\langle v_{ij}, w_{kl} \rangle\big]$$

for $[v_{ij}] \in M_n(V)$ and $[w_{kl}] \in M_m(W)$.

A *graded set* $\mathbf{S} = (S_n)$ is a sequence of sets $S_n (n \in \mathbb{N})$. If V is a locally convex topological vector space, then the canonical topology on $M_n(V)$ $(n \in \mathbb{N})$ is that determined by the natural linear isomorphism $M_n(V) \cong V^{n^2}$, that is, the product topology. A graded set $\mathbf{S} = (S_n)$ with $S_n \subseteq M_n(V)$ is *closed* or *compact* if that is the case for each set S_n in the product topology in $M_n(V)$. Given a vector space V, we say that a graded set $\mathbf{B} = (B_n)$ with $B_n \subseteq M_n(V)$ is *absolutely matrix convex* if for all $m, n \in \mathbb{N}$:

(1) $B_m \oplus B_n \subseteq B_{m+n}$;
(2) $\alpha B_m \beta \subseteq B_n$ for any contractions $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$.

A *matrix convex set* in V is a graded set $\mathbf{K} = (K_n)$ of subsets $K_n \subseteq M_n(V)$ such that

$$\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \in K_n$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, 2, \ldots, k$ satisfying $\sum_{i=1}^{k} \gamma_i^* \gamma_i = 1_n$. Let V and W be vector spaces in duality, and let $\mathbf{S} = (S_n)$ be a graded set with $S_n \subseteq M_n(V)$. The *absolute operator polar* $\mathbf{S}^{\circledcirc} = (S_n^{\circledcirc})$ with $S_n^{\circledcirc} \subseteq M_n(W)$, is defined by $S_n^{\circledcirc} = \{w \in M_n(W)\colon \|\langle\langle v, w\rangle\rangle\| \leqslant 1$ for all $v \in S_r, r \in \mathbb{N}\}$. The *matrix polar* $\mathbf{S}^{\pi} = (S_n^{\pi})$ with $S_n^{\pi} \subseteq M_n(W)$, is defined by $S_n^{\pi} = \{w \in M_n(W)\colon \operatorname{Re}\langle\langle v, w\rangle\rangle \leqslant 1_{r\times n}$ for all $v \in S_r, r \in \mathbb{N}\}$. Given a subset $S \subseteq V$, the *absolute polar* of S is defined by $S^{\circ} = \{w \in W\colon |\langle v, w\rangle| \leqslant 1$ for all $v \in S\}$.

A *gauge* on a vector space V is a function $g : V \mapsto [0, +\infty]$ such that:

(1) $g(v + w) \leqslant g(v) + g(w)$;
(2) $g(\alpha v) \leqslant |\alpha| g(v)$

for all $v, w \in V$ and $\alpha \in \mathbb{C}$. We say that a gauge g is a *seminorm* on V if $g(v) < +\infty$ for all $v \in V$. Given an arbitrary vector space V, a *matrix gauge* $\mathcal{G} = (g_n)$ on V is a sequence of gauges

$$g_n : M_n(V) \mapsto [0, +\infty]$$

such that:

(1) $g_{m+n}(v \oplus w) = \max\{g_m(v),\ g_n(w)\}$;
(2) $g_n(\alpha v \beta) \leqslant \|\alpha\| g_m(v) \|\beta\|$

for any $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$. A matrix gauge $\mathcal{G} = (g_n)$ is a *matrix seminorm* on V if for any $n \in \mathbb{N}$, $g_n(v) < +\infty$ for all $v \in M_n(V)$. If each g_n is a norm on $M_n(V)$, we say that $\mathcal{G}$ is a *matrix norm*. An *operator space* is a vector space together with a matrix norm on it. For a matrix order unit space $(\mathcal{V}, 1)$, it is an operator space with the matrix norm determined by the matrix order on it.

3. Quantized metric space

First we recall the following definitions given in [20,21].

Definition 3.1. Given a matrix order unit space $(\mathcal{V}, 1)$, a *matrix Lipschitz seminorm* $\mathcal{L}$ on $(\mathcal{V}, 1)$ is a sequence of seminorms

$$L_n : M_n(\mathcal{V}) \mapsto [0, +\infty)$$

such that:

(1) the null space of each L_n is $M_n(\mathbb{C}1)$;
(2) $L_{m+n}(v \oplus w) = \max\{L_m(v), L_n(w)\}$;
(3) $L_n(\alpha v \beta) \leqslant \|\alpha\| L_m(v) \|\beta\|$;
(4) $L_m(v^*) = L_m(v)$

for any $v \in M_m(\mathcal{V})$, $w \in M_n(\mathcal{V})$, $\alpha \in M_{n,m}$ and $\beta \in M_{m,n}$.

Let $(\mathcal{V}, 1)$ be a matrix order unit space. The *matrix state space* of $(\mathcal{V}, 1)$ is the collection $\mathcal{CS}(\mathcal{V}) = (CS_n(\mathcal{V}))$ of *matrix states*

$$CS_n(\mathcal{V}) = \{\varphi\colon \varphi \text{ is a unital completely positive linear mapping from } \mathcal{V} \text{ into } M_n\}.$$

If $\mathcal{L} = (L_n)$ is a matrix Lipschitz seminorm on $(\mathcal{V}, 1)$, we have a collection $\mathcal{D}_{\mathcal{L}} = (D_{L_n})$ of metrics on $\mathcal{CS}(\mathcal{V})$ given by

$$D_{L_n}(\varphi, \psi) = \sup\bigl\{\bigl\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\bigr\|\colon a \in M_r(\mathcal{V}),\ L_r(a) \leqslant 1,\ r \in \mathbb{N}\bigr\}$$

for $\varphi, \psi \in CS_n(\mathcal{V})$ (notice that it may take value $+\infty$). And in turn we obtain a sequence $\mathcal{L}_{\mathcal{D}_{\mathcal{L}}} = (L_{D_{L_n}})$ of gauges on $(\mathcal{V}, 1)$ by

$$L_{D_{L_n}}(a) = \sup\biggl\{\frac{\|\langle\langle\varphi, a\rangle\rangle - \langle\langle\psi, a\rangle\rangle\|}{D_{L_r}(\varphi, \psi)}\colon \varphi, \psi \in CS_r(\mathcal{V}),\ \varphi \neq \psi,\ r \in \mathbb{N}\biggr\}$$

for all $a \in M_n(\mathcal{V})$.

Definition 3.2. Let $(\mathcal{V}, 1)$ be a matrix order unit space. By a *matrix Lip-norm* on $(\mathcal{V}, 1)$ we mean a matrix Lipschitz seminorm $\mathcal{L} = (L_n)$ on $(\mathcal{V}, 1)$ such that the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V})$ agrees with the BW-topology.

We are now prepared to make:

Definition 3.3. By a *quantized metric space* we mean a pair $(\mathcal{V}, \mathcal{L})$ consisting of a matrix order unit space $(\mathcal{V}, 1)$ with a matrix Lip-norm $\mathcal{L}$ defined on it.

Example 3.4. Let (X, ρ) be an ordinary compact metric space, let $\mathcal{A}$ denote the set of Lipschitz functions on X, and let L_ρ denote the Lipschitz seminorm on $\mathcal{A}$. Then $\mathcal{A} \subseteq C(X)$, and for $f, g \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, we have

$$L_\rho(f^*) = L_\rho(f), \qquad L_\rho(\alpha f) = |\alpha| L_\rho(f), \qquad L_\rho(f + g) \leqslant L_\rho(f) + L_\rho(g).$$

Thus $\mathcal{A}$ is a self-adjoint linear subspace of $C(X)$ which contains constant functions, and so $\mathcal{A}$ is a matrix order unit space by Theorem 4.4 in [2].

Since L_ρ is lower semicontinuous, $K = \{f \in \mathcal{A}\colon L_\rho(f) \leqslant 1\}$ is an absolutely convex normed-closed (and hence is weakly closed) set in $\mathcal{A}$. K determines a graded set

$$K_n = \begin{cases} K, & \text{if } n = 1, \\ \{0\}, & \text{if } n > 1. \end{cases}$$

The minimal envelope $\hat{\mathcal{K}}$ of K is the matrix bipolar $\mathcal{K}^{\circledcirc\circledcirc}$ of $\mathcal{K}$. $\hat{\mathcal{K}}$ is an absolutely matrix convex weakly closed graded set. We let $\hat{\mathcal{L}} = (\hat{L}_n)$ be the corresponding matrix gauge of $\hat{\mathcal{K}}$. Since $\hat{L}_1 = L_\rho$ is a Lipschitz seminorm, $\hat{\mathcal{L}}$ is a matrix Lipschitz seminorm. $\rho_{L_\rho} = \rho$ implies that $\hat{\mathcal{L}}$ is also a matrix Lip-norm (see [14, Theorem 1.9] and [20, Proposition 7.5]). Therefore, $(\mathcal{A}, \hat{\mathcal{L}})$ is a quantized metric space. It is called the *minimal quantized metric space* of (X, ρ). The maximal envelope $\check{\mathcal{K}}$ of K is the matrix polar $(\mathcal{K}^\circ)^{\circledcirc}$ of $\mathcal{K}^\circ = (K_n^\circ)$, where

$$K_n^\circ = \begin{cases} K^\circ, & \text{if } n = 1, \\ \{0\}, & \text{if } n > 1. \end{cases}$$

Similarly, $\check{\mathcal{K}}$ is an absolutely matrix convex weakly closed graded set, and the corresponding matrix gauge $\check{\mathcal{L}}$ of $\check{\mathcal{K}}$ makes $\mathcal{A}$ into a quantized metric space. $(\mathcal{A}, \check{\mathcal{L}})$ is called the *maximal quantized metric space* of (X, ρ). Moreover, if $\mathcal{C} = (C_n)$ is an absolutely matrix convex weakly closed graded set with $C_1 = K$, then

$$\hat{\mathcal{K}} \subseteq \mathcal{C} \subseteq \check{\mathcal{K}},$$

and the corresponding matrix gauge $\mathcal{L} = (L_n)$ satisfies

$$\check{L}_n \leqslant L_n \leqslant \hat{L}_n, \quad n \in \mathbb{N}$$

(see [4, p. 181]). So $(\mathcal{A}, \mathcal{L})$ is a quantized metric space. It is called a quantized metric space of (X, ρ).

Example 3.5. Let (A, L) be a compact quantum metric space, that is, an order unit space (A, e) equipped with a seminorm L, called Lip-norm, on A such that $L(a) = 0$ if and only if $a \in \mathbb{R}e$, and the topology on the state space $S(A)$ of A from the metric

$$\rho_L(\mu, \nu) = \sup\bigl\{\bigl|\mu(a) - \nu(a)\bigr|:\ L(a) \leqslant 1\bigr\}$$

is the w^*-topology (see [17, Definition 2.2]). So $(S(A), \rho_L)$ is an ordinary compact metric space. Let $\mathcal{A}$ denote the set of Lipschitz functions on $S(A)$. By Example 3.4, there exists a quantized metric space structure $(\mathcal{A}, \mathcal{L}_1)$ of $(S(A), \rho_L)$, where $\mathcal{L}_1 = (L_{1,n})$. From Lemma 3.2 in [15], $A \subseteq \mathcal{A}$ and $L_{1,1}(a) \leqslant L(a)$ for $a \in A$. Let $\|\cdot\| = (\|\cdot\|_n)$ be the matrix norm determined by the matrix order on $(\mathcal{A}, 1)$. By the basic representation theorem of Kadison [7], we also have that $\|a\| = \|a\|_1$ for $a \in A$. If L is lower semicontinuous, the embedding of A into $\mathcal{A}$ is *isometric*, that is, $\|a\| = \|a\|_1$ and $L(a) = L_{1,1}(a)$ for all $a \in A$ [15, Theorem 4.1].

Set

$$\mathcal{V} = \mathcal{A} \cap (A + iA).$$

We denote the restriction of $\mathcal{L}_1$ on $\mathcal{V}$ by $\mathcal{L} = (L_n)$. Then $\mathcal{V}$ is a self-adjoint linear subspace of $\mathcal{A}$ and contains the order unit of $\mathcal{A}$. So $\mathcal{V}$ is a matrix order unit space. Because the $\mathcal{D}_{\mathcal{L}_1}$-topology on $\mathcal{CS}(\mathcal{A})$ agrees with the BW-topology, the image of $L_{1,1}^1 = \{a \in \mathcal{A}:\ L_{1,1}(a) \leqslant 1\}$ in $\tilde{\mathcal{A}} = \mathcal{A}/\mathbb{C}1$ is totally bounded for $\|\cdot\|_1^\sim$ [21, Theorem 5.3]. Since $L_1^1 \subseteq L_{1,1}^1$, the image of L_1^1 in $\tilde{\mathcal{V}} \subseteq \tilde{\mathcal{A}}$ is totally bounded for $\|\cdot\|_1^\sim$, and so [21, Theorem 5.3], the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V})$ is the BW-topology. Therefore, $(\mathcal{V}, \mathcal{L})$ is a quantized metric space, and the embedding of (A, L) into $(\mathcal{V}, \mathcal{L})$ is an isometry if L is lower semicontinuous.

Let $(\mathcal{V}, 1)$ and $(\mathcal{W}, 1)$ be matrix order unit spaces, and let $\varphi : \mathcal{V} \mapsto \mathcal{W}$ be a unital completely positive linear mapping. Then we have the dual mapping $\varphi' : \mathcal{W}^* \mapsto \mathcal{V}^*$ determined by

$\varphi'(f)(v) = f(\varphi(v))$. Let $\tilde{\mathcal{V}}^*$ denote the dual space of $\tilde{\mathcal{V}} = \mathcal{V}/(\mathbb{C}1)$. $\tilde{\mathcal{V}}^*$ is just the subspace of $\mathcal{V}^*$ consisting of those $f \in \mathcal{V}^*$ such that $f(1) = 0$. For any $v \in M_n(\mathcal{V})$ and $g \in M_m(\mathcal{W}^*)$, we have

$$\langle\langle g, \varphi_n(v)\rangle\rangle = \left[g_{kl}\big(\varphi(v_{ij})\big)\right] = \left[\big(\varphi'(g_{kl})\big)(v_{ij})\right] = \langle\langle (\varphi')_m(g), v\rangle\rangle.$$

So $\|\varphi\|_{cb} = 1$, $(\varphi')_m(M_m(\tilde{\mathcal{W}}^*)) \subseteq M_m(\tilde{\mathcal{V}}^*)$ and $(\varphi')_m(CS_m(\mathcal{W})) \subseteq CS_m(\mathcal{V})$. Moreover, φ' is w^*-continuous. Let $\varphi_n^c = (\varphi')_n|_{CS_n(\mathcal{W})}$ for $n \in \mathbb{N}$. Then for $v \in M_n(\mathcal{V})$, $f_i \in CS_{n_i}(\mathcal{W})$ and $\gamma_i \in M_{n_i,m}$ satisfying $\sum_{i=1}^k \gamma_i^* \gamma_i = 1_m$, we have

$$\begin{aligned}\left\langle\left\langle \varphi_m^c\left(\sum_{i=1}^k \gamma_i^* f_i \gamma_i\right), v\right\rangle\right\rangle &= \left\langle\left\langle \sum_{i=1}^k \gamma_i^* f_i \gamma_i, \varphi_n(v)\right\rangle\right\rangle = \sum_{i=1}^k (\gamma_i \otimes 1_n)^* \langle\langle f_i, \varphi_n(v)\rangle\rangle (\gamma_i \otimes 1_n)\\ &= \sum_{i=1}^k (\gamma_i \otimes 1_n)^* \langle\langle \varphi_{n_i}^c(f_i), v\rangle\rangle (\gamma_i \otimes 1_n)\\ &= \left\langle\left\langle \sum_{i=1}^k \gamma_i^* \varphi_{n_i}^c(f_i)\gamma_i, v\right\rangle\right\rangle.\end{aligned}$$

So $\varphi^c = (\varphi_n^c)$ is a BW-continuous matrix affine mapping of $\mathcal{CS}(\mathcal{W})$ into $\mathcal{CS}(\mathcal{V})$. In particular, $\varphi^c(\mathcal{CS}(\mathcal{W})) = (\varphi_n^c(CS_n(\mathcal{W})))$ is a closed matrix convex subset of $\mathcal{CS}(\mathcal{V})$. Clearly φ_n^c is injective if φ is surjective.

Let $\mathcal{L}$ be a matrix Lipschitz seminorm on $\mathcal{V}$. On $\mathcal{V}^*$, we define the matrix gauge $\mathcal{L}' = (L_n')$ by

$$L_n'(f) = \sup\left\{\|\langle\langle f, a\rangle\rangle\|:\ a \in L_r^1,\ r \in \mathbb{N}\right\}, \quad f \in M_n\big(\mathcal{V}^*\big).$$

Then $L_n'(\varphi - \psi) = D_{L_n}(\varphi, \psi)$ for $\varphi, \psi \in CS_n(\mathcal{V})$ [20, Lemma 4.3].

Proposition 3.6. *Let* $(\mathcal{V}, 1)$ *and* $(\mathcal{W}, 1)$ *be matrix order unit spaces, and let* $\varphi : \mathcal{V} \mapsto \mathcal{W}$ *be a unital completely positive linear mapping which is surjective. Let* $\mathcal{L}$ *be a matrix Lipschitz seminorm on* $\mathcal{V}$*, and let* $\mathcal{L}_{\mathcal{W}} = (L_{\mathcal{W},n})$ *be a sequence of the corresponding quotient seminorms on* $\mathcal{W}$*, defined by*

$$L_{\mathcal{W},n}(b) = \inf\left\{L_n(a):\ \varphi_n(a) = b\right\}, \quad b \in M_n(\mathcal{W}).$$

Then

(1) $\mathcal{L}_{\mathcal{W}}$ *is a matrix Lipschitz seminorm on* $\mathcal{W}$;
(2) φ' *is a complete isometry for the matrix norms* $\mathcal{L}'_{\mathcal{W}}$ *and* $\mathcal{L}'$ *on* $\tilde{\mathcal{W}}^*$ *and* $\tilde{\mathcal{V}}^*$;
(3) φ^c *is a complete isometry for the corresponding matrix metrics* $\mathcal{D}_{\mathcal{L}_{\mathcal{W}}}$ *and* $\mathcal{D}_{\mathcal{L}}$;
(4) *If* $\mathcal{L}$ *is a matrix Lip-norm, then so is* $\mathcal{L}_{\mathcal{W}}$.

Proof. (1) For $b_1 \in M_m(\mathcal{W})$, $b_2 \in M_n(\mathcal{W})$, we have

$$\begin{aligned}L_{\mathcal{W},m+n}(b_1 \oplus b_2) &= \inf\left\{L_{m+n}(a):\ \varphi_{m+n}(a) = b_1 \oplus b_2\right\}\\ &\leqslant \inf\left\{L_{m+n}(a_1 \oplus a_2):\ \varphi_{m+n}(a_1 \oplus a_2) = b_1 \oplus b_2\right\}\end{aligned}$$

$$\begin{aligned}&= \inf\bigl\{\max\bigl\{L_m(a_1), L_n(a_2)\bigr\}\colon\ \varphi_m(a_1) = b_1,\ \varphi_n(a_2) = b_2\bigr\}\\&= \max\bigl\{\inf\bigl\{L_m(a_1)\colon\ \varphi_m(a_1) = b_1\bigr\}, \inf\bigl\{L_n(a_2)\colon\ \varphi_n(a_2) = b_2\bigr\}\bigr\}\\&= \max\bigl\{L_{\mathcal{W},m}(b_1), L_{\mathcal{W},n}(b_2)\bigr\}.\end{aligned}$$

If $\alpha \in M_{m,n}$, $\beta \in M_{n,m}$ and $b \in M_n(\mathcal{W})$, we have

$$\begin{aligned}L_{\mathcal{W},m}(\alpha b\beta) &= \inf\bigl\{L_m(a)\colon\ \varphi_m(a) = \alpha b\beta\bigr\} \leqslant \inf\bigl\{L_m(\alpha a\beta)\colon\ \varphi_n(a) = b\bigr\}\\&\leqslant \|\alpha\|\|\beta\| \inf\bigl\{L_n(a)\colon\ \varphi_n(a) = b\bigr\} = \|\alpha\|\|\beta\| L_{\mathcal{W},n}(b)\end{aligned}$$

and

$$\begin{aligned}L_{\mathcal{W},n}\bigl(b^*\bigr) &= \inf\bigl\{L_n(a)\colon\ \varphi_n(a) = b^*\bigr\} = \inf\bigl\{L_n(a)\colon\ \varphi_n\bigl(a^*\bigr) = b\bigr\}\\&= \inf\bigl\{L_n\bigl(a^*\bigr)\colon\ \varphi_n(a) = b\bigr\} = \inf\bigl\{L_n(a)\colon\ \varphi_n(a) = b\bigr\} = L_{\mathcal{W},n}(b).\end{aligned}$$

Given $[\lambda_{ij}] \in M_n$. We have

$$L_{\mathcal{W},n}\bigl([\lambda_{ij}1]\bigr) = \inf\bigl\{L_n(a)\colon\ \varphi_n(a) = [\lambda_{ij}1]\bigr\} \leqslant L_n\bigl([\lambda_{ij}1]\bigr) = 0$$

and so $L_{\mathcal{W},n}([\lambda_{ij}1]) = 0$. If $b = [b_{ij}] \in M_n(\mathcal{W})$ with $L_{\mathcal{W},n}(b) = 0$, then

$$L_{\mathcal{W},1}(b_{ij}) = 0, \quad i, j = 1, 2, \ldots, n.$$

Letting $b_{ij} = c_{ij} + id_{ij}$ $(i, j = 1, 2, \ldots, n)$, where $c_{ij}^* = c_{ij}$, $d_{ij}^* = d_{ij}$, we get

$$L_{\mathcal{W},1}(c_{ij}) = 0, \quad L_{\mathcal{W},1}(d_{ij}) = 0, \quad i, j = 1, 2, \ldots, n.$$

Since $L_{\mathcal{W},1}(c_{ij}) = \inf\{L_1(a)\colon\ \varphi_1(a) = c_{ij}\}$ and φ is positive, we have

$$L_{\mathcal{W},1}(c_{ij}) = \inf\bigl\{L_1(a)\colon\ \varphi_1(a) = c_{ij},\ a = a^*\bigr\}.$$

Now by Proposition 3.1 in [17], there exists an $\alpha_{ij} \in \mathbb{R}$ such that $c_{ij} = \alpha_{ij}1$. Similarly, there exists a $\beta_{ij} \in \mathbb{R}$ such that $d_{ij} = \beta_{ij}1$. Therefore, $b \in M_n(\mathbb{C}1)$. Thus $\mathcal{L}_{\mathcal{W}}$ is a matrix Lipschitz seminorm on $\mathcal{W}$.

(2) Let $f \in M_m(\tilde{\mathcal{W}}^*)$. For any $a \in M_n(\mathcal{V})$ we clearly have $L_{\mathcal{W},n}(\varphi_n(a)) \leqslant L_n(a)$, and so if $L_n(a) \leqslant 1$ we have

$$\bigl\|\bigl\langle\!\bigl\langle(\varphi')_m(f), a\bigr\rangle\!\bigr\rangle\bigr\| = \bigl\|\bigl\langle\!\bigl\langle f, \varphi_n(a)\bigr\rangle\!\bigr\rangle\bigr\| \leqslant L'_{\mathcal{W},m}(f).$$

Consequently, $L'_m((\varphi')_m(f)) \leqslant L'_{\mathcal{W},m}(f)$. But let $\delta > 0$ be given, and let $b \in M_n(\mathcal{W})$ with $L_{\mathcal{W},m}(b) \leqslant 1$. Then there is an $a \in M_n(\mathcal{V})$ such that $\varphi_n(a) = b$ and $L_n(a) \leqslant 1 + \delta$. Thus, $L_n(a/(1+\delta)) \leqslant 1$. Consequently,

$$L'_m\bigl((\varphi')_m(f)\bigr) \geqslant \bigl\|\bigl\langle\!\bigl\langle(\varphi')_m(f), a/(1+\delta)\bigr\rangle\!\bigr\rangle\bigr\| = \bigl\|\bigl\langle\!\bigl\langle f, \varphi_n(a)\bigr\rangle\!\bigr\rangle\bigr\|/(1+\delta) = \bigl\|\langle\!\langle f, b\rangle\!\rangle\bigr\|/(1+\delta).$$

Taking the supremum over $b \in M_n(\mathcal{W})$ with $L_{\mathcal{W},n}(b) \leqslant 1$, we see that

$$L'_m\big((\varphi')_m(f)\big) \geqslant L'_{\mathcal{W},m}(f)/(1+\delta).$$

Since δ is arbitrary, we obtain that $L'_m((\varphi')_m(f)) \geqslant L'_{\mathcal{W},m}(f)$. Thus φ' is a complete isometry.

(3) By (2), we have

$$D_{L_{\mathcal{W},n}}(\phi,\psi) = L'_{\mathcal{W},n}(\phi-\psi) = L'_n\big((\varphi')_n(\phi-\psi)\big) = D_{L_n}\big(\varphi^c_n(\phi),\varphi^c_n(\psi)\big),$$

where $\phi, \psi \in CS_n(\mathcal{W})$, that is, φ^c is a complete isometry for the corresponding matrix metrics $\mathcal{D}_{\mathcal{L}_\mathcal{W}}$ and $\mathcal{D}_\mathcal{L}$.

(4) Suppose that $\mathcal{L}$ is a matrix Lip-norm. Since φ' is w^*-continuous, φ is surjective, and $CS_n(\mathcal{V})$ is BW-compact, $(\varphi')_n$ is a homeomorphism of $CS_n(\mathcal{W})$ onto $(\varphi')_n(CS_n(\mathcal{W})) \subseteq CS_n(\mathcal{V})$. Because D_{L_n} gives the BW-topology on $CS_n(\mathcal{V})$, $D_{L_n}|_{(\varphi')_n(CS_n(\mathcal{W}))}$ gives the relative topology of $(\varphi')_n(CS_n(\mathcal{W}))$. According to (3), $D_{L_{\mathcal{W},n}}$ gives the BW-topology on $CS_n(\mathcal{W})$. Therefore, $\mathcal{L}_\mathcal{W}$ is a matrix Lip-norm. □

Notation 3.7. Under the conditions of Propositions 3.6 we will say that $\mathcal{L}$ induces $\mathcal{L}_\mathcal{W}$ via φ.

For a matrix convex set $\mathbf{K}$ in a locally convex vector space, let $A(\mathbf{K})$ be the set of all matrix affine mappings from $\mathbf{K}$ to $\mathbb{C}$ (see [20, Section 6]). On the other hand, we have

Proposition 3.8. *Let $(\mathcal{V}, 1)$ be a matrix order unit space, and let $\mathcal{K} = (K_n)$ be a compact matrix convex subset of $\mathcal{CS}(\mathcal{V})$. View the elements of $\mathcal{V}$ as matrix affine mapping from $\mathcal{CS}(\mathcal{V})$ to $\mathbb{C}$* [20, *Proposition* 6.1], *and let $\mathcal{W}$ consists of their restrictions to $\mathcal{K}$, with ϕ the restriction mapping of $\mathcal{V}$ onto $\mathcal{W}$. Then $(\mathcal{W}, \phi(1))$ is a matrix order unit space, and $\mathcal{K} = \phi^c(\mathcal{CS}(\mathcal{W}))$.*

Proof. Clearly, with the natural matrix order structure on $\mathcal{W}$ and the order unit $\phi(1)$, $(\mathcal{W}, \phi(1))$ is a matrix order unit space.

For $\varphi \in K_n \subseteq CS_n(\mathcal{V})$, we define the mapping $\psi : \mathcal{W} \mapsto M_n$ by $\psi(\phi(v)) = \varphi(v)$. Then $\psi \in CS_n(\mathcal{W})$ and $(\phi^c_n(\psi))(v) = \psi(\phi(v)) = \varphi(v)$ for $v \in \mathcal{V}$. Thus $K_n \subseteq \phi^c_n(CS_n(\mathcal{W}))$.

Suppose that $\varphi_0 \in CS_n(\mathcal{V})$ and $\varphi_0 \notin K_n$. By Theorem 1.6 in [19], there is a $v = [v_{ij}] \in M_n(\mathcal{V})$ and a self-adjoint $\alpha = [\alpha_{ij}] \in M_n$ such that

$$\mathrm{Re}\langle\langle \varphi, v\rangle\rangle \leqslant \alpha \otimes 1_r$$

for all $r \in \mathbb{N}$, $\varphi \in K_r$, and

$$\mathrm{Re}\langle\langle \varphi_0, v\rangle\rangle \nleqslant \alpha \otimes 1_n.$$

So we obtain $\varphi_n(\mathrm{Re}[\alpha_{ij}1 - v_{ij}]) \geqslant 0$ for all $r \in \mathbb{N}$ and $\varphi \in K_r$. Thus $\phi_n(\mathrm{Re}[\alpha_{ij}1 - v_{ij}]) \geqslant 0$ in $\mathcal{W}$. If $\varphi_0 = \phi^c_n(\psi_0)$ for some $\psi_0 \in CS_n(\mathcal{W})$, we would then have that $\mathrm{Re}\langle\langle \varphi_0, v\rangle\rangle = \mathrm{Re}\langle\langle \psi_0, \phi_n(v)\rangle\rangle = \alpha \otimes 1_n - \mathrm{Re}\langle\langle \psi_0, \phi_n([\alpha_{ij}1 - v_{ij}])\rangle\rangle = \alpha \otimes 1_n - \langle\langle \psi_0, \phi_n(\mathrm{Re}[\alpha_{ij}1 - v_{ij}])\rangle\rangle \leqslant \alpha \otimes 1_n$. Thus, $\varphi_0 \notin \phi^c(CS_n(\mathcal{W}))$. Therefore, $\mathcal{K} = \phi^c(\mathcal{CS}(\mathcal{W}))$. □

Notation 3.9. We will call the matrix order unit space $(\mathcal{W}, \phi(1))$ in the Proposition 3.8 the *quotient* of $(\mathcal{V}, 1)$ with respect to $\mathcal{K}$, and will identify $\mathcal{CS}(\mathcal{W})$ with $\mathcal{K}$. When $(\mathcal{V}, \mathcal{L})$ is a quantized metric space, $(\mathcal{W}, \mathcal{L}_{\mathcal{W}})$ is a quantized metric space by Proposition 3.6. $(\mathcal{W}, \mathcal{L}_{\mathcal{W}})$ is called the *quotient space* of $(\mathcal{V}, \mathcal{L})$ with respect to $\mathcal{K}$ and ϕ.

Proposition 3.10. *Let $(\mathcal{V}_1, 1)$, $(\mathcal{V}_2, 1)$ and $(\mathcal{V}_3, 1)$ be matrix order unit spaces. Suppose that $\varphi : \mathcal{V}_1 \mapsto \mathcal{V}_2$ and $\psi : \mathcal{V}_2 \mapsto \mathcal{V}_3$ are unital completely positive linear mappings which are surjective. Denote $\phi = \psi \circ \varphi$. If $\mathcal{L}$ is a matrix Lipschitz seminorm on $\mathcal{V}_1$, $\mathcal{L}_{\mathcal{V}_2}$ and $\mathcal{L}_{\mathcal{V}_3}$ are the induced matrix Lipschitz seminorms of $\mathcal{L}$ via φ and ϕ, respectively, and $\mathcal{L}_{\mathcal{V}_{23}}$ is the induced matrix Lipschitz seminorm of $\mathcal{L}_{\mathcal{V}_2}$ via ψ, then $\mathcal{L}_{\mathcal{V}_{23}} = \mathcal{L}_{\mathcal{V}_3}$.*

Proof. This follows by exactly the same argument used for quantum Gromov–Hausdorff distance in [17]. □

4. Quantized Gromov–Hausdorff distance

As in the situation of compact quantum metric spaces, we need a corresponding notion of bridge for estimating distance between quantized metric spaces.

Let $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$ be two quantized metric spaces with the matrix norms $\|\cdot\|_1 = (\|\cdot\|_{1,n})$ and $\|\cdot\|_2 = (\|\cdot\|_{2,n})$ determined by their matrix orders on $(\mathcal{V}_1, 1)$ and $(\mathcal{V}_2, 1)$, respectively. We form the direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ of operator spaces (see [13, Section 2.6]). $(\mathcal{V}_1 \oplus \mathcal{V}_2, (1, 1))$ becomes a matrix order unit space.

Definition 4.1. Let $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$ be quantized metric spaces. A *matrix bridge* between $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$ is a matrix seminorm $\mathcal{N}$ on $\mathcal{V}_1 \oplus \mathcal{V}_2$ such that:

(1) $\mathcal{N}$ is matrix continuous for the matrix norm $\|\cdot\|$ on $\mathcal{V}_1 \oplus \mathcal{V}_2$, that is, each N_n is continuous for $\|\cdot\|_n$ on $M_n(\mathcal{V}_1 \oplus \mathcal{V}_2)$;
(2) $N_n((a, b)^*) = N_n(a, b)$ for $a \in M_n(\mathcal{V}_1)$ and $b \in M_n(\mathcal{V}_2)$ and $n \in \mathbb{N}$;
(3) $N_1(1, 1) = 0$ but $N_1(1, 0) \neq 0$;
(4) for any $n \in \mathbb{N}$, $a \in M_n(\mathcal{V}_1)$ and $\epsilon > 0$, there is a $b \in M_n(\mathcal{V}_2)$ such that

$$\max\{L_{2,n}(b), N_n(a, b)\} \leqslant L_{1,n}(a) + \epsilon,$$

and similarly for $\mathcal{V}_1$ and $\mathcal{V}_2$ interchanged.

Example 4.2. Suppose $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$ are quantized metric spaces. Choose $\varphi_1 \in CS_1(\mathcal{V}_1)$ and $\psi_1 \in CS_1(\mathcal{V}_2)$. For $n \in \mathbb{N}$, we define $N_n : M_n(\mathcal{V}_1 \oplus \mathcal{V}_2) \mapsto [0, +\infty)$ by

$$N_n(a, b) = \big\| \langle\langle \varphi_1, a \rangle\rangle - \langle\langle \psi_1, b \rangle\rangle \big\|.$$

Then $\mathcal{N} = (N_n)$ is a matrix seminorm on $\mathcal{V}_1 \oplus \mathcal{V}_2$, and satisfies the conditions (1)–(3) of Definition 4.1. For any $a \in M_n(\mathcal{V}_1)$ and $\epsilon > 0$, choose $b = [\varphi_1(a_{ij})1] \in M_n(\mathcal{V}_2)$. Then we have

$$\max\{L_{2,n}(b), N_n(a, b)\} = 0 \leqslant L_{1,n}(a) + \epsilon,$$

and similarly if we are given $b \in M_n(\mathcal{V}_2)$. So $\mathcal{N}$ is a matrix bridge between $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$.

Proposition 4.3. *Let $\mathcal{N}$ be a matrix bridge between quantized metric spaces $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$. Define $\mathcal{L} = (L_n)$ on $\mathcal{V}_1 \oplus \mathcal{V}_2$ by*

$$L_n(a,b) = \max\{L_{1,n}(a), L_{2,n}(b), N_n(a,b)\}, \quad a \in M_n(\mathcal{V}_1),\ b \in M_n(\mathcal{V}_2),\ n \in \mathbb{N}.$$

Let π_1 and π_2 be the projections from $\mathcal{V}_1 \oplus \mathcal{V}_2$ onto $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, which are unital completely positive linear surjective mappings. Then $\mathcal{L}$ is a matrix Lip-norm on $(\mathcal{V}_1 \oplus \mathcal{V}_2, (1,1))$, and it induces $\mathcal{V}_1$ and $\mathcal{V}_2$ via π_1 and π_2, respectively. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lower semicontinuous, then so is $\mathcal{L}$.

Proof. For $a_i \in M_n(\mathcal{V}_i)$ and $b_i \in M_m(\mathcal{V}_i)$, $i = 1, 2$, we have

$$\begin{aligned}
&L_{n+m}(a_1 \oplus b_1, a_2 \oplus b_2)\\
&\quad = \max\{L_{1,n+m}(a_1 \oplus b_1), L_{2,n+m}(a_2 \oplus b_2), N_{n+m}(a_1 \oplus b_1, a_2 \oplus b_2)\}\\
&\quad = \max\{\max\{L_{1,n}(a_1), L_{1,m}(b_1)\}, \max\{L_{2,n}(a_2), L_{2,m}(b_2)\}, \max\{N_n(a_1,a_2), N_m(b_1,b_2)\}\}\\
&\quad = \max\{\max\{L_{1,n}(a_1), L_{2,n}(a_2), N_n(a_1,a_2)\}, \max\{L_{1,m}(b_1), L_{2,m}(b_2), N_m(b_1,b_2)\}\}\\
&\quad = \max\{L_n(a_1,a_2), L_m(b_1,b_2)\}
\end{aligned}$$

and

$$\begin{aligned}
L_n\big((a_1,a_2)^*\big) = L_n\big(a_1^*, a_2^*\big) &= \max\{L_{1,n}\big(a_1^*\big), L_{2,n}\big(a_2^*\big), N_n\big(a_1^*, a_2^*\big)\}\\
&= \max\{L_{1,n}(a_1), L_{2,n}(a_2), N_n(a_1,a_2)\} = L_n(a_1,a_2),
\end{aligned}$$

and for $\alpha \in M_{m,n}$ and $\beta \in M_{n,m}$, we have

$$\begin{aligned}
L_m\big(\alpha(a_1,a_2)\beta\big) &= L_m(\alpha a_1 \beta, \alpha a_2 \beta)\\
&= \max\{L_{1,m}(\alpha a_1 \beta), L_{2,m}(\alpha a_2 \beta), N_m(\alpha a_1 \beta, \alpha a_2 \beta)\}\\
&\leqslant \max\{\|\alpha\| L_{1,n}(a_1)\|\beta\|, \|\alpha\| L_{2,n}(a_2)\|\beta\|, \|\alpha\| N_n(a_1,a_2)\|\beta\|\}\\
&= \|\alpha\| L_n(a_1,a_2)\|\beta\|.
\end{aligned}$$

Thus $\mathcal{L}$ is a matrix seminorm. Since

$$L_1(a_{st}) \leqslant L_n\big([a_{ij}]\big) \leqslant \sum_{i,j=1}^{n} L_1(a_{ij})$$

for $s, t = 1, 2, \ldots, n$ and $[a_{ij}] \in M_n(\mathcal{V}_1 \oplus \mathcal{V}_2)$, $L_n([\lambda_{ij}(1,1)]) = 0$ for $[\lambda_{ij}] \in M_n$. If $L_n([(a_{ij}, b_{ij})]) = 0$, then $L_{1,n}([a_{ij}]) = L_{2,n}([b_{ij}]) = 0$, and hence $a_{ij} = \lambda_{ij}1$ and $b_{ij} = \mu_{ij}1$, $i, j = 1, 2, \ldots, n$, where $\lambda_{ij}, \mu_{ij} \in \mathbb{C}$. From $N_n([a_{ij}, b_{ij}]) = 0$ and $N_1(a_{st}, b_{st}) \leqslant N_n([a_{ij}, b_{ij}])$ for $s, t = 1, 2, \ldots, n$, we have

$$N_1(\lambda_{ij}1, \mu_{ij}1) = 0, \quad i, j = 1, 2, \ldots, n,$$

and so for $i, j = 1, 2, \ldots, n$,

$$|\lambda_{ij}-\mu_{ij}|N_1(1,0)=N_1\big((\lambda_{ij}-\mu_{ij})1,0\big)=N_1\big((\lambda_{ij}1,\mu_{ij}1)-(\mu_{ij}1,\mu_{ij}1)\big)$$
$$\leqslant N_1(\lambda_{ij}1,\mu_{ij}1)+N_1(\mu_{ij}1,\mu_{ij}1)=0.$$

Thus $[(a_{ij},b_{ij})]=[(\lambda_{ij}1,\lambda_{ij}1)]=[\lambda_{ij}(1,1)]$. So $\mathcal{L}$ is a matrix Lipschitz seminorm.

Similar to the same argument used in Theorem 5.2 of [17], we have that $\mathcal{L}$ induces $\mathcal{L}_1$ and $\mathcal{L}_2$ via π_1 and π_2, respectively. By Proposition 3.1 in [21], Proposition 7.5 in [20] and Theorem 5.2 in [17] (see also [17, Section 2]), the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V}_1\oplus\mathcal{V}_2)$ agrees with the BW-topology. Therefore, $\mathcal{L}$ is a matrix Lip-norm on $(\mathcal{V}_1\oplus\mathcal{V}_2,(1,1))$.

Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are lower semicontinuous. Clearly, $\mathcal{L}$ is lower semicontinuous since $\mathcal{N}$ is matrix continuous. □

Notation 4.4. We will denote by $\mathcal{M}(\mathcal{L}_1,\mathcal{L}_2)$ the set of matrix Lip-norms on $\mathcal{V}_1\oplus\mathcal{V}_2$ which induce both $\mathcal{L}_1$ and $\mathcal{L}_2$ via π_1 and π_2, respectively. By Proposition 4.3 and Example 4.2, $\mathcal{M}(\mathcal{L}_1,\mathcal{L}_2)\neq\emptyset$. From Proposition 3.6, we can view $\mathcal{CS}(\mathcal{V}_1)$ and $\mathcal{CS}(\mathcal{V}_2)$ as closed matrix convex subsets of $\mathcal{CS}(\mathcal{V}_1\oplus\mathcal{V}_2)$.

Now we introduce our notion of distance for quantized metric spaces.

Definition 4.5. Let $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$ be quantized metric spaces. We define the *quantized Gromov–Hausdorff distance* $\mathrm{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)$ between them by

$$\mathrm{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)=\inf_{\mathcal{L}=(L_n)\in\mathcal{M}(\mathcal{L}_1,\mathcal{L}_2)}\sup_{n\in\mathbb{N}}\big\{n^{-2}\,\mathrm{dist}_H^{D_{L_n}}\big(CS_n(\mathcal{V}_1),CS_n(\mathcal{V}_2)\big)\big\},$$

where $\mathrm{dist}_H^{D_{L_n}}(CS_n(\mathcal{V}_1),CS_n(\mathcal{V}_2))$ is the Hausdorff distance between $CS_n(\mathcal{V}_1)$ and $CS_n(\mathcal{V}_2)$ for D_{L_n}.

Given a quantized metric space $(\mathcal{V},\mathcal{L})$, we define its *diameter* $\mathrm{diam}(\mathcal{V},\mathcal{L})$ to be the diameter of $CS_1(\mathcal{V})$ with respect to D_{L_1}. The following proposition indicates that the quantized Gromov–Hausdorff distance is always finite.

Proposition 4.6. *Let $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$ be quantized metric spaces. Then*

$$\mathrm{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)\leqslant 2\big(\mathrm{diam}(\mathcal{V}_1,\mathcal{L}_1)+\mathrm{diam}(\mathcal{V}_2,\mathcal{L}_2)\big).$$

Proof. Choosing arbitrarily $\alpha>0$, $\varphi_0\in CS_1(\mathcal{V}_1)$, $\psi_0\in CS_1(\mathcal{V}_2)$, we set

$$N_n(a,b)=\alpha^{-1}\big\|\langle\langle\varphi_0,a\rangle\rangle-\langle\langle\psi_0,b\rangle\rangle\big\|,\quad a\in M_n(\mathcal{V}_1),\ b\in M_n(\mathcal{V}_2),\ n\in\mathbb{N}.$$

As Example 4.2, $\mathcal{N}=(N_n)$ is a matrix bridge between $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$. By Proposition 4.3, $\mathcal{L}=(L_n)$, where

$$L_n(a,b)=\max\big\{L_{1,n}(a),L_{2,n}(b),N_n(a,b)\big\},\quad a\in M_n(\mathcal{V}_1),\ b\in M_n(\mathcal{V}_2),\ n\in\mathbb{N},$$

is a matrix Lip-norm in $\mathcal{M}(\mathcal{L}_1,\mathcal{L}_2)$. Then for $\varphi\in CS_n(\mathcal{V}_1)$, $\psi\in CS_n(\mathcal{V}_2)$, and $(a,b)\in M_n(\mathcal{V}_1\oplus\mathcal{V}_2)$ with $L_n(a,b)\leqslant 1$, we have

$$\begin{aligned}\big\|\langle\langle\varphi,a\rangle\rangle-\langle\langle\psi,b\rangle\rangle\big\| &\leqslant \big\|\langle\langle\varphi,a\rangle\rangle-\langle\langle\underbrace{\varphi_0\oplus\cdots\oplus\varphi_0}_{n},a\rangle\rangle\big\| \\ &\quad+\big\|\langle\langle\underbrace{\varphi_0\oplus\cdots\oplus\varphi_0}_{n},a\rangle\rangle-\langle\langle\underbrace{\psi_0\oplus\cdots\oplus\psi_0}_{n},b\rangle\rangle\big\| \\ &\quad+\big\|\langle\langle\underbrace{\psi_0\oplus\cdots\oplus\psi_0}_{n},b\rangle\rangle-\langle\langle\psi,b\rangle\rangle\big\| \\ &\leqslant\sum_{i,j}\big\|\langle\langle\varphi_{ij}-\delta_{ij}\varphi_0,a\rangle\rangle\big\|+\alpha+\sum_{i,j}\big\|\langle\langle\psi_{ij}-\delta_{ij}\psi_0,b\rangle\rangle\big\|.\end{aligned}$$

If $n=1$, we get

$$\big\|\langle\langle\varphi,a\rangle\rangle-\langle\langle\psi,b\rangle\rangle\big\|\leqslant \operatorname{diam}(\mathcal{V}_1,\mathcal{L}_1)+\alpha+\operatorname{diam}(\mathcal{V}_2,\mathcal{L}_2),$$

by Proposition 3.1 in [21]. If $n>1$, similar to the proof of Proposition 4.2 in [21], there are $\varphi_{ij}^{(k)}\in CS_1(\mathcal{V}_1)$, $k=1,2,3,4$, such that

$$\varphi_{ij}-\delta_{ij}\varphi_0=\varphi_{ij}^{(1)}-\varphi_{ij}^{(2)}+i\big(\varphi_{ij}^{(3)}-\varphi_{ij}^{(4)}\big).$$

Since $L_{1,n}(a)\leqslant L_n(a,b)\leqslant 1$, we obtain

$$\begin{aligned}\sum_{i,j}\big\|\langle\langle\varphi_{ij}-\delta_{ij}\varphi_0,a\rangle\rangle\big\| &\leqslant\sum_{i,j}\big(\big\|\langle\langle\varphi_{ij}^{(1)},a\rangle\rangle-\langle\langle\varphi_{ij}^{(2)},a\rangle\rangle\big\|+\big\|\langle\langle\varphi_{ij}^{(3)},a\rangle\rangle-\langle\langle\varphi_{ij}^{(4)},a\rangle\rangle\big\|\big) \\ &\leqslant\sum_{i,j}\big(D_{L_{1,1}}\big(\varphi_{ij}^{(1)},\varphi_{ij}^{(2)}\big)+D_{L_{1,1}}\big(\varphi_{ij}^{(3)},\varphi_{ij}^{(4)}\big)\big) \\ &\leqslant 2n^2\operatorname{diam}(\mathcal{V}_1,\mathcal{L}_1).\end{aligned}$$

Applying the same argument, we have

$$\sum_{i,j}\big\|\langle\langle\psi_{ij}-\delta_{ij}\psi_0,b\rangle\rangle\big\|\leqslant 2n^2\operatorname{diam}(\mathcal{V}_2,\mathcal{L}_2).$$

Hence

$$\big\|\langle\langle\varphi,a\rangle\rangle-\langle\langle\psi,b\rangle\rangle\big\|\leqslant 2n^2\big(\operatorname{diam}(\mathcal{V}_1,\mathcal{L}_1)+\alpha+\operatorname{diam}(\mathcal{V}_2,\mathcal{L}_2)\big).$$

The arbitrariness of α implies that $\operatorname{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)\leqslant 2(\operatorname{diam}(\mathcal{V}_1)+\operatorname{diam}(\mathcal{V}_2))$ [21, Proposition 3.1]. □

It is clear that the quantized Gromov–Hausdorff distance is symmetric in $\mathcal{V}_1$ and $\mathcal{V}_2$. We come to prove that it satisfies the triangle inequality.

Theorem 4.7. *If* $(\mathcal{V}_1,\mathcal{L}_1)$, $(\mathcal{V}_2,\mathcal{L}_2)$ *and* $(\mathcal{V}_3,\mathcal{L}_3)$ *be quantized metric spaces, then*

$$\operatorname{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_3)\leqslant \operatorname{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)+\operatorname{dist}_{NC}(\mathcal{V}_2,\mathcal{V}_3).$$

Proof. Given $\epsilon > 0$. Then there are $\mathcal{L}_{12} \in \mathcal{M}(\mathcal{L}_1, \mathcal{L}_2)$ and $\mathcal{L}_{23} \in \mathcal{M}(\mathcal{L}_2, \mathcal{L}_3)$ such that

$$\sup_{n\in\mathbb{N}}\left\{n^{-2}\operatorname{dist}_H^{D_{L_{12,n}}}\left(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\right)\right\} \leqslant \operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) + \epsilon$$

and

$$\sup_{n\in\mathbb{N}}\left\{n^{-2}\operatorname{dist}_H^{D_{L_{23,n}}}\left(CS_n(\mathcal{V}_2), CS_n(\mathcal{V}_3)\right)\right\} \leqslant \operatorname{dist}_{NC}(\mathcal{V}_2, \mathcal{V}_3) + \epsilon.$$

We define $\mathcal{L} = (L_n)$ on $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ by

$$L_n(a_1, a_2, a_3) = \max\left\{L_{12,n}(a_1, a_2), L_{23,n}(a_2, a_3)\right\}.$$

Then for $a_i \in M_n(\mathcal{V}_i)$ and $b_i \in M_m(\mathcal{V}_i)$, $i = 1, 2, 3$, we have

$$\begin{aligned}
&L_{n+m}(a_1 \oplus b_1, a_2 \oplus b_2, a_3 \oplus b_3)\\
&\quad = \max\left\{L_{12,n+m}(a_1 \oplus b_1, a_2 \oplus b_2), L_{23,n+m}(a_2 \oplus b_2, a_3 \oplus b_3)\right\}\\
&\quad = \max\left\{\max\left\{L_{12,n}(a_1, a_2), L_{12,m}(b_1, b_2)\right\}, \max\left\{L_{23,n}(a_2, a_3), L_{23,m}(b_2, b_3)\right\}\right\}\\
&\quad = \max\left\{\max\left\{L_{12,n}(a_1, a_2), L_{23,n}(a_2, a_3)\right\}, \max\left\{L_{12,m}(b_1, b_2), L_{23,m}(b_2, b_3)\right\}\right\}\\
&\quad = \max\left\{L_n(a_1, a_2, a_3), L_m(b_1, b_2, b_3)\right\}
\end{aligned}$$

and

$$\begin{aligned}
L_n\left((a_1, a_2, a_3)^*\right) &= L_n\left(a_1^*, a_2^*, a_3^*\right) = \max\left\{L_{12,n}\left(a_1^*, a_2^*\right), L_{23,n}\left(a_2^*, a_3^*\right)\right\}\\
&= \max\left\{L_{12,n}(a_1, a_2), L_{23,n}(a_2, a_3)\right\} = L_n(a_1, a_2, a_3),
\end{aligned}$$

and for $\alpha \in M_{m,n}$ and $\beta \in M_{n,m}$, we have

$$\begin{aligned}
L_m\left(\alpha(a_1, a_2, a_3)\beta\right) &= L_m(\alpha a_1\beta, \alpha a_2\beta, \alpha a_3\beta)\\
&= \max\left\{L_{12,m}(\alpha a_1\beta, \alpha a_2\beta), L_{23,m}(\alpha a_2\beta, \alpha a_3\beta)\right\}\\
&\leqslant \max\left\{\|\alpha\| L_{12,n}(a_1, a_2)\|\beta\|, \|\alpha\| L_{23,n}(a_2, a_3)\|\beta\|\right\}\\
&= \|\alpha\| L_n(a_1, a_2, a_3)\|\beta\|.
\end{aligned}$$

$L_n(a_1, a_2, a_3) = 0$ if and only if $L_{12,n}(a_1, a_2) = 0$ and $L_{23,n}(a_2, a_3) = 0$, and this is equivalent to that $(a_1, a_2, a_3) \in M_n(\mathbb{C}(1, 1, 1))$. Therefore, $\mathcal{L}$ is a matrix Lipschitz seminorm.

Similar to the same argument used in [17, Lemma 4.4], we have that $\mathcal{L}$ induces $\mathcal{L}_{12}$, $\mathcal{L}_{23}$, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ for the evident quotient mappings by Proposition 3.10. By Proposition 3.1 in [21], Proposition 7.5 in [20] and Lemma 4.4 in [17] (see also [17, Section 2]), the $\mathcal{D}_{\mathcal{L}}$-topology on $\mathcal{CS}(\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3)$ agrees with the BW-topology. So $\mathcal{L}$ is a matrix Lip-norm on $(\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3, (1, 1, 1))$.

By Proposition 3.6, we have

$$\sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\bigr)\bigr\} \leqslant \operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) + \epsilon,$$

$$\sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_2), CS_n(\mathcal{V}_3)\bigr)\bigr\} \leqslant \operatorname{dist}_{NC}(\mathcal{V}_2, \mathcal{V}_3) + \epsilon,$$

and

$$\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_3) \leqslant \sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_3)\bigr)\bigr\}.$$

So

$$\begin{aligned}
\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_3) &\leqslant \sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_3)\bigr)\bigr\}\\
&\leqslant \sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\bigr) + n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_2), CS_n(\mathcal{V}_3)\bigr)\bigr\}\\
&\leqslant \sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\bigr)\bigr\}\\
&\quad + \sup_{n\in\mathbb{N}}\bigl\{n^{-2}\operatorname{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}_2), CS_n(\mathcal{V}_3)\bigr)\bigr\}\\
&\leqslant \operatorname{dist}_{NC}\bigl(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\bigr) + \operatorname{dist}_{NC}\bigl(CS_n(\mathcal{V}_2), CS_n(\mathcal{V}_3)\bigr) + 2\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we obtain

$$\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_3) \leqslant \operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) + \operatorname{dist}_{NC}(\mathcal{V}_2, \mathcal{V}_3). \qquad \square$$

Proposition 4.8. *Let $(\mathcal{V}, \mathcal{L})$ be a quantized metric space, and let $\mathcal{K}^{(1)} = (K_n^{(1)})$ and $\mathcal{K}^{(2)} = (K_n^{(2)})$ be compact matrix convex subsets of $\mathcal{CS}(\mathcal{V})$. If $(\mathcal{V}_j, \mathcal{L}_j)$ is the quotient space of $(\mathcal{V}, \mathcal{L})$ with respect to $\mathcal{K}^{(j)}$ and $\phi^{(j)}$, $j = 1, 2$, then we have*

$$\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) \leqslant \sup_{k\in\mathbb{N}}\bigl\{k^{-2}\operatorname{dist}_H^{D_{L_k}}\bigl(K_k^{(1)}, K_k^{(2)}\bigr)\bigr\}.$$

Proof. Let p_1 and p_2 be the projections from $\mathcal{V} \oplus \mathcal{V}$ onto the first space $\mathcal{V}$ and the second space $\mathcal{V}$, respectively. Denote

$$G_n^{(j)} = \bigl(p_1^c\bigr)_n\bigl(K_n^{(j)}\bigr), \quad H_n^{(j)} = \bigl(p_2^c\bigr)_n\bigl(K_n^{(j)}\bigr), \quad j = 1, 2,\ n \in \mathbb{N},$$

and set $\mathcal{G}^{(j)} = (G_n^{(j)})$, $\mathcal{H}^{(j)} = (H_n^{(j)})$, $j = 1, 2$, and $\mathcal{K} = (K_n) = \overline{\operatorname{mco}}(\mathcal{G}^{(1)} \cup \mathcal{H}^{(2)})$, the BW-closed matrix convex hull of the graded set $(G_n^{(1)} \cup H_n^{(2)})$. Let $(\mathcal{W}, \phi(1 \oplus 1))$ be the quotient of $(\mathcal{V} \oplus \mathcal{V}, 1 \oplus 1)$ with respect to $\mathcal{K}$. Then $\mathcal{K} = \phi^c(\mathcal{CS}(\mathcal{W}))$ by Proposition 3.8.

For $(a, b) \in \operatorname{Ker}\phi$, we have $\phi(a, b) = 0$, that is, $\langle\langle (a, b), \varphi\rangle\rangle = 0_n$ for $\varphi \in K_n$. This is equivalent to $\langle\langle (a, b), \varphi\rangle\rangle = 0_n$ for $\varphi \in G_n^{(1)} \cup H_n^{(2)}$, $n \in \mathbb{N}$, since $(a, b) \in A(\mathcal{K})$. And this holds if and only if $\langle\langle a, \varphi_1\rangle\rangle = 0_n$ and $\langle\langle b, \varphi_2\rangle\rangle = 0_n$ for $\varphi_1 \in G_n^{(1)}$ and $\varphi_2 \in H_n^{(2)}$, $n \in \mathbb{N}$, that is, if and only if $a \in \operatorname{Ker}\phi^{(1)}$ and $b \in \operatorname{Ker}\phi^{(2)}$. So $\operatorname{Ker}\phi = \operatorname{Ker}\phi^{(1)} \oplus \operatorname{Ker}\phi^{(2)}$. And thus there is a complete order isomorphism Ψ from $\mathcal{W}$ onto $\mathcal{V}_1 \oplus \mathcal{V}_2$.

Given $\epsilon > 0$. We define a matrix seminorm $\mathcal{N} = (N_n)$ on $\mathcal{V} \oplus \mathcal{V}$ by

$$N_n(a, b) = \epsilon^{-1} \|a - b\|_n, \quad a, b \in M_n(\mathcal{V}).$$

Then $\mathcal{N}$ is a matrix bridge between $(\mathcal{V}, \mathcal{L})$ and $(\mathcal{V}, \mathcal{L})$, and $\mathcal{Q} = (Q_n) \in \mathcal{M}(\mathcal{L}, \mathcal{L})$ by Proposition 4.3, where

$$Q_n(a, b) = \max\bigl\{L_n(a), L_n(b), N_n(a, b)\bigr\}, \quad a, b \in M_n(\mathcal{V}),\ n \in \mathbb{N}.$$

Thus $\mathcal{Q}$ is a matrix Lip-norm on $(\mathcal{V} \oplus \mathcal{V}, (1, 1))$. Let $\mathcal{P} = (P_n)$ and $(\mathcal{W}, \mathcal{P})$ be the quotient space of $(\mathcal{V} \oplus \mathcal{V}, \mathcal{Q})$ with respect to $\mathcal{K}$ and ϕ. Then $\mathcal{P} \in \mathcal{M}(\mathcal{L}_1, \mathcal{L}_2)$ by Proposition 3.10.

Since $D_{P_k}(\varphi_1, \varphi_2) = D_{Q_k}(\phi_k^c(\varphi_1), \phi_k^c(\varphi_2))$ for $\varphi_1, \varphi_2 \in CS_k(\mathcal{W})$, we have that

$$\operatorname{dist}_H^{D_{P_k}}\bigl(CS_k(\mathcal{V}_1), CS_k(\mathcal{V}_2)\bigr) = \operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(1)}, H_k^{(2)}\bigr).$$

For $\psi \in K_k^{(2)}$, we have

$$\begin{aligned}
&D_{Q_k}\bigl(\bigl(p_1^c\bigr)_k(\psi), \bigl(p_2^c\bigr)_k(\psi)\bigr)\\
&\quad = \sup\bigl\{\bigl\|\bigl\langle\!\bigl\langle\bigl(p_1^c\bigr)_k(\psi), (a, b)\bigr\rangle\!\bigr\rangle - \bigl\langle\!\bigl\langle\bigl(p_2^c\bigr)_k(\psi), (a, b)\bigr\rangle\!\bigr\rangle\bigr\|\colon\ Q_r(a, b) \leqslant 1,\ r \in \mathbb{N}\bigr\}\\
&\quad \leqslant \sup\bigl\{\bigl\|\langle\!\langle\psi, a\rangle\!\rangle - \langle\!\langle\psi, b\rangle\!\rangle\bigr\|\colon\ N_r(a, b) \leqslant 1,\ r \in \mathbb{N}\bigr\} \leqslant \epsilon,
\end{aligned}$$

that is, $\operatorname{dist}_H^{D_{Q_k}}(G_k^{(2)}, H_k^{(2)}) \leqslant \epsilon$. Because $\mathcal{Q} \in \mathcal{M}(\mathcal{L}, \mathcal{L})$, we get that

$$\operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(1)}, G_k^{(2)}\bigr) = \operatorname{dist}_H^{D_{L_k}}\bigl(K_k^{(1)}, K_k^{(2)}\bigr).$$

So

$$\begin{aligned}
\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) &\leqslant \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{P_k}}\bigl(CS_k(\mathcal{V}_1), CS_k(\mathcal{V}_2)\bigr)\bigr\}\\
&= \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(1)}, H_k^{(2)}\bigr)\bigr\}\\
&= \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(1)}, G_k^{(2)}\bigr)\bigr\} + \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(2)}, H_k^{(2)}\bigr)\bigr\}\\
&\leqslant \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{DQ_{P_k}}\bigl(G_k^{(1)}, G_k^{(2)}\bigr) + k^{-2}\epsilon\bigr\}\\
&\leqslant \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\bigl(G_k^{(1)}, G_k^{(2)}\bigr)\bigr\} + \epsilon\\
&= \sup_{k \in \mathbb{N}}\bigl\{k^{-2} \operatorname{dist}_H^{D_{L_k}}\bigl(K_k^{(1)}, K_k^{(2)}\bigr)\bigr\} + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we obtain the desired inequality. □

Let (A, L_A) and (B, L_B) be compact quantum metric spaces. The quantum Gromov–Hausdorff distance between them is defined by

$$\mathrm{dist}_q(A, B) = \inf \mathrm{dist}_H^{\rho_L}\big(S(A), S(B)\big),$$

where the infimum is taken over all Lip-norms L on $A \oplus B$ which induce L_A and L_B (see [17, Definition 4.2]).

Proposition 4.9. *Let (A_j, L_j) for $j = 1, 2$ be compact quantum metric spaces, and let $(\mathcal{V}_j, \mathcal{L}_j)$ be an associated quantized metric space of (A_j, L_j) (see Example 3.5). Then*

$$\mathrm{dist}_q(A_1, A_2) \leqslant \mathrm{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2).$$

Proof. Suppose $\mathcal{Q} \in \mathcal{M}(\mathcal{L}_1, \mathcal{L}_2)$. Then $Q_{\mathcal{V}_j,1} = L_{j,1}$ for $j = 1, 2$ and $L_{j,1}(a) = L_j^s(a)$ for $a \in A_j$, where $L_j^s = L_{\rho_{L_j}}$ (see Example 3.5). So for $a \in A_1$, we have

$$\begin{aligned}
L_1^s(a) = L_{1,1}(a) &= Q_{\mathcal{V}_j,1}(a) \\
&= \inf\big\{Q_1(a_1, b_1)\colon\ \pi_1(a_1, b_1) = a,\ (a_1, b_1) = \mathcal{V}_1 \oplus \mathcal{V}_2\big\} \\
&= \inf\big\{Q_1(a, b_1)\colon\ b_1 \in \mathcal{V}_2\big\} \leqslant \inf\big\{Q_1(a, b)\colon\ b \in A_2\big\} \\
&= \inf\big\{R(a, b)\colon\ b \in A_2\big\} = R_{A_1}(a),
\end{aligned}$$

where π_1 is the projection from $\mathcal{V}_1 \oplus \mathcal{V}_2$ onto $\mathcal{V}_1$ and R is the restriction of Q_1 to $A_1 \oplus A_2$. Denote $c = \inf\{Q_1(a, b)\colon b \in \mathcal{V}_2\}$. Let $\epsilon > 0$ be given. Then there is a $y \in \mathcal{V}_2$ such that $Q_1(a, y) \leqslant c + \epsilon$. Setting $x = \frac{1}{2}(y + y^*)$, we have that $x \in A_2$ and

$$\begin{aligned}
R(a, x) = Q_1(a, x) = Q_1\left(a, \frac{1}{2}\big(y + y^*\big)\right) &\leqslant \frac{1}{2} Q_1\big((a, y) + (a, y)^*\big) \\
&\leqslant \frac{1}{2}\big(Q_1(a, y) + Q_1\big((a, y)^*\big)\big) = Q_1(a, y) \leqslant c + \epsilon.
\end{aligned}$$

Thus $L_1^s(a) = R_{A_1}(a)$ for $a \in A_1$. Similarly, we have that $L_2^s(b) = R_{A_2}(b)$ for $b \in A_2$. So $R \in \mathcal{M}(L_1^s, L_2^s)$.

For $\varphi \in CS_1(\mathcal{V}_1)$ and $\psi \in CS_1(\mathcal{V}_2)$, let $\varphi_1 = \varphi|_{A_1}$ and $\psi_1 = \psi|_{A_2}$. Then $\varphi_1 \in S(A_1)$ and $\psi_1 \in S(A_2)$. Since $Q_1((a, b)^*) = Q_1(a, b)$, we obtain

$$\begin{aligned}
D_{Q_1}(\varphi, \psi) &= \sup\big\{\big|\varphi(c) - \psi(d)\big|\colon\ Q_1(c, d) \leqslant 1,\ (c, d) \in \mathcal{V}_1 \oplus \mathcal{V}_2\big\} \\
&= \sup\big\{\big|\varphi(c) - \psi(d)\big|\colon\ Q_1(c, d) \leqslant 1,\ (c, d) = (c, d)^* \in \mathcal{V}_1 \oplus \mathcal{V}_2\big\} \\
&= \sup\big\{\big|\varphi_1(c) - \psi_1(d)\big|\colon\ R(c, d) \leqslant 1,\ (c, d) \in A_1 \oplus A_2\big\} \\
&= \rho_R(\varphi_1, \psi_1)
\end{aligned}$$

(see [17, Section 2]). So

$$\mathrm{dist}_H^{\rho_R}\big(S(A_1), S(A_2)\big) = \mathrm{dist}_H^{D_{Q_1}}\big(CS_1(\mathcal{V}_1), CS_1(\mathcal{V}_2)\big).$$

Therefore [17, Theorem 4.3 and Proposition 7.1], we have

$$\begin{aligned}\operatorname{dist}_q(A_1, A_2) &\leqslant \operatorname{dist}_q\big((A_1, L_1), \big(A_1, L_1^s\big)\big) + \operatorname{dist}_q\big(\big(A_1, L_1^s\big), \big(A_2, L_2^s\big)\big)\\ &\quad + \operatorname{dist}_q\big(\big(A_2, L_2^s\big), (A_2, L_2)\big)\\ &= \operatorname{dist}_q\big(\big(A_1, L_1^s\big), \big(A_2, L_2^s\big)\big)\\ &\leqslant \operatorname{dist}_H^{\rho_R}\big(S(A_1), S(A_2)\big)\\ &= \operatorname{dist}_H^{D_{Q_1}}\big(CS_1(\mathcal{V}_1), CS_1(\mathcal{V}_2)\big)\\ &\leqslant \sup_{n\in\mathbb{N}}\big\{n^{-2}\operatorname{dist}_H^{D_{Q_n}}\big(CS_n(\mathcal{V}_1), CS_n(\mathcal{V}_2)\big)\big\}.\end{aligned}$$

Consequently, $\operatorname{dist}_q(A_1, A_2) \leqslant \operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2)$. □

5. Distance zero

In this section, we show that $\operatorname{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) = 0$ is equivalent to the existence of a complete isometry between them in the following sense.

If $(\mathcal{V}, \mathcal{L})$ is a quantized metric space, then $\mathcal{L}_{\mathcal{D}_\mathcal{L}}$ is the largest lower semicontinuous matrix Lip-norm smaller than $\mathcal{L}$ [20, Corollary 4.5]. From Proposition 7.1 in [20], $\mathcal{L}_{\mathcal{D}_\mathcal{L}}$ extends uniquely to a closed matrix Lip-norm $\mathcal{L}^c$ on the subspace $\mathcal{V}^c = \{a \in \bar{\mathcal{V}}\colon\ L_1^c(a) < +\infty\}$, where $\bar{\mathcal{V}}$ is the completion of $\mathcal{V}$ for its matrix norm.

Definition 5.1. Let $(\mathcal{V}_1, \mathcal{L}_1)$ and $(\mathcal{V}_2, \mathcal{L}_2)$ be quantized metric spaces. By a *complete isometry* from $(\mathcal{V}_1, \mathcal{L}_1)$ onto $(\mathcal{V}_2, \mathcal{L}_2)$ we mean a unital complete order isomorphism Φ from $\mathcal{V}_1^c$ onto $\mathcal{V}_2^c$ such that $\mathcal{L}_1^c = \mathcal{L}_2^c \circ \Phi$, that is, $L_{1,n}^c = L_{2,n}^c \circ \Phi_n$ for all $n \in \mathbb{N}$.

Lemma 5.2. *Let $(\mathcal{V}, \mathcal{L})$ be a quantized metric space. Then*

$$\operatorname{dist}_{NC}\big(\mathcal{V}, \mathcal{V}^c\big) = 0, \qquad \operatorname{dist}_{NC}\big((\mathcal{V}, \mathcal{L}), (\mathcal{V}, \mathcal{L}_{\mathcal{D}_\mathcal{L}})\big) = 0.$$

Proof. Let $\epsilon > 0$ be given, and define

$$N_n(a_1, a_2) = \epsilon^{-1}\|a_1 - a_2\|_n$$

for $a_1 \in M_n(\mathcal{V})$, $a_2 \in M_n(\mathcal{V}^c)$ and $n \in \mathbb{N}$. Clearly $\mathcal{N} = (N_n)$ is a matrix continuous matrix seminorm on $\mathcal{V} \oplus \mathcal{V}^c$, and $N_1(1, 1) = 0$ and $N_1(1, 0) = \epsilon^{-1} \neq 0$.

For $n \in \mathbb{N}$ and $a_1 \in M_n(\mathcal{V})$ and $\delta > 0$, setting $a_2 = a_1 \in M_n(\mathcal{V}^c)$, we have

$$\max\big\{L_n^c(a_2), N_n(a_1, a_2)\big\} = L_n^c(a_2) = L_{D_{L_n}}(a_2) \leqslant L_n(a_2) < L_n(a_2) + \delta$$

by Propositions 3.6, 7.1 and 3.4 in [20]. Given $n \in \mathbb{N}$ and $a_2 \in M_n(\mathcal{V}^c)$ and $\delta > 0$. By Lemma 7.3 in [20], there is a sequence $\{a_1^{(k)}\}$ of elements in $M_n(\mathcal{V})$ such that $L_n(a_1^{(k)}) \leqslant L_n^c(a_2)$ and $\{a_1^{(k)}\}$ converges to a_2 in norm. Consequently, we can find an $a_1^{(k_0)}$ such that $\epsilon^{-1}\|a_1^{(k_0)} - a_2\|_n \leqslant L_n^c(a_2) + \delta$. So $\max\{L_n(a_1^{(k_0)}), N_n(a_1^{(k_0)}, a_2)\} \leqslant L_n^c(a_2) + \delta$. Thus $\mathcal{N}$ is a matrix bridge between $(\mathcal{V}, \mathcal{L})$ and $(\mathcal{V}^c, \mathcal{L}^c)$.

Define

$$L_n(a_1,a_2)=\max\bigl\{L_n(a_1),\,L_n^c(a_2),\,N_n(a_1,a_2)\bigr\}$$

for $a_1\in M_n(\mathcal{V})$, $a_2\in M_n(\mathcal{V}^c)$ and $n\in\mathbb{N}$. By Proposition 4.3, $\mathcal{L}=(L_n)\in\mathcal{M}(\mathcal{V},\mathcal{V}^c)$. For $n\in\mathbb{N}$ and $\varphi\in CS_n(\mathcal{V}^c)$, we have that $\psi=\varphi|_{\mathcal{V}}\in CS_n(\mathcal{V})$, and hence

$$\begin{aligned}
D_{L_n}(\psi,\varphi)&=\sup\bigl\{\bigl\|\bigl\langle\!\bigl\langle(\pi_1)_n^c(\psi),(a_1,a_2)\bigr\rangle\!\bigr\rangle-\bigl\langle\!\bigl\langle(\pi_2)_n^c(\varphi),(a_1,a_2)\bigr\rangle\!\bigr\rangle\bigr\|:\\
&\qquad\qquad L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&=\sup\bigl\{\bigl\|\langle\!\langle\psi,a_1\rangle\!\rangle-\langle\!\langle\varphi,a_2\rangle\!\rangle\bigr\|:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&=\sup\bigl\{\bigl\|\langle\!\langle\varphi,a_1-a_2\rangle\!\rangle\bigr\|:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&\leqslant\sup\bigl\{\|a_1-a_2\|_r:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&\leqslant\epsilon,
\end{aligned}$$

where π_1 and π_2 are the projections from $\mathcal{V}\oplus\mathcal{V}^c$ onto $\mathcal{V}$ and $\mathcal{V}^c$, respectively. For $n\in\mathbb{N}$ and $\varphi\in CS_n(\mathcal{V})$, there is a $\psi\in CS_n(\mathcal{V}^c)$ such that $\psi|_{\mathcal{V}}=\varphi$ by Arveson's extension theorem. So

$$\begin{aligned}
D_{L_n}(\varphi,\psi)&=\sup\bigl\{\bigl\|\bigl\langle\!\bigl\langle(\pi_1)_n^c(\varphi),(a_1,a_2)\bigr\rangle\!\bigr\rangle-\bigl\langle\!\bigl\langle(\pi_2)_n^c(\psi),(a_1,a_2)\bigr\rangle\!\bigr\rangle\bigr\|:\\
&\qquad\qquad L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&=\sup\bigl\{\bigl\|\langle\!\langle\varphi,a_1\rangle\!\rangle-\langle\!\langle\psi,a_2\rangle\!\rangle\bigr\|:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&=\sup\bigl\{\bigl\|\langle\!\langle\psi,a_1-a_2\rangle\!\rangle\bigr\|:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&\leqslant\sup\bigl\{\|a_1-a_2\|_r:\ L_r(a_1,a_2)\leqslant 1,\ (a_1,a_2)\in M_r\bigl(\mathcal{V}\oplus\mathcal{V}^c\bigr),\ r\in\mathbb{N}\bigr\}\\
&\leqslant\epsilon.
\end{aligned}$$

Thus $\mathrm{dist}_H^{D_{L_n}}(CS_n(\mathcal{V}),CS_n(\mathcal{V}^c))\leqslant\epsilon$ for $n\in\mathbb{N}$, and so

$$\sup_{n\in\mathbb{N}}\bigl\{\mathrm{dist}_H^{D_{L_n}}\bigl(CS_n(\mathcal{V}),CS_n\bigl(\mathcal{V}^c\bigr)\bigr)\bigr\}\leqslant\epsilon.$$

Therefore, $\mathrm{dist}_{NC}(\mathcal{V},\mathcal{V}^c)\leqslant\epsilon$. By the arbitrariness of ϵ, we obtain

$$\mathrm{dist}_{NC}\bigl(\mathcal{V},\mathcal{V}^c\bigr)=0.$$

By Proposition 3.4 in [20] and the proof of Theorem 4.4 in [20], we can prove that $\mathrm{dist}_{NC}((\mathcal{V},\mathcal{L}),(\mathcal{V},\mathcal{L}_{\mathcal{D}_{\mathcal{L}}}))=0$ similarly. □

Theorem 5.3. *Suppose $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$ are quantized metric spaces. If there exists a complete isometry Φ from $(\mathcal{V}_1,\mathcal{L}_1)$ onto $(\mathcal{V}_2,\mathcal{L}_2)$, then*

$$\mathrm{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)=0.$$

Proof. For $\epsilon > 0$, we define

$$N_n(a_1, a_2) = \epsilon^{-1} \left\| \Phi_n(a_1) - a_2 \right\|_n$$

for $a_1 \in M_n(\mathcal{V}_1^c)$, $a_2 \in M_n(\mathcal{V}_2^c)$ and $n \in \mathbb{N}$. Clearly $\mathcal{N} = (N_n)$ is a matrix seminorm on $\mathcal{V}_1^c \oplus \mathcal{V}_2^c$ and $N_n((a_1, a_2)^*) = N_n(a_1, a_2)$ for $a_1 \in M_n(\mathcal{V}_1^c)$, $a_2 \in M_n(\mathcal{V}_2^c)$ and $n \in \mathbb{N}$. And we have that

$$N_1(1, 1) = \epsilon^{-1} \left\| \Phi(1) - 1 \right\|_1 = 0 \quad \text{and} \quad N_1(1, 0) = \epsilon^{-1} \left\| \Phi(1) - 0 \right\|_1 = \epsilon^{-1}.$$

If $\{a_1^{(k)}\} \subseteq M_n(\mathcal{V}_1^c)$ and $\{a_2^{(k)}\} \subseteq M_n(\mathcal{V}_2^c)$ with $\lim_{k\to\infty} a_1^{(k)} = a_1 \in M_n(\mathcal{V}_1^c)$ and $\lim_{k\to\infty} a_2^{(k)} = a_2 \in M_n(\mathcal{V}_2^c)$, we have that

$$\lim_{k\to\infty} N_n\big(a_1^{(k)}, a_2^{(k)}\big) = \lim_{k\to\infty} \epsilon^{-1} \left\| \Phi_n\big(a_1^{(k)}\big) - a_2^{(k)} \right\|_n = \epsilon^{-1} \left\| \Phi_n(a_1) - a_2 \right\|_n = N_n(a_1, a_2)$$

since Φ is completely bounded (see [12, Proposition 3.5]).

Given $a_1 \in M_n(\mathcal{V}_1^c)$ and $\delta > 0$. Taking $a_2 = \Phi_n(a_1)$, we have

$$\begin{aligned} \max\big\{L_{2,n}^c(a_2), N_n(a_1, a_2)\big\} &= \max\big\{L_{2,n}^c\big(\Phi_n(a_1)\big), \epsilon^{-1} \left\| \Phi_n(a_1) - a_2 \right\|\big\} \\ &= L_{1,n}^c(a_1) < L_{1,n}^c(a_1) + \delta. \end{aligned}$$

While if $a_2 \in M_n(\mathcal{V}_2^c)$ and $\delta > 0$, we can take $a_1 \in M_n(\mathcal{V}_1^c)$ such that $\Phi_n(a_1) = a_2$, and hence we have

$$\begin{aligned} \max\big\{L_{1,n}^c(a_1), N_n(a_1, a_2)\big\} &= \max\big\{L_{2,n}^c\big(\Phi_n(a_1)\big), \epsilon^{-1} \left\| \Phi_n(a_1) - a_2 \right\|\big\} \\ &= L_{2,n}^c(a_2) < L_{2,n}^c(a_2) + \delta. \end{aligned}$$

Therefore, $\mathcal{N}$ is a matrix bridge between $(\mathcal{V}_1^c, \mathcal{L}_1^c)$ and $(\mathcal{V}_2^c, \mathcal{L}_2^c)$.

Define

$$L_n(a_1, a_2) = \max\big\{L_{1,n}^c(a_1), L_{2,n}^c(a_2), N_n(a_1, a_2)\big\}$$

for $a_1 \in M_n(\mathcal{V}_1^c)$, $a_2 \in M_n(\mathcal{V}_2^c)$ and $n \in \mathbb{N}$. By Proposition 4.3, $\mathcal{L} = (L_n) \in \mathcal{M}(\mathcal{L}_1^c, \mathcal{L}_2^c)$. For $n \in \mathbb{N}$ and $\varphi \in CS_n(\mathcal{V}_2^c)$, we have that $\varphi \circ \Phi \in CS_n(\mathcal{V}_1^c)$, and so

$$\begin{aligned} &D_{L_n}(\varphi \circ \Phi, \varphi) \\ &\quad = \sup\big\{ \left\| \langle\!\langle \varphi \circ \Phi, a_1 \rangle\!\rangle - \langle\!\langle \varphi, a_2 \rangle\!\rangle \right\| \colon L_r(a_1, a_2) \leqslant 1,\ (a_1, a_2) \in M_r\big(\mathcal{V}_1^c \oplus \mathcal{V}_2^c\big),\ r \in \mathbb{N}\big\} \\ &\quad = \sup\big\{ \left\| \big\langle\!\big\langle \varphi, \Phi_r(a_1) - a_2 \big\rangle\!\big\rangle \right\| \colon L_r(a_1, a_2) \leqslant 1,\ (a_1, a_2) \in M_r\big(\mathcal{V}_1^c \oplus \mathcal{V}_2^c\big),\ r \in \mathbb{N}\big\} \\ &\quad \leqslant \sup\big\{ \left\| \Phi_r(a_1) - a_2 \right\|_r \colon L_r(a_1, a_2) \leqslant 1,\ (a_1, a_2) \in M_r\big(\mathcal{V}_1^c \oplus \mathcal{V}_2^c\big),\ r \in \mathbb{N}\big\} \\ &\quad \leqslant \epsilon. \end{aligned}$$

Similarly, for $n \in \mathbb{N}$ and $\psi \in CS_n(\mathcal{V}_1^c)$, we have that $D_{L_n}(\psi, \psi \circ \Phi^{-1}) \leqslant \epsilon$. Thus we obtain that $\mathrm{dist}_H^{D_{L_n}}(CS_n(\mathcal{V}_1^c), CS_n(\mathcal{V}_2^c)) \leqslant \epsilon$ for $n \in \mathbb{N}$, and so

$$\sup_{n\in\mathbb{N}}\left\{n^{-2}\,\mathrm{dist}_H^{D_{L_n}}\left(CS_n\left(\mathcal{V}_1^c\right), CS_n\left(\mathcal{V}_2^c\right)\right)\right\} \leqslant \epsilon.$$

Therefore, $\mathrm{dist}_{NC}(\mathcal{V}_1^c, \mathcal{V}_2^c) \leqslant \epsilon$. Since ϵ is arbitrary, we conclude

$$\mathrm{dist}_{NC}\left(\mathcal{V}_1^c, \mathcal{V}_2^c\right) = 0.$$

Now, by Theorem 4.7 and Lemma 5.2 we have

$$\mathrm{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) \leqslant \mathrm{dist}_{NC}\left(\mathcal{V}_1, \mathcal{V}_1^c\right) + \mathrm{dist}_{NC}\left(\mathcal{V}_1^c, \mathcal{V}_2^c\right) + \mathrm{dist}_{NC}\left(\mathcal{V}_2^c, \mathcal{V}_2\right) = 0.$$

So $\mathrm{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) = 0$. □

Given a quantized metric space $(\mathcal{V}, \mathcal{L})$. From Proposition 6.1 in [20] and the proof of Proposition 3.5 in [19], the mapping $\Psi : \mathcal{V} \mapsto A(\mathcal{CS}(\mathcal{V}))$, defined by $\Psi(a)(\varphi) = \varphi(a)$ for $\varphi \in CS_r(\mathcal{V})$, is a unital complete order isomorphism from $\mathcal{V}$ into $A(\mathcal{CS}(\mathcal{V}))$, and Ψ can be extended to a unital complete order isomorphism $\bar{\Psi}$ from the completion $\bar{\mathcal{V}}$ of $\mathcal{V}$ onto $A(\mathcal{CS}(\mathcal{V}))$. Define

$$L_{D_{L_n}}\left(\mathbf{F}^{(n)}\right) = \sup\left\{\frac{\|F_r^{(n)}(\varphi) - F_r^{(n)}(\psi)\|}{D_{L_r}(\varphi, \psi)}\colon\ \varphi \neq \psi,\ \varphi, \psi \in CS_r(\mathcal{V}),\ r \in \mathbb{N}\right\},$$

where $\mathbf{F}^{(n)} \in A(\mathcal{CS}(\mathcal{V}), M_n)$. Then $\mathcal{L}_{\mathcal{D}_{\mathcal{L}}} = (L_{D_{L_n}})$ is a matrix gauge on $A(\mathcal{CS}(\mathcal{V}))$. Denote

$$K_n = \left\{\mathbf{F}^{(n)} \in A\left(\mathcal{CS}(\mathcal{V}), M_n\right)\colon\ L_{D_{L_n}}\left(\mathbf{F}^{(n)}\right) < +\infty\right\}.$$

Let $L_n^1 = \{a \in \mathcal{V}\colon\ L_n(a) \leqslant 1\}$ and $\bar{L}_n^1$ be the norm closure of L_n^1 in $\bar{\mathcal{V}}$. Denote $\mathcal{L}^1 = (L_n^1)$ and $\bar{\mathcal{L}}^1 = (\bar{L}_n^1)$. The matrix gauge $\bar{\mathcal{L}} = (\bar{L}_n)$ on $(\bar{\mathcal{V}}, 1)$ determined by $\bar{\mathcal{L}}^1$ is called the *closure* of $\mathcal{L}$. $\mathcal{L}$ is *closed* if $\mathcal{L} = \bar{\mathcal{L}}$ on the subspace where $\bar{\mathcal{L}}$ is finite (see [20, Definition 7.2]).

Lemma 5.4. *If $\mathcal{L}$ is closed, then $\Psi_n(M_n(\mathcal{V})) = K_n$ for $n \in \mathbb{N}$.*

Proof. Denote

$$M_n^1 = \left\{\Psi_n(a)\colon\ a \in M_n(\mathcal{V}),\ L_n(a) \leqslant 1\right\}, \quad n \in \mathbb{N},$$
$$L_{D_{L_n}}^1 = \left\{\mathbf{F}^{(n)} \in A\left(\mathcal{CS}(\mathcal{V}), M_n\right)\colon\ L_{D_{L_n}}\left(\mathbf{F}^{(n)}\right) \leqslant 1\right\}, \quad n \in \mathbb{N},$$

and set $\mathcal{M}^1 = (M_n^1)$. Define

$$L_n'(f) = \sup\left\{\left\|\left\langle\!\left\langle f, \Psi_r(a)\right\rangle\!\right\rangle\right\|\colon\ L_r(a) \leqslant 1,\ a \in M_r(\mathcal{V}),\ r \in \mathbb{N}\right\}$$

for $f \in M_n((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$, where $\mathbf{I}$ is the order unit of $A(\mathcal{CS}(\mathcal{V}))$. Here we view $M_n((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$ as the subspace of $M_n((A(\mathcal{CS}(\mathcal{V})))^*)$ consisting of those $f \in M_n((A(\mathcal{CS}(\mathcal{V})))^*)$ with $f(a) = 0_n$ for $a \in \mathbb{C}\mathbf{I}$. Clearly, $\mathcal{L}' = (L_n')$ is a matrix gauge on

$(A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*$ and $L'_n(f^*) = L'_n(f)$ for all $f \in M_n((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$. The generalized bipolar theorem says that $(\mathcal{M}^1)^{\circledcirc\circledcirc}$ is the smallest weakly closed absolutely matrix convex set containing $\mathcal{M}^1$ (see [4, Proposition 4.1]). Since $\mathcal{L} = (L_n)$ is a matrix gauge and Ψ is a unital complete order isomorphism, $\mathcal{M}^1$ is absolutely matrix convex. The closeness of $\mathcal{L}$ implies that $\mathcal{M}^1$ is normed-closed [20, Lemma 7.4]. So $\mathcal{M}^1$ is weakly closed. Thus

$$\left(\mathcal{M}^1\right)^{\circledcirc\circledcirc} = \mathcal{M}^1.$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left(M_n^1\right)^{\circledcirc} &= \left\{\Psi_n(a)\colon\ a \in M_n(\mathcal{V}),\ L_n(a) \leqslant 1\right\}^{\circledcirc} \\
&= \left\{f \in M_n\left(\left(A\left(\mathcal{CS}(\mathcal{V})\right)\right)^*\right)\colon\ \left\|\left\langle\!\left\langle f, \Psi_r(a)\right\rangle\!\right\rangle\right\| \leqslant 1 \text{ for all } a \in M_r(\mathcal{V}),\ L_r(a) \leqslant 1,\ r \in \mathbb{N}\right\} \\
&= \left\{f \in M_n\left(\left(A\left(\mathcal{CS}(\mathcal{V})\right)/(\mathbb{C}\mathbf{I})\right)^*\right)\colon\ \left\|\left\langle\!\left\langle f, \Psi_r(a)\right\rangle\!\right\rangle\right\| \leqslant 1 \right. \\
&\qquad \left. \text{for all } a \in M_r(\mathcal{V}),\ L_r(a) \leqslant 1,\ r \in \mathbb{N}\right\} \\
&= \left\{f \in M_n\left(\left(A\left(\mathcal{CS}(\mathcal{V})\right)/(\mathbb{C}\mathbf{I})\right)^*\right)\colon\ L'_n(f) \leqslant 1\right\}
\end{aligned}$$

and

$$\begin{aligned}
\left(M_n^1\right)^{\circledcirc\circledcirc} &= \left\{f \in M_n\left(\left(A\left(\mathcal{CS}(\mathcal{V})\right)/(\mathbb{C}\mathbf{I})\right)^*\right)\colon\ L'_n(f) \leqslant 1\right\}^{\circledcirc} \\
&= \left\{\mathbf{F}^{(n)} \in M_n\left(A\left(\mathcal{CS}(\mathcal{V})\right)\right)\colon\ \left\|\left\langle\!\left\langle f, \mathbf{F}^{(n)}\right\rangle\!\right\rangle\right\| \leqslant 1 \right. \\
&\qquad \left. \text{for all } f \in M_r\left(\left(A\left(\mathcal{CS}(\mathcal{V})\right)/(\mathbb{C}\mathbf{I})\right)^*\right),\ L'_r(f) \leqslant 1,\ r \in \mathbb{N}\right\}.
\end{aligned}$$

So $\mathbf{F}^{(n)} \in (M_n^1)^{\circledcirc\circledcirc}$ if and only if

$$\left\|\left\langle\!\left\langle f, \mathbf{F}^{(n)}\right\rangle\!\right\rangle\right\| \leqslant L'_r(f)$$

for all $f \in M_r((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$ and $r \in \mathbb{N}$.

Suppose that $\|\langle\!\langle f, \mathbf{F}^{(n)}\rangle\!\rangle\| \leqslant L'_r(f)$ for all $f = f^* \in M_r((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$ and $r \in \mathbb{N}$. Then for $g \in M_r((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$, we have

$$\begin{aligned}
\left\|\left\langle\!\left\langle g, \mathbf{F}^{(n)}\right\rangle\!\right\rangle\right\| &= \left\|[1 \quad 0]\begin{bmatrix} 0 & \langle\!\langle g, \mathbf{F}^{(n)}\rangle\!\rangle \\ \langle\!\langle g^*, \mathbf{F}^{(n)}\rangle\!\rangle & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\| \\
&\leqslant \left\|\left\langle\!\!\left\langle \begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}, \mathbf{F}^{(n)}\right\rangle\!\!\right\rangle\right\| \leqslant L'_{2r}\left(\begin{bmatrix} 0 & g \\ g^* & 0 \end{bmatrix}\right) \\
&\leqslant L'_{2r}\left(\begin{bmatrix} g & 0 \\ 0 & g^* \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \leqslant L'_{2r}\left(\begin{bmatrix} g & 0 \\ 0 & g^* \end{bmatrix}\right) = L'_r(g).
\end{aligned}$$

Thus $\mathbf{F}^{(n)} \in (M_n^1)^{\circledcirc\circledcirc}$ exactly if $\|\langle\!\langle f, \mathbf{F}^{(n)}\rangle\!\rangle\| \leqslant L'_r(f)$ for all $f = f^* \in M_r((A(\mathcal{CS}(\mathcal{V}))/(\mathbb{C}\mathbf{I}))^*)$ and $r \in \mathbb{N}$. According to Lemma 4.1 in [20], $\mathbf{F}^{(n)} \in (M_n^1)^{\circledcirc\circledcirc}$ exactly if

$$\left\|\left\langle\!\left\langle \varphi, \mathbf{F}^{(n)}\right\rangle\!\right\rangle - \left\langle\!\left\langle \psi, \mathbf{F}^{(n)}\right\rangle\!\right\rangle\right\| \leqslant L'_r(\varphi - \psi) \quad \text{for all } \varphi, \psi \in CS_r\left(A\left(\mathcal{CS}(\mathcal{V})\right)\right) \text{ and } r \in \mathbb{N}.$$

So $\mathbf{F}^{(n)} \in (M_n^1)^{\circledcirc\circledcirc}$ exactly if

$$\begin{aligned}
&\big\|\big\langle\!\big\langle\varphi\circ\Psi^{-1},\mathbf{F}^{(n)}\big\rangle\!\big\rangle-\big\langle\!\big\langle\psi\circ\Psi^{-1},\mathbf{F}^{(n)}\big\rangle\!\big\rangle\big\|\\
&\quad\leqslant L'_r\big(\varphi\circ\Psi^{-1}-\psi\circ\Psi^{-1}\big)\\
&\quad=\sup\big\{\big\|\big\langle\!\big\langle(\varphi-\psi)\circ\Psi^{-1},\Psi_k(a)\big\rangle\!\big\rangle\big\|:\ L_k(a)\leqslant 1,\ a\in M_k(\mathcal{V}),\ k\in\mathbb{N}\big\}\\
&\quad=\sup\big\{\big\|\langle\!\langle\varphi-\psi,a\rangle\!\rangle\big\|:\ L_k(a)\leqslant 1,\ a\in M_k(\mathcal{V}),\ k\in\mathbb{N}\big\}\\
&\quad=D_{L_r}(\varphi,\psi)
\end{aligned}$$

for all $\varphi,\psi\in CS_r(A(\mathcal{CS}(\mathcal{V})))$ and $r\in\mathbb{N}$. Because $\|\langle\!\langle\varphi\circ\Psi^{-1},\mathbf{F}^{(n)}\rangle\!\rangle-\langle\!\langle\psi\circ\Psi^{-1},\mathbf{F}^{(n)}\rangle\!\rangle\|=\|F_r^{(n)}(\varphi)-F_r^{(n)}(\psi)\|$, we have

$$\mathbf{F}^{(n)}\in\big(M_n^1\big)^{\circledcirc\circledcirc}\quad\text{if and only if}\quad\big\|F_r^{(n)}(\varphi)-F_r^{(n)}(\psi)\big\|\leqslant D_{L_r}(\varphi,\psi)$$

for all $\varphi,\psi\in CS_r(A(\mathcal{CS}(\mathcal{V})))$ and $r\in\mathbb{N}$. And this says exactly that $\mathbf{F}^{(n)}\in L^1_{D_{L_n}}$. Therefore, $M_n^1=L^1_{D_{L_n}}$ for $n\in\mathbb{N}$. So $\Psi_n(M_n(\mathcal{V}))=K_n$ for $n\in\mathbb{N}$. □

Lemma 5.5. *Let $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$ be quantized metric spaces such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed. For every matrix affine mapping $\alpha=(\alpha_n)$ from $\mathcal{CS}(\mathcal{V}_1)$ onto $\mathcal{CS}(\mathcal{V}_2)$ which is completely isometric for $\mathcal{D}_{\mathcal{L}_1}$ and $\mathcal{D}_{\mathcal{L}_2}$, there is a unital complete order isomorphism Ψ from $\mathcal{V}_1$ onto $\mathcal{V}_2$ such that $\mathcal{L}_1=\mathcal{L}_2\circ\Psi$.*

Proof. Define $\Phi:A(\mathcal{CS}(\mathcal{V}_2))\mapsto A(\mathcal{CS}(\mathcal{V}_1))$ by

$$\big(\Phi\big(\mathbf{F}^{(1)}\big)\big)_r(\varphi)=F_r^{(1)}\big(\alpha_r(\varphi)\big)$$

for $\mathbf{F}^{(1)}\in A(\mathcal{CS}(\mathcal{V}_2))$ and $\varphi\in CS_r(\mathcal{V}_1)$. Since α is isometric and matrix affine, Φ is well defined. Clearly, Φ is unital and surjective. On the level of matrices, we have

$$\big(\Phi_n\big(\mathbf{F}^{(n)}\big)\big)_r(\varphi)=F_r^{(n)}\big(\alpha_r(\varphi)\big)$$

for $\mathbf{F}^{(n)}\in M_n(A(\mathcal{CS}(\mathcal{V}_2)))$ and $\varphi\in CS_r(\mathcal{V}_1)$. Since $\mathbf{F}^{(n)}\geqslant 0$ in $M_n(A(\mathcal{CS}(\mathcal{V}_2)))$ if and only if $F_r^{(n)}(\varphi)\geqslant 0$ for all $r\in\mathbb{N}$ and $\varphi\in CS_r(\mathcal{V}_2)$, Φ is a unital complete order isomorphism from $A(\mathcal{CS}(\mathcal{V}_2))$ onto $A(\mathcal{CS}(\mathcal{V}_1))$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed, Φ is a unital complete order isomorphism from $\mathcal{V}_2$ onto $\mathcal{V}_1$ by Lemma 5.4. That α is completely isometric for $\mathcal{D}_{\mathcal{L}_1}$ and $\mathcal{D}_{\mathcal{L}_2}$ implies that $L_{D_{L_{1,n}}}(\Phi_n(a_2))=L_{D_{L_{2,n}}}(a_2)$ for all $a_2\in M_n(\mathcal{V}_2)$ and $n\in\mathbb{N}$. Because $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed, they are lower semicontinuous, so that $\mathcal{L}_{\mathcal{D}_{\mathcal{L}_1}}=\mathcal{L}_1$ on $\mathcal{V}_1$ [20, Theorem 4.4], and similarly for $\mathcal{V}_2$. Thus Φ^{-1} is a unital complete order isomorphism from $\mathcal{V}_1$ onto $\mathcal{V}_2$ such that $\mathcal{L}_1=\mathcal{L}_2\circ\Psi$. □

Theorem 5.6. *Suppose $(\mathcal{V}_1,\mathcal{L}_1)$ and $(\mathcal{V}_2,\mathcal{L}_2)$ are quantized metric spaces. If*

$$\mathrm{dist}_{NC}(\mathcal{V}_1,\mathcal{V}_2)=0,$$

then there exists a complete isometry Φ from $(\mathcal{V}_1,\mathcal{L}_1)$ onto $(\mathcal{V}_2,\mathcal{L}_2)$.

Proof. Since $\mathrm{dist}_{NC}(\mathcal{V}_1, \mathcal{V}_2) = 0$, we have

$$\mathrm{dist}_{NC}\big(\mathcal{V}_1^c, \mathcal{V}_2^c\big) = 0$$

by Lemma 5.2 and Theorem 4.7. From that $\mathrm{dist}_{NC}(\mathcal{V}_1^c, \mathcal{V}_2^c) = 0$, there is a sequence $\{\mathcal{L}^{(k)}\} \subseteq \mathcal{M}(\mathcal{V}_1^c \oplus \mathcal{V}_2^c)$ of matrix Lip-norms such that

$$\sup_{n\in\mathbb{N}}\Big\{n^{-2}\,\mathrm{dist}_H^{D_{L_n^{(k)}}}\big(CS_n\big(\mathcal{V}_1^c\big), CS_n\big(\mathcal{V}_2^c\big)\big)\Big\} < \frac{1}{k}.$$

So for each $n \in \mathbb{N}$, we have

$$n^{-2}\,\mathrm{dist}_H^{D_{L_n^{(k)}}}\big(CS_n\big(\mathcal{V}_1^c\big), CS_n\big(\mathcal{V}_2^c\big)\big) < \frac{1}{k}.$$

And for $\varphi, \psi \in CS_n(\mathcal{V}_i^c)$, $i = 1, 2$, by Proposition 3.6 we have

$$D_{L_n^{(k)}}\big((\pi_i)_n^c(\varphi), (\pi_i)_n^c(\psi)\big) = D_{L_{i,n}^c}(\varphi, \psi),$$

where π_i, $i = 1, 2$, is the projection from $\mathcal{V}_1^c \oplus \mathcal{V}_2^c$ onto $\mathcal{V}_i^c$. Therefore, for each $n \in \mathbb{N}$ we get

$$\mathrm{dist}_{GH}\big(CS_n\big(\mathcal{V}_1^c\big), CS_n\big(\mathcal{V}_2^c\big)\big) = 0,$$

where $\mathrm{dist}_{GH}(CS_n(\mathcal{V}_1^c), CS_n(\mathcal{V}_2^c))$ is the Gromov–Hausdorff distance (see [6, Definition 3.4]) between $(CS_n(\mathcal{V}_1^c), D_{L_{1,n}^c})$ and $(CS_n(\mathcal{V}_2^c), D_{L_{2,n}^c})$. As in the proofs of Theorems 7.6 and 7.7 in [17], there is a subsequence $\{D_{L_1^{(k_{j_1})}}\}$ which converges uniformly on the disjoint union $CS_1(\mathcal{V}_1^c) \sqcup CS_1(\mathcal{V}_2^c)$ to a semi-metric σ_1 and σ_1 determines an isometry α_1 from $CS_1(\mathcal{V}_1^c)$ onto $CS_1(\mathcal{V}_2^c)$ by the condition that $\sigma_1(\varphi, \alpha_1(\varphi)) = 0$. Similarly, there is a subsequence $\{D_{L_2^{(k_{j_1,j_2})}}\}$ of $\{D_{L_2^{(k_{j_1})}}\}$ which converges uniformly on $CS_2(\mathcal{V}_1^c) \sqcup CS_2(\mathcal{V}_2^c)$ to a semi-metric σ_2 and σ_2 determines an isometry α_2 from $CS_2(\mathcal{V}_1^c)$ onto $CS_2(\mathcal{V}_2^c)$ by the condition that $\sigma_2(\varphi, \alpha_2(\varphi)) = 0$. In general, once

$$\big\{D_{L_2^{(k_{j_1})}}\big\}, \quad \big\{D_{L_2^{(k_{j_1,j_2})}}\big\}, \quad \ldots, \quad \big\{D_{L_2^{(k_{j_1,j_2,\ldots,j_n})}}\big\}$$

have been chosen, there is a subsequence $\{D_{L_2^{(k_{j_1,j_2,\ldots,j_n,j_{n+1}})}}\}$ of $\{D_{L_2^{(k_{j_1,j_2,\ldots,j_n})}}\}$ which converges uniformly on $CS_{n+1}(\mathcal{V}_1^c) \sqcup CS_{n+1}(\mathcal{V}_2^c)$ to a semi-metric σ_{n+1} and σ_{n+1} determines an isometry α_{n+1} from $CS_{n+1}(\mathcal{V}_1^c)$ onto $CS_{n+1}(\mathcal{V}_2^c)$ by the condition that $\sigma_{n+1}(\varphi, \alpha_{n+1}(\varphi)) = 0$.

Given $\varphi_i \in CS_{n_i}(\mathcal{V}_1^c)$ and $\gamma_i \in M_{n_i,n}$, $i = 1, 2, \ldots, m$, satisfying $\sum_{i=1}^m \gamma_i^*\gamma_i = 1_n$, and $\epsilon > 0$. Let $s = \max\{n, n_1, \ldots, n_m\}$. We can find $K \in \mathbb{N}$ such that if $k_{j_1,\ldots,j_s} > K$ then

$$\big\|\sigma_l - D_{L_l^{(k_{j_1,\ldots,j_s})}}\big\| < \frac{\epsilon}{2} \quad \text{for } l = n, n_1, \ldots, n_m.$$

Now for $k_{j_1,\ldots,j_s} > K$ we have

$$\begin{aligned}
&\sigma_n\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i, \sum_{i=1}^m \gamma_i^*\alpha_{n_i}(\varphi_i)\gamma_i\right)\\
&\leqslant D_{L_n^{(k_{j_1,\dots,j_s})}}\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i, \sum_{i=1}^m \gamma_i^*\alpha_{n_i}(\varphi_i)\gamma_i\right) + \frac{\epsilon}{2}\\
&\leqslant D_{L_{n_1+\dots+n_m}^{(k_{j_1,\dots,j_s})}}\big(\varphi_1\oplus\dots\oplus\varphi_m, \alpha_{n_1}(\varphi_1)\oplus\dots\oplus\alpha_{n_m}(\varphi_m)\big) + \frac{\epsilon}{2}\\
&= \max\big\{D_{L_{n_1}^{(k_{j_1,\dots,j_s})}}\big(\varphi_1, \alpha_{n_1}(\varphi_1)\big), \dots, D_{L_{n_m}^{(k_{j_1,\dots,j_s})}}\big(\varphi_m, \alpha_{n_m}(\varphi_m)\big)\big\} + \frac{\epsilon}{2}\\
&< \max\left\{\sigma_{n_1}\big(\varphi_1, \alpha_{n_1}(\varphi_1)\big) + \frac{\epsilon}{2}, \dots, \sigma_{n_m}\big(\varphi_m, \alpha_{n_m}(\varphi_m)\big) + \frac{\epsilon}{2}\right\} + \frac{\epsilon}{2}\\
&= \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we have

$$\sigma_n\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i, \sum_{i=1}^m \gamma_i^*\alpha_{n_i}(\varphi_i)\gamma_i\right) = 0.$$

But

$$\sigma_n\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i, \alpha_n\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i\right)\right) = 0.$$

By Lemma 7.4 in [17], we obtain

$$\alpha_n\left(\sum_{i=1}^m \gamma_i^*\varphi_i\gamma_i\right) = \sum_{i=1}^m \gamma_i^*\alpha_{n_i}(\varphi_i)\gamma_i.$$

So $\alpha = (\alpha_n)$ is matrix affine.

Now, by Lemma 5.5 we conclude that there exists a unital complete order isomorphism Φ from $\mathcal{V}_1^c$ onto $\mathcal{V}_2^c$ such that $\mathcal{L}_1^c = \mathcal{L}_2^c \circ \Phi$, that is, Φ is a complete isometry from $(\mathcal{V}_1, \mathcal{L}_1)$ onto $(\mathcal{V}_2, \mathcal{L}_2)$. □

6. Completeness

For the metric space of complete isometry classes of quantized metric spaces with the quantized Gromov–Hausdorff distance, we show in this section that it is complete.

Let $\{(\mathcal{V}_i, 1)\}$ be a sequence of matrix order unit space. We will denote by $\bigoplus_{i\in\mathbb{N}} \mathcal{V}_i$ the operator space direct sum that is formed of all sequences $\{a_i\}$ with $a_i \in \mathcal{V}_i$ and $\sup_{i\in\mathbb{N}} \|a_i\| < +\infty$, and by $\bigoplus_{i=1}^n \mathcal{V}_i$ the operator space direct sum of $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n$ (see [13, Section 2.6]). Then $(\bigoplus_{i\in\mathbb{N}} \mathcal{V}_i, \{1\})$ and $(\bigoplus_{i=1}^n \mathcal{V}_i, (\underbrace{1,\dots,1}_{n}))$ are matrix order unit spaces.

Suppose we have a sequence $\{(\mathcal{V}_i, \mathcal{L}_i)\}$ of quantized metric spaces. Suppose further that we have a sequence $\{\mathcal{M}_i\}$ of matrix Lip-norms with $\mathcal{M}_i \in \mathcal{M}(\mathcal{L}_i, \mathcal{L}_{i+1})$. Define $\mathcal{Q} = (Q_k)$ on $\prod_{i\in\mathbb{N}} \mathcal{V}_i$, the full product, by

$$Q_k\big(\{a_i\}\big) = \sup_{i\in\mathbb{N}}\big\{M_{i,k}(a_i, a_{i+1})\big\}, \quad \{a_i\} \in M_k\bigg(\prod_{i\in\mathbb{N}} \mathcal{V}_i\bigg),$$

and set

$$\mathcal{E}_1 = \bigg\{\{a_i\} \in \bigoplus_{i\in\mathbb{N}} \mathcal{V}_i\colon\ Q_1\big(\{a_i\}\big) < +\infty\bigg\}.$$

It is easy to check that $\mathcal{E}_1$ is a self-adjoint subspace of $\bigoplus_{i\in\mathbb{N}} \mathcal{V}_i$ containing $\{1\}$, and so is a matrix order unit space, and that $\mathcal{Q}$ is a matrix Lipschitz seminorm on $\mathcal{E}_1$.

For the evident identifications, we have

$$\mathcal{CS}(\mathcal{V}_i) \subseteq \mathcal{CS}\bigg(\bigoplus_{j=1}^{i} \mathcal{V}_j\bigg) \subseteq \mathcal{CS}\bigg(\bigoplus_{j=1}^{n} \mathcal{V}_j\bigg) \subseteq \mathcal{CS}(\mathcal{E}_1), \quad 0 \leqslant i \leqslant n.$$

Given a family of graded sets $\mathbf{S}_i = (S_{i,n}),\ i \in I$. We denote by $\bigcup_{i\in I} \mathbf{S}_i$ the graded set $(\bigcup_{i\in I} S_{i,n})$. If $\mathbf{S} = (S_n)$ is a graded set in a vector space, we denote by $\mathrm{mco}(\mathbf{S})$ the matrix convex hull of $\mathbf{S}$. Let $\mathcal{Z} = (Z_n) = \mathrm{mco}(\bigcup_{i\in\mathbb{N}} \mathcal{CS}(\mathcal{V}_i))$, $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{CS}(\bigoplus_{j=1}^n \mathcal{V}_j)$ and $\mathcal{W}_n = \mathrm{mco}(\bigcup_{i=1}^n \mathcal{CS}(\mathcal{V}_i))$.

Proposition 6.1. *$\mathcal{Z}$ and $\mathcal{U}$ are BW-dense in $\mathcal{CS}(\mathcal{E}_1)$. In particular, $\mathcal{W}_n$ is BW-dense in $\mathcal{CS}(\bigoplus_{j=1}^n \mathcal{V}_j)$ for $n \in \mathbb{N}$.*

Proof. Since the matrix polar

$$\begin{aligned}
Z_n^\pi &= \big\{a \in M_n(\mathcal{E}_1)\colon\ \mathrm{Re}\langle\!\langle a, \varphi\rangle\!\rangle \leqslant 1_{r\times n} \text{ for all } \varphi \in Z_r,\ r \in \mathbb{N}\big\}\\
&= \big\{a \in M_n(\mathcal{E}_1)\colon\ \varphi_n\big(\mathrm{Re}(a)\big) \leqslant 1_{r\times n} \text{ for all } \varphi \in Z_r,\ r \in \mathbb{N}\big\}\\
&= \big\{a \in M_n(\mathcal{E}_1)\colon\ \varphi_n\big(1_n - \mathrm{Re}(a)\big) \geqslant 0 \text{ for all } \varphi \in Z_r,\ r \in \mathbb{N}\big\}\\
&= \big\{a \in M_n(\mathcal{E}_1)\colon\ 1_n - \mathrm{Re}(a) \geqslant 0\big\}\\
&= \big\{a \in M_n(\mathcal{E}_1)\colon\ \|a_+\| \leqslant 1\big\},
\end{aligned}$$

we have

$$Z_n^{\pi\pi} = \bigg\{f \in M_n\big(\mathcal{E}_1^*\big)\colon\ \mathrm{Re}\langle\!\langle f, a\rangle\!\rangle \leqslant 1_{r\times n} \text{ when } 1_r - \mathrm{Re}(a) \geqslant 0,\ a \in M_r\bigg(\bigoplus_{i\in\mathbb{N}} \mathcal{V}_i\bigg),\ r \in \mathbb{N}\bigg\}.$$

For $f \in Z_n^{\pi\pi}$, $\lambda \in \mathbb{R}$ and $a = a^* \in M_r(\mathcal{E}_1)$, we have that

$$1_r - \mathrm{Re}(i\lambda a) \geqslant 0, \quad \text{and so} \quad \mathrm{Re}\langle\!\langle i\lambda a, f\rangle\!\rangle \leqslant 1_{r\times n}.$$

Thus

$$0 = \mathrm{Re}\langle\langle ia, f\rangle\rangle = \mathrm{Re}\big(i\langle\langle a, f\rangle\rangle\big) = -\mathrm{Im}\langle\langle a, f\rangle\rangle,$$

that is, $\mathrm{Im}\langle\langle a, f\rangle\rangle = 0$. If $a \in M_r(\mathcal{E}_1)$, $a \geqslant 0$ and $\lambda \leqslant 0$, then $1_r - \mathrm{Re}(\lambda a) \geqslant 0$ and so

$$\langle\langle \lambda a, f\rangle\rangle = \mathrm{Re}\langle\langle \lambda a, f\rangle\rangle \leqslant 1_{r\times n}.$$

Thus $\langle\langle a, f\rangle\rangle \geqslant 0$. Clearly, $\langle\langle 1_r, f\rangle\rangle \leqslant 1_{r\times n}$. Therefore,

$$Z_n^{\pi\pi} = \big\{f \in M_n\big(\mathcal{E}_1^*\big)\colon\ f \text{ is completely positive and } f(1) \leqslant 1_n\big\}.$$

By the bipolar theorem in matrix convexity (see [5, Corollary 5.5]), $\mathrm{mco}(\mathcal{Z} \cup \{0\})$ is BW-dense in $\mathcal{Z}^{\pi\pi} = (Z_n^{\pi\pi})$. Evaluating the completely positive mappings at 1, we see that $\mathcal{Z}$ is BW-dense in $\mathcal{CS}(\mathcal{E}_1)$. Because $\mathcal{Z} \subseteq \mathcal{U} \subseteq \mathcal{CS}(\mathcal{E}_1)$, $\mathcal{U}$ is BW-dense in $\mathcal{CS}(\mathcal{E}_1)$. □

Define $\mathcal{P}_n = (P_{n,k})$ on $\bigoplus_{i=1}^n \mathcal{V}_i$ by

$$P_{n,k}(a_1, \ldots, a_n) = \max\big\{M_{i,k}(a_i, a_{i+1})\colon\ 1 \leqslant i \leqslant n-1\big\},$$

for $(a_1, \ldots, a_n) \in M_k(\bigoplus_{i=1}^n \mathcal{V}_i)$. Similar to the proof of Proposition 4.3, we have

Proposition 6.2. *$\mathcal{P}_n$ is a matrix Lip-norm on $\bigoplus_{i=1}^n \mathcal{V}_i$, and induces $\mathcal{L}_j$, $1 \leqslant j \leqslant n$ and $\mathcal{M}_i$ and $\mathcal{P}_i$, $1 \leqslant i \leqslant n-1$, via the evident projections.*

For $b \in M_i(\bigoplus_{j=1}^n \mathcal{V}_j)$ and $\epsilon > 0$, set $b_n = b$. Since $\mathcal{P}_{n+1}$ induces $\mathcal{P}_n$, we can find $b_{n+1} \in M_i(\bigoplus_{j=1}^{n+1} \mathcal{V}_j)$ such that $(\pi_n)_i(b_{n+1}) = b_n$ and $P_{n+1,i}(b_{n+1}) < P_{n,i}(b_n) + \frac{\epsilon}{2^n}$, where π_n is the evident projection from $\bigoplus_{j=1}^{n+1} \mathcal{V}_j$ onto $\bigoplus_{j=1}^n \mathcal{V}_j$. Similarly, we can find $b_{n+2} \in M_i(\bigoplus_{j=1}^{n+2} \mathcal{V}_j)$ such that $(\pi_{n+1})_i(b_{n+2}) = b_{n+1}$ and $P_{n+2,i}(b_{n+2}) < P_{n+1,i}(b_{n+1}) + \epsilon/2^{n+1}$. Continuing in this way, for $t \geqslant n$ we get $b_{t+1} \in M_i(\bigoplus_{j=1}^{t+1} \mathcal{V}_j)$ such that $(\pi_t)_i(b_{t+1}) = b_t$ and $P_{t+1,i}(b_{t+1}) < P_{t,i}(b_t) + \epsilon/2^t$. We let $c = \{c_j\}$ be the unique element of $M_i(\prod_{j\in\mathbb{N}} \mathcal{V}_j)$ such that $(\pi_t)_i(c) = b_t$ for $t \geqslant n$. Then $Q_i(c) \leqslant P_{n,i}(b) + \epsilon$. So, $P_{n,i}(b) = Q_{\bigoplus_{j=1}^n \mathcal{V}_j, i}(b)$. Set $d_k = (c_1, \ldots, c_k)$, $k \in \mathbb{N}$.

Since $\mathcal{P}_n$ induces $\mathcal{M}_i$ $(1 \leqslant i \leqslant n-1)$, via the evidence projections,

$$\mathrm{dist}_H^{D_{P_{n,i}}}\big(CS_i(\mathcal{V}_k), CS_i(\mathcal{V}_{k+1})\big) = \mathrm{dist}_H^{D_{M_{k,i}}}\big(CS_i(\mathcal{V}_k), CS_i(\mathcal{V}_{k+1})\big).$$

For $m, n \in \mathbb{N}$ with $m < n$ and $\varphi_n \in CS_i(\mathcal{V}_n)$, we can find $\varphi_{n-1} \in CS_i(\mathcal{V}_{n-1})$ with

$$D_{P_{n,i}}(\varphi_{n-1}, \varphi_n) \leqslant \mathrm{dist}_H^{D_{M_{n-1,i}}}\big(CS_i(\mathcal{V}_{n-1}), CS_i(\mathcal{V}_n)\big).$$

Similarly, we can find $\varphi_{n-2} \in CS_i(\mathcal{V}_{n-2})$ with

$$D_{P_{n,i}}(\varphi_{n-2}, \varphi_{n-1}) \leqslant \mathrm{dist}_H^{D_{M_{n-2,i}}}\big(CS_i(\mathcal{V}_{n-2}), CS_i(\mathcal{V}_{n-1})\big).$$

Inductively, we can find $\varphi_m, \ldots, \varphi_{n-1}$ with $\varphi_k \in CS_i(\mathcal{V}_k)$ and

$$D_{P_{n,i}}(\varphi_k, \varphi_{k+1}) \leqslant \operatorname{dist}_H^{D_{M_{k,i}}}\big(CS_i(\mathcal{V}_k), CS_i(\mathcal{V}_{k+1})\big)$$

for $m \leqslant k \leqslant n-1$. Consequently,

$$D_{P_{n,i}}(\varphi_m, \varphi_n) \leqslant \sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,i}}}\big(CS_i(\mathcal{V}_j), CS_i(\mathcal{V}_{j+1})\big), \quad 2 \leqslant k \leqslant n-1.$$

Similarly, for $\varphi_m \in CS_i(\mathcal{V}_m)$ we can find a $\varphi_n \in CS_i(\mathcal{V}_n)$ such that the inequality above holds. Thus by Proposition 6.1, we have

$$\operatorname{dist}_H^{D_{P_{n,i}}}\left(CS_i\left(\bigoplus_{j=1}^{m} \mathcal{V}_j\right), CS_i(\mathcal{V}_n)\right) \leqslant \sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,i}}}\big(CS_i(\mathcal{V}_j), CS_i(\mathcal{V}_{j+1})\big).$$

Proposition 6.3. *For $m < n$, we have*

$$\operatorname{dist}_H^{D_{P_{n,i}}}\left(CS_i\left(\bigoplus_{j=1}^{m} \mathcal{V}_j\right), CS_i\left(\bigoplus_{j=1}^{n} \mathcal{V}_j\right)\right) \leqslant \sum_{k=1}^{i}\sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,k}}}\big(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\big).$$

Proof. For $\varphi \in CS_i(\bigoplus_{j=1}^{m} \mathcal{V}_j)$, we can find a $\psi \in CS_i(\mathcal{V}_n) \subseteq CS_i(\bigoplus_{j=1}^{n} \mathcal{V}_j)$ such that

$$D_{P_{n,i}}(\varphi, \psi) \leqslant \sum_{k=1}^{i}\sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,k}}}\big(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\big)$$

from the discussion before the proposition. Suppose $\varphi \in CS_i(\bigoplus_{j=1}^{n} \mathcal{V}_j)$ and $\epsilon > 0$. For each $i \in \mathbb{N}$, $\bigcup_{j=1}^{n} CS_i(\mathcal{V}_j)$ is a BW-closed subset of $CS_i(\bigoplus_{j=1}^{n} \mathcal{V}_j)$, and $\gamma^*(\bigcup_{j=1}^{n} CS_i(\mathcal{V}_j))\gamma \subseteq \bigcup_{j=1}^{n} CS_k(\mathcal{V}_j)$ for all isometries $\gamma \in M_{i,k}$. From Proposition 6.1, the BW-closure $\overline{\mathrm{mco}}(\bigcup_{j=1}^{n} CS_i(\mathcal{V}_j))$ of $\mathrm{mco}(\bigcup_{j=1}^{n} CS_i(\mathcal{V}_j))$ is $\mathcal{CS}(\bigoplus_{j=1}^{n} \mathcal{V}_j)$, and so by Theorems 4.6 and 4.3 in [19], there exist $\varphi_k \in CS_{l_k}(\mathcal{V}_{j_k})$ and $\gamma_k \in M_{l_k,i}$ for $k = 1, 2, \ldots, s$, $1 \leqslant j_k \leqslant n$, $1 \leqslant l_k \leqslant i$ satisfying $\sum_{k=1}^{s} \gamma_k^* \gamma_k = 1_i$ such that

$$D_{P_{n,i}}\left(\varphi, \sum_{k=1}^{s} \gamma_k^* \varphi_k \gamma_k\right) < \epsilon.$$

For each φ_k, we can find $\psi_k \in CS_{l_k}(\bigoplus_{j=1}^{m} \mathcal{V}_j)$ so that

$$D_{P_{n,l_k}}(\varphi_k, \psi_k) \leqslant \sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,l_k}}}\big(CS_{l_k}(\mathcal{V}_j), CS_{l_k}(\mathcal{V}_{j+1})\big).$$

Thus

$$\begin{aligned} D_{P_{n,i}}\left(\varphi, \sum_{k=1}^{s} \gamma_k^* \psi_k \gamma\right) &\leqslant D_{P_{n,i}}\left(\varphi, \sum_{k=1}^{s} \gamma_k^* \varphi_k \gamma\right) + D_{P_{n,i}}\left(\sum_{k=1}^{s} \gamma_k^* \varphi_k \gamma, \sum_{k=1}^{s} \gamma_k^* \psi_k \gamma\right) \\ &< \epsilon + D_{P_{n,i}}\left(\sum_{k=1}^{s} \gamma_k^* \varphi_k \gamma, \sum_{k=1}^{s} \gamma_k^* \psi_k \gamma\right) \\ &\leqslant \epsilon + D_{P_{n,\sum_{k=1}^{s} l_k}}(\varphi_1 \oplus \cdots \oplus \varphi_s, \psi_1 \oplus \cdots \oplus \psi_s) \\ &= \epsilon + \max\left\{D_{P_{n,l_1}}(\varphi_1, \psi_1), \ldots, D_{P_{n,l_s}}(\varphi_s, \psi_s)\right\} \\ &\leqslant \epsilon + \sum_{k=1}^{i} \sum_{j=m}^{n-1} \operatorname{dist}_H^{D_{M_{j,k}}}\left(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\right), \end{aligned}$$

because $\mathcal{D}_{\mathcal{P}_n} = (D_{P_{n,k}})$ is a matrix metric (see [20, Example 5.2]). Since ϵ is arbitrary, we obtain the desired inequality. □

Now for $\varphi, \psi \in CS_i(\bigoplus_{j=1}^{n} \mathcal{V}_j)$, there are $\varphi_1, \psi_1 \in CS_i(\mathcal{V}_1)$ with

$$D_{P_{n,i}}(\varphi_1, \varphi) \leqslant \sum_{j=1}^{n-1} \operatorname{dist}_H^{D_{M_{j,i}}}\left(CS_i(\mathcal{V}_j), CS_i(\mathcal{V}_{j+1})\right)$$

and

$$D_{P_{n,i}}(\psi_1, \psi) \leqslant \sum_{j=1}^{n-1} \operatorname{dist}_H^{D_{M_{j,i}}}\left(CS_i(\mathcal{V}_j), CS_i(\mathcal{V}_{j+1})\right).$$

So

$$\begin{aligned} D_{P_{n,i}}(\varphi, \psi) &\leqslant D_{P_{n,i}}(\varphi, \varphi_1) + D_{P_{n,i}}(\varphi_1, \psi_1) + D_{P_{n,i}}(\psi_1, \psi) \\ &\leqslant \operatorname{diam}\left(CS_i(\mathcal{V}_1), D_{L_{1,i}}\right) + 2\sum_{j=1}^{n-1} \operatorname{dist}_H^{D_{M_{j,i}}}\left(CS_i(\mathcal{V}_j), CS_i(\mathcal{V}_{j+1})\right) \triangleq h_i, \end{aligned}$$

where $\operatorname{diam}(CS_i(\mathcal{V}_1), D_{L_{1,i}})$ is the diameter of $CS_i(\mathcal{V}_1)$ with respect to $D_{L_{1,i}}$.

By Propositions 5.2 and 3.8 in [21], we have

$$\|\tilde{d}_n\|_i^{\sim} \leqslant h_i P_{n,i}(d_n) \leqslant h_i Q_i(c) < h_i Q_i(c) + \epsilon,$$

where $\epsilon > 0$. So there is an $\alpha_{n,i} = [\alpha_{st}^{(n,i)}] \in M_i$ such that

$$\left\|d_n - \left[\alpha_{st}^{(n,i)}(1, \ldots, 1)\right]\right\|_i \leqslant h_i Q_i(c) + \epsilon, \quad n \in \mathbb{N}.$$

Set

$$G_{n,i} = \left\{\beta_{n,i} = \left[\beta_{st}^{(n,i)}\right] \in M_i\colon \left\|d_n - \left[\beta_{st}^{(n,i)}(1, \ldots, 1)\right]\right\|_i \leqslant h_i Q_i(c) + \epsilon\right\}.$$

Then $G_{n,i}$ is a non-empty closed bounded subset of M_i. Clearly, $G_{n+1,i} \subseteq G_{n,i}$. So there exists a $\beta_0 \in \bigcap_{n=1}^{\infty} G_{n,i}$. We have

$$\|d_n\|_i \leqslant \|\beta_0\| + h_i Q_i(c) + \epsilon, \quad n \in \mathbb{N}.$$

Thus $c \in M_i(\bigoplus_{j\in\mathbb{N}} \mathcal{V}_j)$, and we obtain

Proposition 6.4. *For $n \in \mathbb{N}$, $\mathcal{Q}$ induces $\mathcal{P}_n$ via the evident projection.*

Theorem 6.5. *The metric space $\mathfrak{R}$ of complete isometry classes of quantized metric spaces, with the metric dist_{NC}, is complete.*

Proof. Let $\{(\mathcal{V}_n, \mathcal{L}_n)\}$ be a sequence in $\mathfrak{R}$ which is Cauchy with respect to the quantized Gromov–Hausdorff distance dist_{NC}. To show that $\{(\mathcal{V}_n, \mathcal{L}_n)\}$ converges it suffices to show that a subsequence converges. Since $\{(\mathcal{V}_n, \mathcal{L}_n)\}$ is Cauchy, we can choose a subsequence, still denoted by $\{(\mathcal{V}_n, \mathcal{L}_n)\}$, such that

$$\mathrm{dist}_{NC}(\mathcal{V}_n, \mathcal{V}_{n+1}) < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. By definition, there exists $\mathcal{M}_n = (M_{n,k}) \in \mathcal{M}(\mathcal{L}_n, \mathcal{L}_{n+1})$ with

$$\sup_{k\in\mathbb{N}}\left\{k^{-2}\,\mathrm{dist}_H^{D_{M_{n,k}}}\big(CS_k(\mathcal{V}_n), CS_k(\mathcal{V}_{n+1})\big)\right\} < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty}\sup_{k\in\mathbb{N}}\left\{k^{-2}\,\mathrm{dist}_H^{D_{M_{n,k}}}\big(CS_k(\mathcal{V}_n), CS_k(\mathcal{V}_{n+1})\big)\right\} < +\infty.$$

Let $\epsilon > 0$ be given. Then there is an $m \in \mathbb{N}$ such that

$$\sum_{n=m}^{\infty}\sup_{k\in\mathbb{N}}\left\{k^{-2}\,\mathrm{dist}_H^{D_{M_{n,k}}}\big(CS_k(\mathcal{V}_n), CS_k(\mathcal{V}_{n+1})\big)\right\} < \epsilon.$$

By Propositions 6.2–6.4, we have

$$\begin{aligned}
&\mathrm{dist}_H^{D_{Q_i}}\left(CS_i\left(\bigoplus_{j=1}^{m}\mathcal{V}_j\right), CS_i\left(\bigoplus_{j=1}^{n}\mathcal{V}_j\right)\right)\\
&\quad\leqslant \sum_{k=1}^{i}\sum_{j=m}^{n-1}\mathrm{dist}_H^{D_{M_{j,k}}}\big(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\big)\\
&\quad\leqslant \sum_{k=1}^{i}k^2\sum_{j=m}^{n-1}\sup_{k\in\mathbb{N}}\left\{k^{-2}\,\mathrm{dist}_H^{D_{M_{j,k}}}\big(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\big)\right\} < \left(\sum_{k=1}^{i}k^2\right)\epsilon
\end{aligned}$$

for $n > m$. This says that $CS_i(\bigoplus_{j=1}^m \mathcal{V}_j)$ is $(\sum_{k=1}^i k^2)\epsilon$-dense for D_{Q_i} in Z_i. But $CS_i(\bigoplus_{j=1}^m \mathcal{V}_j)$ is BW-compact for the topology from $D_{Q_i} = D_{P_{m,i}}$ by Proposition 6.2. Thus $CS_i(\bigoplus_{j=1}^m \mathcal{V}_j)$ is totally bounded for D_{Q_i}, and so Z_i is totally bounded for D_{Q_i}. Let $\hat{\mathcal{Z}} = (\hat{Z}_n)$ be the completion of $\mathcal{Z}$ for $\mathcal{D}_Q$. We let $\mathcal{D}_Q$ denote also the extension of $\mathcal{D}_Q$ to $\hat{\mathcal{Z}}$. Then $\hat{\mathcal{Z}}$ is a compact matrix convex set.

For $\{a_i\} \in M_n(\mathcal{E}_1)$, we have

$$\left\langle\!\!\left\langle \sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j, \{a_i\} \right\rangle\!\!\right\rangle = \sum_{j=1}^m (\gamma_j \otimes 1_n)^* \langle\!\langle \varphi_j, \{a_i\} \rangle\!\rangle (\gamma_j \otimes 1_n)$$

and

$$\begin{aligned} &\left\| \left\langle\!\!\left\langle \sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j, \{a_i\} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \sum_{k=1}^p \lambda_k^* \psi_k \lambda_k, \{a_i\} \right\rangle\!\!\right\rangle \right\| \\ &\quad \leqslant L_{D_{Q_n}}\big(\{a_i\}\big) D_{Q_r}\left(\sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j, \sum_{k=1}^p \lambda_k^* \psi_k \lambda_k \right) \\ &\quad \leqslant Q_n\big(\{a_i\}\big) D_{Q_r}\left(\sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j, \sum_{k=1}^p \lambda_k^* \psi_k \lambda_k \right), \end{aligned}$$

where $\varphi_j \in CS_{n_j}(\mathcal{V}_{q_j})$, $\psi_k \in CS_{m_k}(\mathcal{V}_{l_k})$, and $\gamma_j \in M_{n_j,r}$, and $\lambda_k \in M_{m_k,r}$ satisfying $\sum_{j=1}^m \gamma_j^* \times \gamma_j = 1_r$ and $\sum_{k=1}^p \lambda_k^* \lambda_k = 1_r$. So the map $\Phi : \mathcal{E}_1 \mapsto A(\mathcal{Z})$, given by

$$\big(\Phi\big(\{a_i\}\big)\big)\left(\sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j \right) = \sum_{j=1}^m \gamma_j^* \varphi_j (a_{q_j}) \gamma_j$$

for $\{a_i\} \in \mathcal{E}_1$, $\varphi_j \in CS_{n_j}(\mathcal{V}_{q_j})$ and $\gamma_j \in M_{n_j,r}$ satisfying $\sum_{j=1}^m \gamma_j^* \gamma_j = 1_r$, is well defined and $\Phi(\{a_i\})$ can be extended to an element $\widehat{\Phi(\{a_i\})} \in A(\hat{\mathcal{Z}})$. Moreover, if $\{a_i\} \geqslant 0$ in $\mathcal{E}_1$ then $\widehat{\Phi(\{a_i\})} \geqslant 0$ in $A(\hat{\mathcal{Z}})$ and $\widehat{\Phi(\{1\})} = \mathbf{I}$. Thus $\mathcal{E}_1$ can be regarded as a matrix order unit subspace of $A(\hat{\mathcal{Z}})$.

Define the map $\Psi_r : \hat{Z}_r \mapsto CS_r(\mathcal{E}_1)$, $r \in \mathbb{N}$, by

$$\Psi_r(z)\big(\{a_i\}\big) = \widehat{\Psi\big(\{a_i\}\big)}(z)$$

for $z \in \hat{Z}_r$ and $\{a_i\} \in \mathcal{E}_1$. Clearly, Ψ is continuous. For $z = \sum_{j=1}^m \gamma_j^* \varphi_j \gamma_j \in Z_r$ with $\varphi_j \in CS_{n_j}(\mathcal{V}_{q_j})$, $\gamma_j \in M_{n_j,r}$ satisfying $\sum_{j=1}^m \gamma_j^* \gamma_j = 1_r$, we have

$$\Psi_r(z)\big(\{a_i\}\big) = \widehat{\Psi\big(\{a_i\}\big)}(z) = \Psi\big(\{a_i\}\big)(z) = z\big(\{a_i\}\big),$$

that is, $\Psi_r(z) = z$. Since Z_r is dense in $CS_r(\mathcal{E}_1)$ and $\hat{Z}_r$ is compact, we obtain that $\Psi_r(\hat{Z}_r) = CS_r(\mathcal{E}_1)$.

If $z_1, z_2 \in \hat{Z}_r$ with $z_1 \neq z_2$ and $k = D_{Q_r}(z_1, z_2)$, we can find $y_1, y_2 \in Z_r$ such that $D_{Q_r}(z_i, y_i) < k/4$, $i = 1, 2$. Thus $D_{Q_r}(y_1, y_2) > k/2$. So we can find $\{w_i\} \in M_r(\mathcal{E}_1)$ with $Q_r(\{w_i\}) \leqslant 1$ and $\|\langle\langle \{w_i\}, y_1\rangle\rangle - \langle\langle \{w_i\}, y_2\rangle\rangle\| > k/2$. But $L_{D_{Q_r}}(\widehat{\Phi_r(\{w_i\}}) \leqslant 1$ so that

$$\left\|\left\langle\!\left\langle \widehat{\Phi_r(\{w_i\})}, z_i\right\rangle\!\right\rangle - \left\langle\!\left\langle \widehat{\Phi_r(\{w_i\})}, y_i\right\rangle\!\right\rangle\right\| < \frac{k}{4}, \quad i = 1, 2.$$

Thus $\|\langle\langle \widehat{\Phi_r(\{w_i\}}), z_1\rangle\rangle - \langle\langle \widehat{\Phi_r(\{w_i\}}), z_2\rangle\rangle\| > 0$. Therefore, Ψ_r is injective. So Ψ_r is a homeomorphism of $\hat{Z}_r$ onto $CS_r(\mathcal{E}_1)$ for $r \in \mathbb{N}$. From this we see that the $\mathcal{D}_{\mathcal{Q}}$-topology on $\mathcal{CS}(\mathcal{E}_1)$ agrees with the BW-topology. Hence $\mathcal{Q}$ is a matrix Lip-norm on $\mathcal{E}_1$.

By Propositions 6.2 and 6.4, we obtain

$$\sum_{n=1}^{\infty} \sup_{k\in\mathbb{N}} \left\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_n), CS_k(\mathcal{V}_{n+1})\right)\right\} < +\infty,$$

which indicate that, for $k \in \mathbb{N}$, $\{CS_k(\mathcal{V}_n)\}$ is a Cauchy sequence for $\operatorname{dist}_H^{D_{Q_k}}$, and has a limit $K_k \subseteq CS_k(\mathcal{E}_1)$. Clearly $\mathcal{K} = (K_k)$ is a compact matrix convex set.

Because $\mathcal{E}_1$ is completely order isomorphic to a dense subspace of $A(\mathcal{CS}(\mathcal{E}_1))$ [20, Proposition 6.1(1)], we can view $\mathcal{E}_1$ as a dense subspace of $A(\mathcal{CS}(\mathcal{E}_1))$. Let ϕ be the map which restricts the elements of $A(\mathcal{CS}(\mathcal{E}_1))$ to $\mathcal{K}$ and $\mathcal{V} = \phi(\mathcal{E}_1)$. Then $(\mathcal{V}, \mathcal{Q}_{\mathcal{V}})$ is a quantized metric space.

Given $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that

$$\sum_{j=n}^{\infty} \sup_{k\in\mathbb{N}} \left\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\right)\right\} < \epsilon, \quad n \geqslant N.$$

For $k, p \in \mathbb{N}$, we have

$$\begin{aligned} k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_n), CS_k(\mathcal{V}_{n+p})\right) &\leqslant \sum_{j=n}^{n+p-1} k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\right) \\ &\leqslant \sum_{j=n}^{\infty} \sup_{k\in\mathbb{N}} \left\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_j), CS_k(\mathcal{V}_{j+1})\right)\right\} \\ &< \epsilon \end{aligned}$$

for $n \geqslant \mathbb{N}$. Letting $p \to +\infty$, we obtain

$$k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_n), K_k\right) \leqslant \epsilon$$

for $k \in \mathbb{N}$, and so $\sup_{k\in\mathbb{N}}\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}(CS_k(\mathcal{V}_n), K_k)\} \leqslant \epsilon$. By Proposition 4.8, for $n \geqslant N$ we have

$$\operatorname{dist}_{NC}(\mathcal{V}_n, \mathcal{V}) \leqslant \sup_{k\in\mathbb{N}} \left\{k^{-2} \operatorname{dist}_H^{D_{Q_k}}\left(CS_k(\mathcal{V}_n), K_k\right)\right\} \leqslant \epsilon.$$

Therefore, $\lim_{n\to\infty} \operatorname{dist}_{NC}(\mathcal{V}_n, \mathcal{V}) = 0$. □

7. Matrix approximability

In this section, we establish a matrix approximability theorem for 1-exact matrix order unit spaces.

Lemma 7.1. *Let $(\mathcal{V}, \mathcal{L})$ be a quantized metric space and let $x = [x_{st}] \in M_k(\mathcal{V})$, $x_{st} = x_{st}^{(1)} + i x_{st}^{(2)}$ with $(x_{st}^{(p)})^* = x_{st}^{(p)}$ for $p = 1, 2$, $s, t = 1, 2, \ldots, k$. Suppose $\lambda_{st}^{(p)} \in \sigma(x_{st}^{(p)})$ for $p = 1, 2$, $s, t = 1, 2, \ldots, k$. Then*

$$\left\| x - \left[\left(\lambda_{st}^{(1)} + i\lambda_{st}^{(2)}\right)1\right]\right\|_k \leqslant 2k^2 L_k(x)\, \mathrm{diam}(\mathcal{V}, \mathcal{L}).$$

Proof. By Proposition 2.11 in [8], we have

$$\begin{aligned}
\left\| x - \left[\left(\lambda_{st}^{(1)} + i\lambda_{st}^{(2)}\right)1\right]\right\|_k &\leqslant \left\|\left[x_{st}^{(1)} - \lambda_{st}^{(1)}1\right]\right\|_k + \left\|\left[x_{st}^{(2)} - \lambda_{st}^{(2)}1\right]\right\|_k \\
&\leqslant \sum_{p=1}^{2}\sum_{s,t=1}^{k} \left\| x_{st}^{(p)} - \lambda_{st}^{(p)}1\right\|_1 \\
&\leqslant \sum_{p=1}^{2}\sum_{s,t=1}^{k} L_1\left(x_{st}^{(p)}\right)\mathrm{diam}(\mathcal{V}, \mathcal{L}) \\
&\leqslant \sum_{s,t=1}^{k} 2L_k(x)\mathrm{diam}(\mathcal{V}, \mathcal{L}) \\
&= 2k^2 L_k(x)\mathrm{diam}(\mathcal{V}, \mathcal{L}). \qquad \square
\end{aligned}$$

An operator space $\mathcal{X}$ is said to be 1-*exact* if for every finite-dimensional subspace $\mathcal{E} \subseteq \mathcal{X}$ and $\lambda > 1$ there is an isomorphism α from $\mathcal{E}$ onto a subspace of a matrix algebra such that $\|\alpha\|_{cb}\|\alpha^{-1}\|_{cb} \leqslant \lambda$. A matrix order unit space $(\mathcal{V}, 1)$ is said to be 1-*exact* if it is 1-exact as an operator space.

Theorem 7.2. *Let $(\mathcal{V}, \mathcal{L})$ be a quantized metric space. If $\mathcal{V}$ is* 1-exact*, then for every $\epsilon > 0$, there is a quantized metric space $(M_{n_{\lambda_\epsilon}}, \mathcal{N})$ such that*

$$\mathrm{dist}_{NC}(\mathcal{V}, M_{n_{\lambda_\epsilon}}) < \epsilon.$$

Proof. Since $\mathcal{V}$ is 1-exact, by Lemma 5.1 in [9] there is a unital complete order embedding $\iota : \mathcal{V} \mapsto \mathcal{B}(\mathcal{H})$ and a net

$$\mathcal{V} \xrightarrow{\varphi_\lambda} M_{n_\lambda} \xrightarrow{\psi_\lambda} \mathcal{B}(\mathcal{H})$$

of unital completely positive mappings through matrix algebras such that $\psi_\lambda \circ \varphi_\lambda$ converges pointwise to ι. Given $\epsilon > 0$. By Lemma 7.2, we have

$$L_1^1 = B_1^{2\,\mathrm{diam}(\mathcal{V},\mathcal{L})} + \mathbb{C}1,$$

where

$$B_1^{2\operatorname{diam}(\mathcal{V},\mathcal{L})} = \big\{a \in \mathcal{V}\colon\ L_1(a) \leqslant 1,\ \|a\|_1 \leqslant 2\operatorname{diam}(\mathcal{V},\mathcal{L})\big\}.$$

From Proposition 7.5 in [20], $B_1^{2\operatorname{diam}(\mathcal{V},\mathcal{L})}$ is totally bounded for $\|\cdot\|_1$. So there is a λ_ϵ such that

$$\big\|(\psi_{\lambda_\epsilon} \circ \varphi_{\lambda_\epsilon})(x) - x\big\| < \frac{\epsilon}{5}, \quad x \in L_1^1.$$

Denote $\mathcal{W} = \varphi_{\lambda_\epsilon}(\mathcal{V})$ and $Q_k(y) = \inf\{L_k(x)\colon (\varphi_{\lambda_\epsilon})_k(x) = y\}$ for $y \in M_k(\mathcal{W})$ and $k \in \mathbb{N}$.

We define

$$N_k(x,y) = \frac{5}{\epsilon}\big\|(\varphi_{\lambda_\epsilon})_k(x) - y\big\|_k, \quad (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W}),\ k \in \mathbb{N}.$$

It is clear that $\mathcal{N} = (N_k)$ is a matrix seminorm on $\mathcal{V} \oplus \mathcal{W}$ and satisfies the conditions (1)–(3) of Definition 4.1. For $x \in M_k(\mathcal{V})$ and $\delta > 0$, we can choose $y = (\varphi_{\lambda_\epsilon})_k(x)$. Then

$$\max\big\{Q_k(y), N_k(x,y)\big\} = Q_k(y) \leqslant L_k(x) \leqslant L_k(x) + \delta.$$

For $y \in M_k(\mathcal{W})$ and $\delta > 0$, we can take $x \in M_k(\mathcal{V})$ such that $y = (\varphi_{\lambda_\epsilon})_k(x)$ and $L_k(x) \leqslant Q_k(y) + \delta$. Then

$$\max\big\{L_k(x), N_k(x,y)\big\} = L_k(x) \leqslant Q_k(y) + \delta.$$

So $\mathcal{N}$ is a matrix bridge between $(\mathcal{V},\mathcal{L})$ and $(\mathcal{W},\mathcal{Q})$. Define

$$P_k(x,y) = \max\big\{L_k(x), Q_k(y), N_k(x,y)\big\}, \quad (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W}),\ k \in \mathbb{N}.$$

Then $\mathcal{P} = (P_k) \in \mathcal{M}(\mathcal{L},\mathcal{Q})$ by Proposition 4.3. If $f \in CS_k(\mathcal{W})$, we have $f \circ \varphi_{\lambda_\epsilon} \in CS_k(\mathcal{V})$ and

$$\begin{aligned}
&D_{P_k}(f, f \circ \varphi_{\lambda_\epsilon}) \\
&\quad = \sup\big\{\big\|\langle\langle f, y\rangle\rangle - \langle\langle f \circ \varphi_{\lambda_\epsilon}, x\rangle\rangle\big\|\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \\
&\quad = \sup\big\{\big\|\big\langle\big\langle f, y - (\varphi_{\lambda_\epsilon})_k(x)\big\rangle\big\rangle\big\|\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \\
&\quad \leqslant \sup\big\{\big\|y - (\varphi_{\lambda_\epsilon})_k(x)\big\|_k\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \leqslant \frac{\epsilon}{5}.
\end{aligned}$$

On the other hand, if $g \in CS_k(\mathcal{V})$, g can be extended to a $\bar{g} \in CS_k(\mathcal{B}(\mathcal{H}))$ by Arveson's extension theorem. We have $\bar{g} \circ \psi_{\lambda_\epsilon} \in CS_k(\mathcal{W})$ and

$$\begin{aligned}
&D_{P_k}(g, \bar{g} \circ \psi_{\lambda_\epsilon}) \\
&\quad = \sup\big\{\big\|\langle\langle g, x\rangle\rangle - \langle\langle \bar{g} \circ \psi_{\lambda_\epsilon}, y\rangle\rangle\big\|\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \\
&\quad = \sup\big\{\big\|\big\langle\big\langle \bar{g}, x - (\psi_{\lambda_\epsilon})_k(y)\big\rangle\big\rangle\big\|\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \\
&\quad \leqslant \sup\big\{\big\|\big\langle\big\langle \bar{g}, x - (\psi_{\lambda_\epsilon} \circ \varphi_{\lambda_\epsilon})_k(x)\big\rangle\big\rangle\big\| + \big\|\big\langle\big\langle \bar{g} \circ \psi_{\lambda_\epsilon}, (\varphi_{\lambda_\epsilon})_k(x) - y\big\rangle\big\rangle\big\|\colon \\
&\qquad\quad P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\} \\
&\quad \leqslant \sup\big\{\big\|x - (\psi_{\lambda_\epsilon} \circ \varphi_{\lambda_\epsilon})_k(x)\big\|_k + \big\|(\varphi_{\lambda_\epsilon})_k(x) - y\big\|_k\colon P_k(x,y) \leqslant 1,\ (x,y) \in M_k(\mathcal{V} \oplus \mathcal{W})\big\}
\end{aligned}$$

$$\leqslant \sup\left\{ \sum_{i,j=1}^{k} \left\| x_{ij} - (\psi_{\lambda_\epsilon} \circ \varphi_{\lambda_\epsilon})(x_{ij}) \right\|_1 \colon P_k(x, y) \leqslant 1,\ (x, y) \in M_k(\mathcal{V} \oplus \mathcal{W}) \right\} + \frac{\epsilon}{5}$$
$$\leqslant \left(k^2 + 1\right)\frac{\epsilon}{5}.$$

So we obtain that $\mathrm{dist}_{NC}(\mathcal{V}, \mathcal{W}) < \epsilon/2$.

Since $\mathcal{W} \subseteq M_{n_{\lambda_\epsilon}}$ is finite-dimensional, $K = Q_1^1$ is a normed-closed (and hence weakly closed) absolutely convex set in $M_{n_{\lambda_\epsilon}}$, and $\mathcal{Q}^1 = (Q_k^1)$ is a normed-closed (and hence weakly closed) absolutely matrix convex set in $M_{n_{\lambda_\epsilon}}$. Then for the corresponding matrix seminorm $\check{\mathcal{R}} = (\check{R}_k)$ of the maximal envelope $\check{\mathcal{K}}$ of K in $M_{n_{\lambda_\epsilon}}$ (see Example 3.4), we have

$$\check{R}_1^1 = Q_1^1, \qquad \check{R}_k\big|_{M_k(\mathcal{W})} \leqslant Q_k\big|_{M_k(\mathcal{W})}, \quad k \in \mathbb{N}$$

(see [4, p. 181]). It is clear that $\check{\mathcal{R}}$ is a matrix Lipschitz seminorm. Since the image of $Q_1^1 = \check{R}_1^1$ in $\mathcal{W}/(\mathbb{C}1)$ is totally bounded for $\|\cdot\|^\sim$ and $\mathcal{W} \subseteq M_{n_{\lambda_\epsilon}}$, the image of $\check{R}_1^1$ in $M_{n_{\lambda_\epsilon}}/(\mathbb{C}1)$ is totally bounded for $\|\cdot\|^\sim$. By Theorem 5.3 in [21], $\mathcal{D}_{\mathcal{R}}$-topology on $\mathcal{CS}(M_{n_{\lambda_\epsilon}})$ agrees with the BW-topology. So $\check{\mathcal{R}}$ is a matrix Lip-norm on $(M_{n_{\lambda_\epsilon}}, 1)$. By Lemma 3.2.3 in [1], there is a (real linear) projection T from $(M_{n_{\lambda_\epsilon}})_{sa}$ onto $(\mathcal{W})_{sa}$ with $\|T\| \leqslant n_{\lambda_\epsilon}$. We define $S : M_{n_{\lambda_\epsilon}} \mapsto \mathcal{W}$ by $S(a + ib) = T(a) + iT(b)$ for $a, b \in (M_{n_{\lambda_\epsilon}})_{sa}$. Then S is a bounded linear mapping with $\|S\| \leqslant 2n_{\lambda_\epsilon}$. Define

$$N_k(x) = \max\left\{ Q_k\big(S_k(x)\big), \check{R}_k(x), \frac{4}{\epsilon}\left\|x - S_k(x)\right\|_k \right\}, \quad x \in M_k(M_{n_{\lambda_\epsilon}}),\ k \in \mathbb{N}.$$

It is clear that $\mathcal{N} = (N_k)$ is a matrix Lip-norm on $M_{n_{\lambda_\epsilon}}$ since $\check{R}_k \leqslant N_k$ for all $k \in \mathbb{N}$ and $\check{\mathcal{R}}$ is a matrix Lip-norm. And for $y \in M_k(\mathcal{W})$, we have

$$N_k(y) = \max\left\{ Q_k\big(S_k(y)\big), \check{R}_k(y), \frac{4}{\epsilon}\left\|y - S_k(y)\right\|_k \right\} = \max\big\{Q_k(y), \check{R}_k(y)\big\} = Q_k(y).$$

Define

$$X_k(x, y) = \frac{4}{\epsilon}\left\|y - S_k(x)\right\|_k, \quad (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ k \in \mathbb{N}.$$

It is clear that $\mathcal{N} = (N_k)$ is a matrix seminorm on $M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}$ and satisfies the conditions (1)–(3) of Definition 4.1. For $x \in M_k(M_{n_{\lambda_\epsilon}})$ and $\delta > 0$, we choose $y = S_k(x)$. Then we have

$$\max\big\{Q_k(y), X_k(x, y)\big\} = Q_k\big(S_k(x)\big) \leqslant N_k(x) \leqslant N_k(x) + \delta.$$

For $y \in M_k(\mathcal{W})$ and $\delta > 0$, we choose $x = y$. Then we have

$$\max\big\{N_k(x), X_k(x, y)\big\} = N_k(y) = Q_k(y) \leqslant Q_k(x) + \delta.$$

So $\mathcal{X} = (X_k)$ is a matrix bridge between $(M_k(M_{n_{\lambda_\epsilon}}), \mathcal{N})$ and $(\mathcal{W}, \mathcal{Q})$. Define

$$Y_k(x, y) = \max\big\{N_k(x), Q_k(y), X_k(x, y)\big\}, \quad (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ k \in \mathbb{N}.$$

Then $\mathcal{Y} = (Y_k) \in \mathcal{M}(\mathcal{N}, \mathcal{Q})$ by Proposition 4.3. For $\varphi \in CS_k(M_{n_{\lambda_\epsilon}})$, $\psi = \varphi|_{\mathcal{W}} \in CS_k(\mathcal{W})$ and

$$\begin{aligned}
D_{Y_k}(\varphi, \psi) &= \sup\bigl\{ \| \langle\langle \varphi, x\rangle\rangle - \langle\langle \psi, y\rangle\rangle \| \colon (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \sup\bigl\{ \| \langle\langle \varphi, x\rangle\rangle - \bigl\langle\bigl\langle \varphi, S_k(x)\bigr\rangle\bigr\rangle + \bigl\langle\bigl\langle \varphi, S_k(x)\bigr\rangle\bigr\rangle - \langle\langle \varphi, y\rangle\rangle \| \colon \\
&\qquad (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \sup\bigl\{ \bigl\| x - S_k(x)\bigr\|_k + \bigl\| S_k(x) - y\bigr\|_k \colon (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \frac{\epsilon}{2}.
\end{aligned}$$

For $\psi \in CS_k(\mathcal{W})$, ψ can be extended to a $\varphi \in CS_k(M_{n_{\lambda_\epsilon}})$ by Arveson's extension theorem. We have

$$\begin{aligned}
D_{Y_k}(\varphi, \psi) &= \sup\bigl\{ \| \langle\langle \varphi, x\rangle\rangle - \langle\langle \psi, y\rangle\rangle \| \colon (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \sup\bigl\{ \| \langle\langle \varphi, x\rangle\rangle - \bigl\langle\bigl\langle \varphi, S_k(x)\bigr\rangle\bigr\rangle + \bigl\langle\bigl\langle \varphi, S_k(x)\bigr\rangle\bigr\rangle - \langle\langle \varphi, y\rangle\rangle \| \colon \\
&\qquad (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \sup\bigl\{ \bigl\| x - S_k(x)\bigr\| + \bigl\| S_k(x) - y\bigr\| \colon (x, y) \in M_k(M_{n_{\lambda_\epsilon}} \oplus \mathcal{W}),\ Y_k(x, y) \leqslant 1\bigr\} \\
&\leqslant \frac{\epsilon}{2}.
\end{aligned}$$

So $\mathrm{dist}_{NC}(\mathcal{W}, M_{n_{\lambda_\epsilon}}) < \epsilon/2$. Therefore,

$$\mathrm{dist}_{NC}(\mathcal{V}, M_{n_{\lambda_\epsilon}}) \leqslant \mathrm{dist}_{NC}(\mathcal{V}, \mathcal{W}) + \mathrm{dist}_{NC}(\mathcal{W}, M_{n_{\lambda_\epsilon}}) < \epsilon. \qquad \square$$

8. Sphere as the limit of matrix algebras

Let G be a connected compact semisimple Lie group with a continuous length function l on G, which satisfies the additional condition $l(xyx^{-1}) = l(y)$ for all $x, y \in G$. Fix an irreducible unitary representation $(U, \mathcal{H})$ of G. Then $(U, \mathcal{H})$ have a highest weight vector ξ, of norm 1, unique up to a scalar multiple. Let P be the rank-one projection for ξ. Denote by H the stability subgroup for P under α. For any $n \in \mathbb{N}$, we form the nth inner tensor power $(U^{\otimes n}, \mathcal{H}^{\otimes n})$ of $(U, \mathcal{H})$. Let $(U^{(n)}, \mathcal{H}^{(n)})$ denote the subrepresentation generated by $\xi^{(n)} = \xi^{\otimes n}$. Then $(U^{(n)}, \mathcal{H}^{(n)})$ is irreducible with $\xi^{(n)}$ as highest weight vector. We let $\mathcal{B}^{(n)} = \mathcal{B}(\mathcal{H}^{(n)})$. The action of G on $\mathcal{B}^{(n)}$ by conjugation by $U^{(n)}$ is denoted by $\alpha^{(n)}$. We let λ denote the action of G on G/H, and so on $\mathcal{A} = C(G/H)$, by left-translation. We denote the corresponding Lip-norm for $\alpha^{(n)}$ and l on $\mathcal{B}^{(n)}$ by $L^{(n)}$, that is,

$$L^{(n)}(T) = \sup\left\{ \frac{\|\alpha_x^{(n)}(T) - T\|}{l(x)} \colon x \neq e,\ x \in G \right\}, \quad T \in \mathcal{B}^{(n)},$$

and we denote the Lip-norm for λ and l on $\mathcal{A}$ by L, that is,

$$L(f) = \sup\left\{ \frac{\|\lambda_x(f) - f\|_\infty}{l(x)} \colon x \neq e,\ x \in G \right\}, \quad f \in \mathcal{A},$$

here we view $C(G/H)$ as a subalgebra of $C(G)$. By Theorem 3.2 in [18], the quantum metric spaces $(\mathcal{B}^{(n)}, L^{(n)})$ converge to $(\mathcal{A}, L)$ for quantum Gromov–Hausdorff distance as n goes to ∞. In this section, a more general statement is established.

Let $\|\cdot\|_\infty = (\|\cdot\|_{\infty,k})$ be the matrix norm on $\mathcal{A}$. Set $\mathcal{L}^{(n)} = (L_k^{(n)})$, where

$$L_k^{(n)}(T) = \sup\left\{\frac{\|[\alpha_x^{(n)}(T_{ij}) - T_{ij}]\|}{l(x)}\colon x \neq e,\ x \in G\right\}, \quad T = [T_{ij}] \in M_k\big(\mathcal{B}^{(n)}\big),\ k \in \mathbb{N},$$

and $\mathcal{L} = (L_k)$, where

$$L_k(f) = \sup\left\{\frac{\|[\lambda_x(f_{ij}) - f_{ij}]\|_{\infty,k}}{l(x)}\colon x \neq e,\ x \in G\right\}, \quad f = [f_{ij}] \in M_k(\mathcal{A}),\ k \in \mathbb{N}.$$

Then $(\mathcal{B}^{(n)}, \mathcal{L}^{(n)})$ and $(\mathcal{A}, \mathcal{L})$ are quantized metric spaces for all $n \in \mathbb{N}$ [20, Example 6.5]. As in [18], we will not restrict $\mathcal{L}$ to the Lipschitz functions. Let $P^{(n)}$ denote the rank-one projection for $\xi^{(n)}$. We denote the corresponding Berezin symbol mapping from $\mathcal{B}^{(n)}$ to $\mathcal{A}$ by $\sigma^{(n)}$. Then $\sigma^{(n)}$ is unital, positive, norm-nonincreasing and $\alpha^{(n)}$-λ-equivariant (see [18, p. 73]). For $k \in \mathbb{N}$ and $T = [T_{ij}] \in M_k(\mathcal{B}^{(n)})$, define

$$\sigma_T^{(n)}(x) = \big[\sigma_{T_{ij}}^{(n)}(x)\big], \quad x \in G.$$

For $\epsilon > 0$, define

$$N_k(f, T) = \epsilon^{-1}\big\|f - \sigma_T^{(n)}\big\|_{\infty,k}, \quad f \in M_k(\mathcal{A}),\ T \in M_k\big(\mathcal{B}^{(n)}\big),$$

and denote $\mathcal{N} = (N_k)$.

Lemma 8.1. *For any $T \in M_k(\mathcal{B}^{(n)})$, we have*

$$L_k\big(\sigma_T^{(n)}\big) < L_k^{(n)}(T) + \epsilon.$$

Proof. Since $\sigma^{(n)}$ is a unital positive mapping from $\mathcal{B}^{(n)}$ to $\mathcal{A}$, $\sigma^{(n)}$ is unital completely positive and hence $\|\sigma^{(n)}\|_{cb} = 1$ [12, Theorem 3.8 and Proposition 3.5]. So we have

$$\begin{aligned}
L_k\big(\sigma_T^{(n)}\big) &= \sup\left\{\frac{\|[\lambda_x(\sigma_{T_{ij}}^{(n)}) - \sigma_{T_{ij}}^{(n)}]\|_{\infty,k}}{l(x)}\colon x \neq e,\ x \in G\right\} \\
&= \sup\left\{\frac{\|[\sigma^{(n)}_{(\alpha_x^{(n)}(T_{ij}) - T_{ij})}]\|_{\infty,k}}{l(x)}\colon x \neq e,\ x \in G\right\} \\
&\leqslant \sup\left\{\frac{\|[\alpha_x^{(n)}(T_{ij}) - T_{ij}]\|}{l(x)}\colon x \neq e,\ x \in G\right\} \\
&= L_k^{(n)}(T) < L_k^{(n)}(T) + \epsilon
\end{aligned}$$

by the $\alpha^{(n)}$-λ-equivariation of $\sigma^{(n)}$. □

Put on $\mathcal{A}$ the inner product from $L^2(G/H)$, while on $\mathcal{B}^{(n)}$ its Hilbert–Schmidt inner product. Then the mapping $\sigma^{(n)}$ from $\mathcal{B}^{(n)}$ to $\mathcal{A}$ has an adjoint operator $\breve{\sigma}^{(n)}$ from $\mathcal{A}$ to $\mathcal{B}^{(n)}$. For any $T \in \mathcal{B}^{(n)}$, a function $f \in \mathcal{A}$ such that $\breve{\sigma}_f^{(n)} = T$ is called a Berezin contravariant symbol for T. Moreover, $\breve{\sigma}^{(n)}$ is unital, positive, norm-nonincreasing, and λ-$\alpha^{(n)}$-equivariant (see [18, p. 75]). From Theorem 3.10 and Proposition 3.5 in [12], $\breve{\sigma}^{(n)}$ is unital completely positive and $\|\breve{\sigma}^{(n)}\|_{cb} = 1$. So by the same argument as in the proof of Lemma 8.1, we obtain:

Lemma 8.2. *For any $f = [f_{ij}] \in M_k(\mathcal{A})$, we have*

$$L_k^{(n)}\big(\breve{\sigma}_f^{(n)}\big) < L_k(f) + \epsilon,$$

where $\breve{\sigma}_f^{(n)} = [\breve{\sigma}_{f_{ij}}^{(n)}] \in M_k(\mathcal{B}^{(n)})$.

Denote

$$D_{L_k}(\varphi, \psi) = \sup\big\{\big\|\langle\langle f, \varphi\rangle\rangle - \langle\langle f, \psi\rangle\rangle\big\|:\ L_r(f) \leqslant 1,\ f \in M_r(\mathcal{A}),\ r \in \mathbb{N}\big\},$$

for $\varphi, \psi \in CS_k(\mathcal{A}),\ k \in \mathbb{N}$, and

$$h_{P^{(n)}}(x) = d^{(n)}\tau^{(n)}\big(P^{(n)}\alpha_x^{(n)}\big(P^{(n)}\big)\big), \quad x \in G/H,$$

where $\tau^{(n)}$ denotes the usual (un-normalized) trace on $\mathcal{B}^{(n)}$ and $d^{(n)}$ is the dimension of $\mathcal{H}^{(n)}$. Set

$$\gamma^{(n)} = \int_{G/H} D_{L_1}(\hat{e}, \hat{y}) h_{P^{(n)}}(y)\, dy,$$

where every $y \in G/H$ is naturally identified with an element $\hat{y}$ of $CS_1(\mathcal{A})$. Then:

Lemma 8.3. *For $f \in M_k(\mathcal{A})$, we have*

$$\big\|f - \sigma^{(n)}\big(\breve{\sigma}_f^{(n)}\big)\big\|_{\infty,k} \leqslant \gamma^{(n)} L_k(f).$$

Proof. Suppose $f = [f_{ij}]$. Then for any $x \in G/H$, we have

$$\begin{aligned}
\big\|f(x) - \big(\sigma^{(n)}\big(\breve{\sigma}_f^{(n)}\big)\big)(x)\big\| &= \bigg\|\bigg[\int_{G/H} \big(f_{ij}(x) - f_{ij}(y)\big) h_{P^{(n)}}\big(y^{-1}x\big)\, dy\bigg]\bigg\| \\
&= \bigg\|\int_{G/H} \big[f_{ij}(x) - f_{ij}(y)\big] h_{P^{(n)}}\big(y^{-1}x\big)\, dy\bigg\| \\
&\leqslant \int_{G/H} \big\|\big[f_{ij}(x) - f_{ij}(y)\big]\big\| h_{P^{(n)}}\big(y^{-1}x\big)\, dy
\end{aligned}$$

$$\leqslant L_k(f) \int_{G/H} D_{L_1}(\hat{x}, \hat{y}) h_{P^{(n)}}\big(y^{-1}x\big)\, dy$$
$$= L_k(f) \int_{G/H} D_{L_1}(\hat{e}, \hat{y}) h_{P^{(n)}}(y)\, dy$$
$$\leqslant \gamma^{(n)} L_k(f)$$

by the formula (2.2) in [18]. So

$$\big\| f - \sigma^{(n)}\big(\breve{\sigma}_f^{(n)}\big)\big\|_{\infty,k} = \max\big\{\big\| f(x) - \big(\sigma^{(n)}\big(\breve{\sigma}_f^{(n)}\big)\big)(x)\big\|\colon x \in G/H\big\} \leqslant \gamma^{(n)} L_k(f). \qquad \square$$

Since the sequence $\{\gamma^{(n)}\}$ converges to 0 as $n \to \infty$ (see [18, p. 80]), there is an $N_1 \in \mathbb{N}$ such that $\gamma^{(n)} < \epsilon/2$ for $n > N_1^{(n)}$. So we obtain:

Proposition 8.4. *For $n > N_1$, $\mathcal{N}$ is a matrix bridge between $(\mathcal{B}^{(n)}, \mathcal{L}^{(n)})$ and $(\mathcal{A}, \mathcal{L})$, and hence $\mathcal{Q} = (Q_k) \in \mathcal{M}(\mathcal{L}^{(n)}, \mathcal{L})$, where*

$$Q_k(f, T) = \max\big\{L_k^{(n)}(T), L_k(f), N_k(f, T)\big\}, \quad (f, T) \in M_r\big(\mathcal{B}^{(n)} \oplus \mathcal{A}\big).$$

From Theorem 6.1 in [18], we have:

Lemma 8.5. *There is an $N_2 \in \mathbb{N}$ such that*

$$\big\|T - \breve{\sigma}^{(n)}\big(\sigma_T^{(n)}\big)\big\| < \frac{\epsilon}{2} L_1^{(n)}(T),$$

for all $T \in \mathcal{B}^{(n)}$ and $n > N_2$.

Theorem 8.6. *With notation as above, the quantized metric spaces $(\mathcal{B}^{(n)}, \mathcal{L}^{(n)})$ converge to $(\mathcal{A}, \mathcal{L})$ for quantized Gromov–Hausdorff distance as n goes to ∞.*

Proof. Given $\epsilon > 0$. Choose $N = \max\{N_1, N_2\}$. Then for $n > N$, we have that $\mathcal{Q} \in \mathcal{M}(\mathcal{L}^{(n)}, \mathcal{L})$ by Proposition 8.4. Given $\varphi \in CS_k(\mathcal{A})$. we have $\varphi \circ \sigma^{(n)} \in CS_k(\mathcal{B}^{(n)})$, and

$$\begin{aligned} &D_{L_k}\big(\varphi, \varphi \circ \sigma^{(n)}\big) \\ &\quad = \sup\big\{\big\|\langle\!\langle \varphi, f\rangle\!\rangle - \big\langle\!\big\langle \varphi \circ \sigma^{(n)}, T\big\rangle\!\big\rangle\big\|\colon L_r(f, T) \leqslant 1,\ (f, T) \in M_r\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big),\ r \in \mathbb{N}\big\} \\ &\quad = \sup\big\{\big\|\big\langle\!\big\langle \varphi, f - \sigma_T^{(n)}\big\rangle\!\big\rangle\big\|\colon L_r(f, T) \leqslant 1,\ (f, T) \in M_r\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big),\ r \in \mathbb{N}\big\} \\ &\quad \leqslant \sup\big\{\big\| f - \sigma_T^{(n)}\big\|_{\infty,r}\colon L_r(f, T) \leqslant 1,\ (f, T) \in M_r\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big),\ r \in \mathbb{N}\big\} \\ &\quad \leqslant \epsilon. \end{aligned}$$

On the other hand, if $\psi \in CS_k(\mathcal{B}^{(n)})$, then $\psi \circ \breve{\sigma}^{(n)} \in CS_k(\mathcal{A})$, and

$$\begin{aligned}
&D_{L_k}\big(\psi \circ \breve{\sigma}^{(n)}, \psi\big)\\
&\quad = \sup\big\{\big\|\big\langle\!\big\langle \psi \circ \breve{\sigma}^{(n)}, f\big\rangle\!\big\rangle - \langle\!\langle \psi, T\rangle\!\rangle\big\|\colon L_k(f,T) \leqslant 1,\ (f,T) \in M_k\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big)\big\}\\
&\quad = \sup\big\{\big\|\big\langle\!\big\langle \psi, \breve{\sigma}_f^{(n)} - T\big\rangle\!\big\rangle\big\|\colon L_k(f,T) \leqslant 1,\ (f,T) \in M_k\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big)\big\}\\
&\quad \leqslant \sup\big\{\big\|\breve{\sigma}_f^{(n)} - T\big\|\colon L_k(f,T) \leqslant 1,\ (f,T) \in M_k\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big)\big\}\\
&\quad \leqslant \sup\big\{\big\|\breve{\sigma}_f^{(n)} - \breve{\sigma}^{(n)}\big(\sigma_T^{(n)}\big)\big\| + \big\|\breve{\sigma}^{(n)}\big(\sigma_T^{(n)}\big) - T\big\|\colon L_k(f,T) \leqslant 1,\ (f,T) \in M_k\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big)\big\}\\
&\quad \leqslant \big\|f - \sigma_T^{(n)}\big\|_{\infty,k} + \sup\big\{\big\|\breve{\sigma}^{(n)}\big(\sigma_T^{(n)}\big) - T\big\|\colon L_k(f,T) \leqslant 1,\ (f,T) \in M_k\big(\mathcal{A} \oplus \mathcal{B}^{(n)}\big)\big\}\\
&\quad \leqslant \frac{\epsilon}{2} + \frac{1}{2}k^2\epsilon\\
&\quad \leqslant k^2\epsilon
\end{aligned}$$

by Lemma 8.5. Therefore, for $n > N$, we have

$$\mathrm{dist}_{NC}\big(\mathcal{B}^{(n)}, \mathcal{A}\big) \leqslant \epsilon,$$

that is, $\lim_{n\to\infty} \mathrm{dist}_{NC}(\mathcal{B}^{(n)}, \mathcal{A}) = 0$. □

Acknowledgments

I am grateful to Marc Rieffel for valuable discussions. I also thank Hanfeng Li and the referee for helpful comments. The Project sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry, and the National Natural Science Foundation of China.

References

[1] B. Blackadar, E. Kirchberg, Generalized inductive limits of finite-dimensional C^*-algebras, Math. Ann. 307 (1997) 343–380.
[2] M.-D. Choi, E.G. Effros, Injectivity and operator spaces, J. Funct. Anal. 24 (1977) 156–209.
[3] A. Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness, Ergodic Theory Dynam. Systems 9 (2) (1989) 207–220.
[4] E.G. Effros, C. Webster, Operator analogues of locally convex spaces, in: Operator Algebras and Applications, Samos, 1996, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 495, Kluwer Acad. Publ., Dordrecht, 1997, pp. 163–207.
[5] E.G. Effros, S. Winkler, Matrix convexity: Operator analogues of the bipolar and Hahn–Banach theorems, J. Funct. Anal. 144 (1997) 117–152.
[6] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhäuser Boston, Boston, MA, 1999.
[7] R.V. Kadison, A representation theory for commutative topological algebra, Mem. Amer. Math. Soc. 7 (1951).
[8] D. Kerr, Matricial quantum Gromov–Hausdorff distance, J. Funct. Anal. 205 (2003) 132–167.
[9] D. Kerr, H. Li, On Gromov–Hausdorff convergence for operator metric spaces, math.OA/0411157, 2004, v2.
[10] H. Li, θ-Deformations as compact quantum metric spaces, Comm. Math. Phys. 256 (1) (2005) 213–238.
[11] N. Ozawa, M.A. Rieffel, Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces, Canad. J. Math. 57 (5) (2005) 1056–1079.
[12] V.I. Paulsen, Completely Bounded Maps and Dilations, Pitman Res. Notes Math. Ser., vol. 146, Longman Sci./Wiley, New York, 1986.
[13] G. Pisier, Introduction to Operator Space Theory, London Math. Soc. Lecture Note Ser., vol. 294, Cambridge Univ. Press, Cambridge, 2003.

[14] M.A. Rieffel, Metrics on states from actions of compact groups, Doc. Math. 3 (1998) 215–229.
[15] M.A. Rieffel, Metrics on state spaces, Doc. Math. 4 (1999) 559–600.
[16] M.A. Rieffel, Group C^*-algebras as compact quantum metric spaces, Doc. Math. 7 (2002) 605–651.
[17] M.A. Rieffel, Gromov–Hausdorff distance for quantum metric spaces, Mem. Amer. Math. Soc. 168 (2004) 1–65.
[18] M.A. Rieffel, Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance, Mem. Amer. Math. Soc. 168 (2004) 67–91.
[19] C. Webster, S. Winkler, The Krein–Milman theorem in operator convexity, Trans. Amer. Math. Soc. 351 (1999) 307–322.
[20] W. Wu, Non-commutative metrics on matrix state spaces, J. Ramanujan Math. Soc. 20 (3) (2005) 215–254.
[21] W. Wu, Non-commutative metric topology on matrix state spaces, Proc. Amer. Math. Soc. 134 (2) (2006) 443–453.