Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Combinatorial Theory, Series A 114 (2007) 1515–1525
Journal of Combinatorial Theory
Series A
www.elsevier.com/locate/jcta

Diameter graphs of polygons and the proof of a conjecture of Graham

Jim Foster, Tamas Szabo

Department of Mathematics, Weber State University, Ogden, UT 84408-1702, USA

Received 11 October 2006

Available online 19 April 2007

Abstract

We show that for an n-gon with unit diameter to have maximum area, its diameter graph must contain a cycle, and we derive an isodiametric theorem for such n-gons in terms of the length of the cycle. We then apply this theorem to prove Graham's 1975 conjecture that the diameter graph of a maximal $2m$-gon ($m \geqslant 3$) must be a cycle of length $2m-1$ with one additional edge attached to it.

Keywords: Discrete geometry; Extremal polygons; Diameter graph; Isodiametric theorem; Reuleaux polygons

1. Introduction

Consider all polygons with unit diameter and n sides. Which of these have the largest area? In 1922, Reinhardt [9] proved that for n odd the regular n-gon has the maximum area, but he noticed that this is not the case for n even. For $n = 4$, although the square is maximal, there are infinitely many quadrilaterals with unit diameter and the same area of $1/2$. Bieri [2] in 1961 found the largest hexagon with unit diameter, but the proof of this is due to Graham [5] who proved that there is a unique hexagon with maximal area, about 4% larger than the regular one. He also conjectured that the diameter graph of a maximal n-gon for n even must have a specific form, which we will state shortly. The first open case of this conjecture ($n = 8$) was proved by Audet et al. [1] in 2002 following Graham's method of examining all possible diameter graphs. Recently Mossinghoff [8] has provided numerical evidence for this conjecture up to $n = 20$ (see

E-mail addresses: jfoster@weber.edu (J. Foster), tszabo@weber.edu (T. Szabo).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.02.006

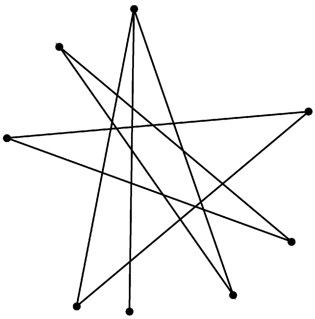

Fig. 1. Conjectured shape of the diameter graph.

also his expository article [7]). The main result of this paper is a proof of Graham's conjecture, and on the way to proving it, we derive a new isodiametric inequality for polygons.

The *diameter graph* of a set of points X is defined as the graph with vertices corresponding to the points of X, and an edge between two vertices if the distance between the corresponding points of X is equal to the diameter of X. A segment connecting two such vertices is also commonly called a diameter. It is an elementary geometric fact that if X is a set in the plane, then any two of its diameters must intersect in either a common endpoint or a common interior point. It follows then that the diameter graph of a set must have a *linear thrackleation*: a representation where each edge is represented by a line segment and each edge intersects every other edge, possibly at an endpoint.

The *diameter graph of a polygon* is the diameter graph of its vertex set. An old result of Hopf and Panwitz [6] (proved by Sutherland [10]) says that the number of edges of a polygon's diameter graph cannot exceed the number of vertices. Using a convexity result of Fáry and Rédei [3], Graham [5] proved that the diameter graph of a polygon of maximum area with unit diameter must be connected, and made the following conjecture:

Conjecture. *(See Graham [5].) The diameter graph of a unit diameter $2m$-gon with maximal area has a cycle of length $2m-1$ and one additional edge from the remaining vertex.*

Figure 1 shows the conjectured shape of the diameter graph for $m=4$. We will prove this conjecture in the following way. In Section 2 we use a convexity argument to show that the diameter graph cannot be a path. Then we use this result in Section 3 to prove that it cannot even be a tree, and thus must contain a cycle. Section 4 uses the isodiametric theorem for Reuleaux polygons to derive an isodiametric theorem for fixed-diameter polygons whose diameter graphs contain a cycle of given length. In Section 5, we use this theorem to prove Graham's conjecture by constructing a $2m$-gon whose area is larger than that of any $2m$-gon whose diameter graph contains a cycle of length less than $2m-1$.

Since the convex hull of a set of points has the same diameter as the set itself, we can restrict the discussion that follows to convex polygons.

2. The diameter graph cannot be a path

Suppose a unit-diameter convex k-gon P ($k \geqslant 5$) has a path of length $k-1$ as its diameter graph. We will describe P as $p_1p_2p_3\ldots p_k$, where p_ip_{i+1} are the successive diameters. We can

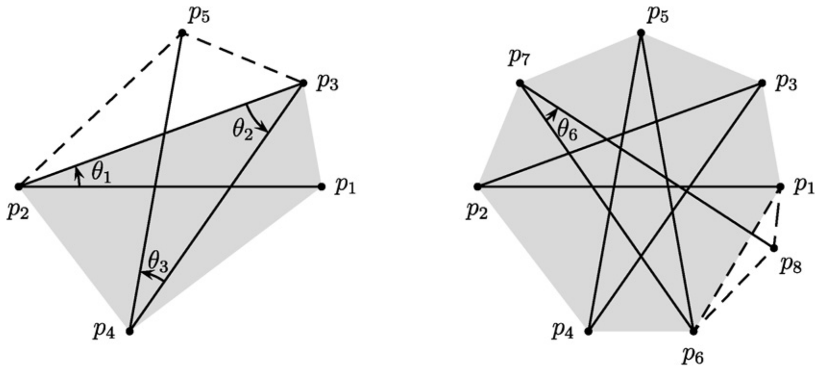

Fig. 2. Polygons whose diameter graphs are paths.

parameterize P by the angles $\theta_i = \angle p_i p_{i+1} p_{i+2}$, $1 \leqslant i \leqslant k-2$, as in Fig. 2. If we imagine each diameter rotating into the next one, the fact that diameters not having a common endpoint must cross implies that the total rotation is less than π:

$$0 < \sum_{i=1}^{k-2} \theta_i < \pi.$$

Adding the vertex p_k to the polygon $p_1 p_2 p_3 \dots p_{k-1}$ increases its area by the area of $\triangle p_1 p_{k-2} p_k$ if k is even or of $\triangle p_2 p_{k-2} p_k$ if k is odd (see Fig. 2 again). Let $a_k(\theta_1, \dots, \theta_{k-2})$ denote twice the area of P, and $b_k(\theta_1, \dots, \theta_{k-2})$ denote twice the area of the last triangle added to P. Clearly, $a_k = a_{k-1} + b_k$.

Place a coordinate system so that $p_1 = (1, 0)$ and $p_2 = (0, 0)$. Then $p_3 = (\cos\theta_1, \sin\theta_1)$, $p_4 = (\cos\theta_1 - \cos(\theta_1 + \theta_2), \sin\theta_1 - \sin(\theta_1 + \theta_2))$ and in general

$$p_k = \left(\sum_{j=1}^{k-2} (-1)^{j+1} \cos\psi_j, \sum_{j=1}^{k-2} (-1)^{j+1} \sin\psi_j \right),$$

where $\psi_j = \sum_{i=1}^{j} \theta_i$.

If k is odd, then

$$\begin{aligned} b_k &= \begin{vmatrix} p_{k-2} - p_2 \\ p_k - p_2 \end{vmatrix} = \begin{vmatrix} p_{k-2} \\ p_k \end{vmatrix} = \begin{vmatrix} p_{k-2} \\ p_k - p_{k-2} \end{vmatrix} \\ &= \sum_{j=1}^{k-4} (-1)^{j+1} \cos\psi_j \left[(-1)^{k-2} \sin\psi_{k-3} + (-1)^{k-1} \sin\psi_{k-2} \right] \\ &\quad - \sum_{j=1}^{k-4} (-1)^{j+1} \sin\psi_j \left[(-1)^{k-2} \cos\psi_{k-3} + (-1)^{k-1} \cos\psi_{k-2} \right] \\ &= (-1)^{k-2} \sum_{j=1}^{k-4} (-1)^{j+1} \left[\sin(\psi_{k-3} - \psi_j) - \sin(\psi_{k-2} - \psi_j) \right]. \end{aligned}$$

If k is even,

$$b_k = \begin{vmatrix} p_{k-2} - p_1 \\ p_k - p_1 \end{vmatrix} = \begin{vmatrix} p_{k-2} - p_1 \\ p_k - p_{k-2} \end{vmatrix} = \begin{vmatrix} p_{k-2} \\ p_k - p_{k-2} \end{vmatrix} - \begin{vmatrix} p_1 \\ p_k - p_{k-2} \end{vmatrix}.$$

The first determinant is the same as for odd k, and the second determinant, including the minus sign in front, is equal to $\sin\psi_{k-2} - \sin\psi_{k-3}$. We can combine them as

$$b_k = (-1)^{k-2}\sum_{j=0}^{k-4}(-1)^{j+1}\big[\sin(\psi_{k-3}-\psi_j) - \sin(\psi_{k-2}-\psi_j)\big],$$

under the convention that $\psi_0 = 0$.

We will show in the following lemma that a_k is a strictly convex function of $(\theta_1, \ldots, \theta_{k-2})$ in the direction $(1, -1, 1, -1, \ldots)$. More precisely, we will show that

$$f_k(t) = a_k\big(\theta_1 + t, \theta_2 - t, \theta_3 + t, \theta_4 - t, \ldots, \theta_{k-2} + (-1)^{k-1}t\big)$$

is a strictly convex function of t, where t is restricted to a sufficiently small interval about 0.

Lemma 1. *If $k \geqslant 5$, then $f_k(t)$ is a strictly convex function of t for t sufficiently small.*

Proof. The proof is by induction on k. First, $f_4(t) = \sin(\theta_1 + \theta_2)$ and $f_4''(t) = 0$. Now let $g_k(t) = b_k(\theta_1 + t, \theta_2 - t, \ldots, \theta_{k-2} + (-1)^{k-1}t)$. Then $f_k(t) = f_{k-1}(t) + g_k(t)$, and the lemma will be true if we can show that $g_k''(t) > 0$ for $k \geqslant 5$. From the formula for b_k,

$$g_k''(t) = \sum\nolimits' \sin\big(\psi_{k-3} - \psi_j + (-1)^k t\big) + \sum\nolimits'' \sin\big(\psi_{k-2} - \psi_j + (-1)^{k+1} t\big)$$

where $\sum'$ is the sum over $j \in \{1, 3, 5, \ldots, k-4\}$ if k is odd and over $j \in \{0, 2, 4, \ldots, k-4\}$ if k is even, and $\sum''$ is the sum over $j \in \{2, 4, 6, \ldots, k-5\}$ if k is odd and over $j \in \{1, 3, 5, \ldots, k-5\}$ if k is even. If we take t small enough that the arguments of all the sine functions are in $(0, \pi)$, each of the above sums is positive, and $g_k''(t) > 0$. □

Theorem 1. *If $n \geqslant 5$, the diameter graph of an n-gon with unit diameter and maximal area cannot be a path.*

Proof. Suppose such a polygon P has a path with k vertices as its diameter graph. By the connectedness of the diameter graphs of polygons of maximal area, we must have $k = n$. If we perturb the angles of the path by $\pm t$, as in the definition of the function $f_k(t)$, where t is small enough that the polygon is still convex and has unit diameter, then by Lemma 1 we can find a nearby n-gon with a larger area than P. □

3. The diameter graph cannot be a tree

Let P be a polygon whose diameter graph is a tree. Using the fact that all diameters have to intersect, it is not hard to see that the tree must be a *caterpillar*, a path with possible additional edges hanging on its vertices. For if it were not, it would have a vertex p with at least three neighbors p_1, q_1, r_1 so that each of those have additional neighbors p_2, q_2, r_2. Since the edges represent diameters of P, one of p_1, q_1, r_1 is between the other two, say q_1 is in the acute angle p_1pr_1 (see Fig. 3). Now the diameter q_1q_2 could not cross both diameters pp_1 and pr_1, since those are on two different sides of pq_1. This contradiction shows that if the diameter graph is a tree, it has to be a caterpillar. (This result also follows from a slightly more general theorem of Woodall [11] on linear thrackleations.)

Let $S = p_1p_2 \ldots p_k$ be a longest path of the caterpillar diameter graph of an n-gon P ($n \geqslant 5$) with the additional property that the first and last angles of the path are maximal. In Fig. 4, this

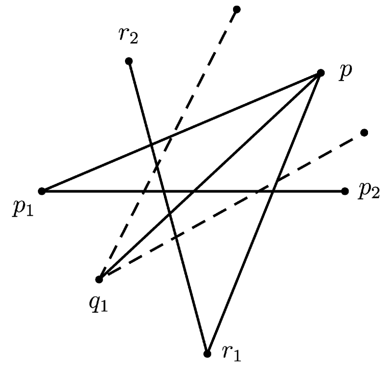

Fig. 3. Why the tree must be a caterpillar.

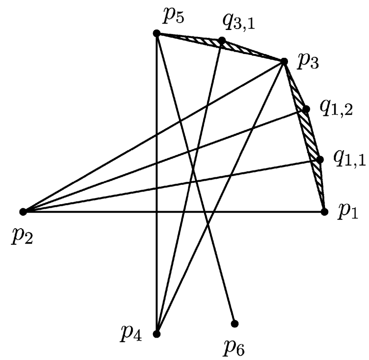

Fig. 4. Increment of the area when the diameter graph is a tree.

would be $p_1p_2p_3p_4p_5p_6$ or its reverse. The maximality of the first and last angles guarantees that any diameter of P that is not in S will lie in an acute angle between two diameters of S. Thus any angle $\theta_i = \angle p_i p_{i+1} p_{i+2}$ of S will contain an additional $j_i \geqslant 0$ diameters $p_i q_{i,1}, \ldots, p_i q_{i,j_i}$, where $\sum_{i=1}^{k-2} j_i \leqslant n-k$.

We now show that P cannot have maximal area. By Graham's connectedness result, we need to consider only those P with $\sum_{i=1}^{k-2} j_i = n-k$. The area of P is equal to the area of the convex hull of S plus the sum of the areas of polygons with vertices $p_i q_{i,1} q_{i,2} \ldots q_{i,j_i} p_{i+2}$, shown shaded in Fig. 4. These polygons have maximal area only if the j_i extra diameters divide the angle θ_i into $j_i + 1$ equal parts, and that area is one half of $(j_i+1)\sin\frac{\theta_i}{j_i+1} - \sin\theta_i$.

We now perturb the angles of S, as in Section 2, to form the function

$$h_i(t) = (j_i+1)\sin\left(\frac{\theta_i + (-1)^{i+1}t}{j_i+1}\right) - \sin\bigl(\theta_i + (-1)^{i+1}t\bigr),$$

representing twice the shaded area. Then

$$h_i''(t) = -\frac{1}{j_i+1}\sin\left(\frac{\theta_i + (-1)^{i+1}t}{j_i+1}\right) + \sin\bigl(\theta_i + (-1)^{i+1}t\bigr) \geqslant 0,$$

for t sufficiently small, with strict inequality for $j_i > 0$. This shows that $h_i(t)$ is a convex function near $t = 0$ (strictly convex if $j_i > 0$). The area of P after perturbation is then half of $f_k(t) +$

$\sum_{i=1}^{k-2} h_i(t)$. If $k \geqslant 5$, this sum is strictly convex by Lemma 1, so P could not have maximum area.

If $k = 4$ then $f_4(t) = \sin(\theta_1 + \theta_2)$ and $f_k''(t) = 0$, while there is at least one $j_i \geqslant 1$, since P has at least five vertices. So in this case also, the sum is strictly convex, and again the area of P cannot be maximal.

If $k = 3$, it is no longer true that $f_3''(t) \geqslant 0$, but in this case our diameter graph is a star and we can clearly increase the area of P by increasing the only angle θ_1. As a summary we can conclude:

Theorem 2. *For $n \geqslant 5$, the diameter graph of an n-gon with unit diameter and maximal area cannot be a tree, and hence must contain a cycle.*

4. An isodiametric theorem for polygons whose diameter graphs have cycles

Any cycle in the diameter graph of a planar set must be of odd length (because of the intersection property of the diameters), and it is not hard to see that there can be at most one cycle in the graph (see Woodall [11] for details). In this section we will establish a bound on the area based on the length of this cycle. As a corollary we will get a bound on the areas of polygons with a cycle of length less than $2m - 1$ in their diameter graphs.

The key ingredient in the proof of Theorem 3 below is the isodiametric theorem for Reuleaux polygons. A *Reuleaux polygon* is a convex figure of constant width bounded by k (necessarily odd) arcs of equal radius d which is then the diameter of the figure. Reuleaux polygons correspond to cycles in the diameter graphs of ordinary polygons, as illustrated in Fig. 5(a), which shows an 8-gon with a 5-cycle in its diameter graph, and the enclosing Reuleaux 5-gon.

Isodiametric Theorem for Reuleaux Polygons. *(See Firey [4].) If a Reuleaux polygon R has k sides and diameter d, then its area A satisfies*

$$A \leqslant \frac{d^2}{2}\left(k \cos\frac{\pi}{k} \tan\frac{\pi}{2k} + \pi - k \sin\frac{\pi}{k}\right),$$

with equality if and only if R is regular.

Theorem 3. *If a polygon P has n sides, diameter d and a k-cycle in its diameter graph, then its area A satisfies*

$$A \leqslant \frac{d^2}{2}\left(k \cos\frac{\pi}{k} \tan\frac{\pi}{2k} - k \sin\frac{\pi}{k} + n \sin\frac{\pi}{n}\right).$$

Equality holds if and only if k divides n, the polygon is equilateral and every vertex is the endpoint of at least one diameter.

Proof. Consider the Reuleaux k-gon R that contains P and whose vertices match our cycle. If some of the vertices of P are not on the boundary of R, we could increase the area of P by pulling them out to the boundary, as in Fig. 5(a). Thus we get the following bound:

$$A \leqslant \text{area}(R) - \text{area}(n \text{ shaded circular segments}).$$

These circular segments may not be equal, but their total area is smallest when they are. This can be demonstrated by putting the segments next to each other around a semicircle of radius d,

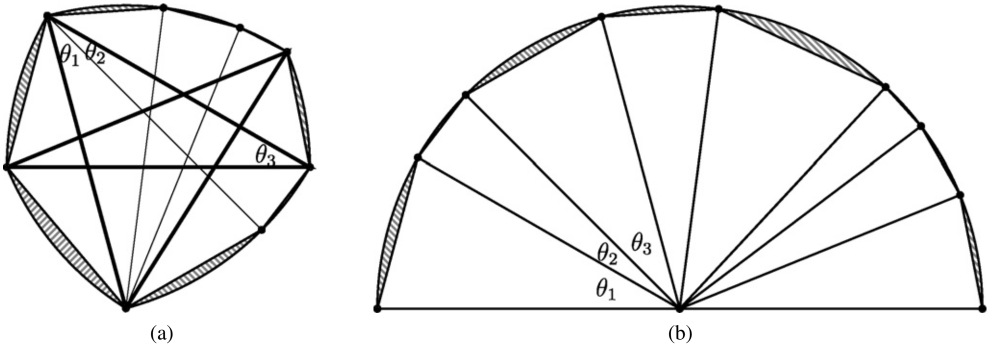

Fig. 5. Placing the n segments around a semicircle.

which is possible since the central angles of the segments must sum to π, and the largest $(n+1)$-gon inscribed in the semicircle of radius one must have n equal sides (see Fig. 5(b)). Applying Firey's theorem, we get

$$\begin{aligned} A(P) &\leqslant \text{area(regular } R) - \text{area(}n \text{ equal circular segments)} \\ &= \frac{d^2}{2}\left(k\cos\frac{\pi}{k}\tan\frac{\pi}{2k} + \pi - k\sin\frac{\pi}{k}\right) - \frac{d^2}{2}\left(\pi - n\sin\frac{\pi}{n}\right) \\ &= \frac{d^2}{2}\left(k\cos\frac{\pi}{k}\tan\frac{\pi}{2k} - k\sin\frac{\pi}{k} + n\sin\frac{\pi}{n}\right). \end{aligned}$$

For equality to hold we need both that the Reuleaux k-gon be regular and that the n circular segments be equal. □

The bound of Theorem 3 is an increasing function of k. This implies that Theorem 3 is a sharpening, for polygons with cycles in their diameter graphs, of the standard isodiametric inequality for polygons, $2A \leqslant d^2 n\cos(\pi/n)\tan(\pi/2n)$ (Mossinghoff [8]). It also leads immediately to the following corollary.

Corollary 1. *For $m \geqslant 3$, if a $2m$-gon of unit diameter has a cycle in its diameter graph of length at most $2m-3$, then its area A satisfies*

$$A \leqslant \frac{2m-3}{2}\cos\frac{\pi}{2m-3}\tan\frac{\pi}{2(2m-3)} - \frac{2m-3}{2}\sin\frac{\pi}{2m-3} + m\sin\frac{\pi}{2m}.$$

5. The diameter graphs of polygons with maximum area

In this section we complete the proof of our main result, the conjecture of Graham:

Theorem 4. *If, for $m \geqslant 3$, a $2m$-gon has unit diameter and maximum area, then its diameter graph must be a cycle of length $2m-1$ with an additional edge attached to a vertex of the cycle.*

In view of Theorem 2, Theorem 4 will be proved if we can find a $2m$-gon of unit diameter whose area exceeds the bound of Corollary 1. We construct such a polygon as follows.

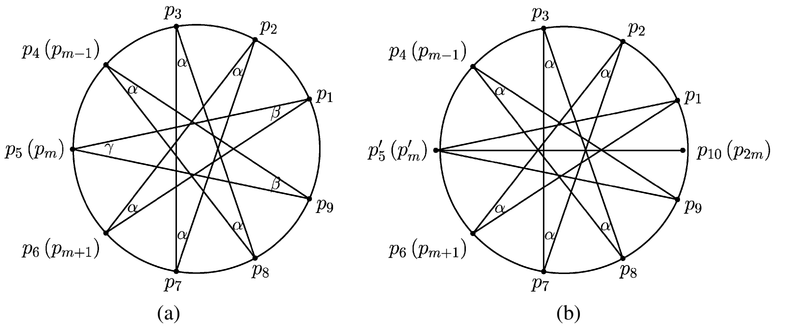

Fig. 6. Construction of P and P'.

Start with a polygon $P = p_1 p_2 \ldots p_{2m-1}$ with vertices on a circle of radius r, such that the angles of the associated star polygon all have value $\alpha = \pi/(2m - 1/2)$, except that the angles at p_1 and p_{2m-1} are $\beta = 9\alpha/8$ and the angle at p_m is then $\gamma = \pi - (2m-4)\alpha - 2\beta = 5\alpha/4$. (Figure 6(a) illustrates the case $m = 5$.) Then P has area

$$\operatorname{area}(P) = r^2\left[(m-2)\sin 2\alpha + \sin 2\beta + \frac{1}{2}\sin 2\gamma\right]$$

and diameter $2r\cos\alpha/2$, which follows from the facts that α is less than both β and γ and that $2\beta = \alpha + \gamma$, causing the minor arcs subtended by any two chords forming an angle α to be equal. Specifying the radius of the circle to be $r = (1/2)\sec\alpha/2$ makes the diameter of P equal to 1.

Next, modify P by replacing p_m with a point p_m' located radially outward from p_m at a distance $1 - 2r\cos\gamma/2$ from it. Finally, place an additional vertex p_{2m} at distance 1 from p_m' and on the opposite side of p_m from p_m'. The $2m$-gon P' constructed this way is illustrated in Fig. 6(b). By the triangle inequality, $p_m' p_1 < p_m' p_m + p_m p_1 = (1 - 2r\cos\gamma/2) + 2r\cos\gamma/2 = 1$, so P' still has diameter 1. Notice that the diameter graph of P' has no cycle and is not even connected, so it cannot have maximal area. However, its area will turn out to be larger than that of any $2m$-gon with a cycle of length at most $2m-3$ in its diameter graph. (By pulling p_m' out slightly further we could get a polygon with a $(2m-1)$-cycle in its diameter graph, but the formula for its area would be much more difficult to analyze.)

The area of P' is equal to that of P increased by the areas of three triangles, $\triangle p_{m-1} p_m' p_m$, $\triangle p_{m+1} p_m' p_m$, and $\triangle p_{2m-1} p_{2m} p_1$. The first two triangles each have area $\frac{1}{2}(1 - 2r\cos\gamma/2) r \times \sin 2\beta = \frac{1}{2} r^2 (1/r - 2\cos\gamma/2)\sin 2\beta$, and the third has area $r^2(2\cos\gamma/2 - \cos\gamma - 1)\sin\gamma$. Adding these to the area of P we get

$$\begin{aligned}\operatorname{area}(P') = r^2\bigg[&(m-2)\sin 2\alpha + \sin 2\beta - \sin\gamma \\ &+ \left(2\cos\frac{\alpha}{2} - 2\cos\frac{\gamma}{2}\right)\sin 2\beta + 2\cos\frac{\gamma}{2}\sin\gamma\bigg].\end{aligned}$$

Substituting $r = (1/2)\sec\alpha/2$, $m = \pi/2\alpha + 1/4$, $\beta = 9\alpha/8$ and $\gamma = 5\alpha/4$ gives the area of P' as a function of α:

$$\begin{aligned}A(\alpha) = \frac{1}{4}\sec^2\frac{\alpha}{2}\bigg[&\pi\frac{\sin 2\alpha}{2\alpha} - \frac{7}{4}\sin 2\alpha + \sin\frac{9}{4}\alpha - \sin\frac{5}{4}\alpha \\ &+ \left(2\cos\frac{\alpha}{2} - 2\cos\frac{5}{8}\alpha\right)\sin\frac{9}{4}\alpha + 2\cos\frac{5}{8}\alpha\sin\frac{5}{4}\alpha\bigg].\end{aligned}$$

Our goal is now to show that $A(\alpha)$ is larger than the bound of Corollary 1 with $m = \pi/2\alpha + 1/4$,

$$B(\alpha) = \frac{\pi}{2} \cdot \frac{\sin \frac{2\pi\alpha}{2\pi+\alpha}}{\frac{2\pi\alpha}{2\pi+\alpha}} - \frac{\pi}{4} \cdot \frac{\tan \frac{\pi\alpha}{2\pi-5\alpha}}{\frac{\pi\alpha}{2\pi-5\alpha}}.$$

Thus we need to prove that the function $E(\alpha) = A(\alpha) - B(\alpha)$ satisfies $E(\alpha) > 0$ for all values of α corresponding to $m \geqslant 3$. At this point we mention that the Maclaurin series of $E(\alpha)$ is $\alpha^3/192 + O(\alpha^4)$. This observation shows already that Theorem 4 is true for m sufficiently large. But to prove it for all $m \geqslant 3$ we have to work a little harder. The technique we will use is to bound $E(\alpha)$ from below by a polynomial.

Lemma 2. *For* $0 < \alpha < \pi$,

$$A(\alpha) > \frac{(4+\alpha^2)(983040\pi - 655360\pi\alpha^2 - 61440\alpha^3 - 2185661\alpha^5)}{15728640}.$$

Proof. First rewrite $A(\alpha)$ using the identity $2\cos x \sin y = \sin(x+y) - \sin(x-y)$ as

$$A(\alpha) = \frac{1}{4}\sec^2\frac{\alpha}{2}\left(\pi\frac{\sin 2\alpha}{2\alpha} - \frac{7}{4}\sin 2\alpha + \sin\frac{9}{4}\alpha - \sin\frac{5}{4}\alpha + \sin\frac{11}{4}\alpha \right.$$
$$\left. + \sin\frac{7}{4}\alpha - \sin\frac{23}{8}\alpha - \sin\frac{13}{8}\alpha + \sin\frac{15}{8}\alpha + \sin\frac{5}{8}\alpha\right).$$

Next apply to the terms of $A(\alpha)$ the following inequalities, which come from the truncated Maclaurin series of $\sec^2\theta$ and $\sin\theta$:

$$\sec^2\theta > 1 + \theta^2,$$
$$\sin\theta > \theta - \frac{\theta^3}{3!},$$
$$-\sin\theta > -\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right).$$

The first inequality holds for $-\pi/2 < \theta < \pi/2$ and the remaining two for $\theta > 0$. After simplifying (we used a computer algebra system), we get the inequality of the lemma. $\square$

Lemma 3. *For* $0 < \alpha < 2\pi/7$,

$$B(\alpha) < \frac{240\pi^3 - 100\pi^3\alpha^2 - 20\pi^2\alpha^3 - (435\pi - 2\pi^3)\alpha^4 + (40 - 8\pi^2)\alpha^5 + 10\pi\alpha^6}{960\pi^2}.$$

Proof. From the third inequality in the proof of Lemma 2,

$$\frac{\sin \frac{2\pi\alpha}{2\pi+\alpha}}{\frac{2\pi\alpha}{2\pi+\alpha}} < 1 - \frac{\alpha^2}{3!}\left(1 + \frac{\alpha}{2\pi}\right)^{-2} + \frac{\alpha^4}{5!}\left(1 + \frac{\alpha}{2\pi}\right)^{-4}$$

which requires only $\alpha > 0$. Then the inequalities

$$\left(1 + \frac{\alpha}{2\pi}\right)^{-2} > 1 - \frac{\alpha}{\pi} + \frac{3\alpha^2}{4\pi^2} - \frac{\alpha^3}{2\pi^3},$$
$$\left(1 + \frac{\alpha}{2\pi}\right)^{-4} < 1 - \frac{2\alpha}{\pi} + \frac{5\alpha^2}{2\pi^2},$$

which require $0 < \alpha < 2\pi$, imply

$$\frac{\sin\frac{2\pi\alpha}{2\pi+\alpha}}{\frac{2\pi\alpha}{2\pi+\alpha}} < 1 - \frac{\alpha^2}{3!}\left(1 - \frac{\alpha}{\pi} + \frac{3\alpha^2}{4\pi^2} - \frac{\alpha^3}{2\pi^3}\right) + \frac{\alpha^4}{5!}\left(1 - \frac{2\alpha}{\pi} + \frac{5\alpha^2}{2\pi^2}\right).$$

For the second term of $B(\alpha)$, observe that $(\tan x)/x$ has a Maclaurin series with positive coefficients, and so does $x = \pi\alpha/(2\pi - 5\alpha)$, so their composition also has this property, and thus can be bounded from below by the first few terms:

$$\frac{\tan\frac{\pi\alpha}{2\pi-5\alpha}}{\frac{\pi\alpha}{2\pi-5\alpha}} > 1 + \frac{\alpha^2}{12} + \frac{5\alpha^3}{12\pi} + \left(\frac{1}{120} + \frac{25}{16\pi^2}\right)\alpha^4.$$

For this inequality we need $\alpha > 0$ and $\pi\alpha/(2\pi - 5\alpha) < \pi/2$, that is $0 < \alpha < 2\pi/7$. Putting these inequalities into the formula for $B(\alpha)$ and simplifying, we arrive at the inequality of Lemma 3. □

We now continue with the proof of Theorem 4. Lemmas 2 and 3 imply, after a little algebra, that

$$E(\alpha) > \frac{\alpha^3}{15728640\pi^2}\, p(\alpha)$$

for $0 < \alpha < 2\pi/7$, where

$$\begin{aligned} p(\alpha) = {} & 81920\pi^2 + 49152\pi\left(145 - 14\pi^2\right)\alpha \\ & - 4\left(163840 + 2168253\pi^2\right)\alpha^2 - 163840\pi\alpha^3 - 2185661\pi^2\alpha^4. \end{aligned}$$

The polynomial $p(\alpha)$ has exactly one positive zero, $\alpha = 0.10294\ldots$. Since $p(0) > 0$, it must be true that $p(\alpha) > 0$, and hence $E(\alpha) > 0$, for $0 < \alpha < 0.10294$. These values of α correspond to $m \geqslant 16$. Table 1 takes care of the remaining values of m, finishing the proof of Theorem 4.

We conclude with two open questions. Is there, for all $m \geqslant 3$, a *unique* $2m$-gon of unit diameter and maximal area? If so, is it *symmetric* about the diameter that is not in the cycle of the diameter graph? The answer to both questions is known to be affirmative for $m = 3$ and $m = 4$ (see [1,5]).

Table 1
$E(\alpha)$ is positive for small m

m	$A(\alpha)$	$B(\alpha)$	$E(\alpha) = A(\alpha) - B(\alpha)$
3	0.67472	0.63397	0.04075
4	0.72666	0.71843	0.00823
5	0.74901	0.74623	0.00278
6	0.76065	0.75944	0.00121
7	0.76748	0.76687	0.00061
8	0.77183	0.77148	0.00035
9	0.77476	0.77455	0.00021
10	0.77684	0.77670	0.00014
11	0.77836	0.77827	0.00009
12	0.77951	0.77945	0.00007
13	0.78040	0.78036	0.00005
14	0.78110	0.78107	0.00004
15	0.78167	0.78164	0.00003

References

[1] C. Audet, P. Hansen, F. Messine, J. Xiong, The largest small octagon, J. Combin. Theory Ser. A 98 (2002) 46–59.
[2] H. Bieri, Ungelöste Probleme: Zweiter Nachtrag zu Nr. 12, Elem. Math. 16 (1961) 105–106.
[3] I. Fáry, L. Rédei, Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern, Math. Ann. 122 (1950) 205–220.
[4] W.J. Firey, Isoperimetric ratios of Reuleaux polygons, Pacific J. Math. 10 (1960) 823–830.
[5] R.L. Graham, The largest small hexagon, J. Combin. Theory Ser. A 18 (1975) 165–170.
[6] H. Hopf, E. Panwitz, Aufgabe Nr. 167, Jahresber. Deutsch. Math.-Verein. 43 (1934) 114.
[7] M.J. Mossinghoff, A $1 problem, Amer. Math. Monthly 113 (2006) 385–402.
[8] M.J. Mossinghoff, Isodiametric problems for polygons, Discrete Comput. Geom. 36 (2006) 363–379.
[9] K. Reinhardt, Extremale Polygone gegebenen Durchmessers, Jahresber. Deutsch. Math.-Verein. 31 (1922) 251–270.
[10] J. Sutherland, Solution to problem 167, Jahresber. Deutsch. Math.-Verein. 45 (1935) 33–34.
[11] D.R. Woodall, Thrackles and deadlock, in: D.J.A. Welsh (Ed.), Combinatorial Mathematics and Its Applications, Academic Press, New York, 1971, pp. 335–347.