JOURNAL OF ALGEBRA 63, 444–458 (1980)

Maximal Duo Algebras of Matrices

R. C. COURTER

300 West 107th Street APT 5 B New York, N.Y. 10025

Communicated by N. Jacobson

Received October 5, 1978

1. INTRODUCTION

Schur proved in 1905 [9] that the maximum number of linearly independent commuting n by n complex matrices is $g(n) = [n^2/4] + 1$, where [ ] is the greatest integer function. This result was generalized to an arbitrary field by Jacobson [8]. We prove in this paper that $g(n)$ is the maximum of the dimensions of duo subalgebras of the full matrix algebra K_n, where K is an arbitrary field. We prove that the duo subalgebras of K_n having dimension $g(n)$ are commutative, so that they form the known class of commutative subalgebras with that dimension (see [8] and [5]). Another result states that each $g(n)$-dimensional commutative subalgebra of K_n is a maximal right duo and maximal left duo subring of the endomorphism ring of the group K^n of n-tuples. Indeed, these maximality properties are enjoyed by a class A_n of local commutative subalgebras of K_n whose maximal commutativity was proved in [1]. A local commutative subalgebra of K_n belongs to A_n if, and only if, its radical consists of all the K-linear transformations of N into N', where N and N' are any nonzero complementary subspaces of K^n; each $g(n)$-dimensional commutative subalgebra belongs to A_n, when $n > 3$.

We present examples of noncommutative maximal duo subalgebras arising from the following theorems: If a ring R possesses a faithful cyclic right module M, R has no proper right duo overring contained in the endomorphism ring of M A dual theorem, concerning uniform representations, is proved for rings $R \subseteq K_n$.

A detailed account of our results and of the mentioned classes of subalgebras is forthcoming, but first some general notation is introduced. K is an arbitrary field. The space of n-tuples over K is denoted by K^n. Dim S denotes the K-dimension of a K-space S. K_n is the full algebra of n by n matrices over K; I_n is the identity matrix. E_{ij} is the matrix with 1 in the (i, j) position and zeros elsewhere. The notation $(0: V)_W$ denotes $\{w \in W \mid vw = 0$ for all $v \in V\}$, assuming a function on $V \times W$ with zero in the range.

Rings have a unity element; modules are unital, and are right modules unless the contrary is stated. The Jacobson radical of a ring R is denoted by rad R

0021-8693/80/040444-15$02.00/0

or by P. For a commutative group M, End M denotes its ring of endomorphisms. A ring is a local ring if, and only if, its nonunits form an ideal. A ring is a duo ring [right duo ring] if, and only if, it has no strictly one-sided [strictly right-sided] ideals.

Some preliminary theorems on duo subalgebras make the opening of section 2: (i) $g(n)$ is an upper bound of the dimensions of duo subalgebras of K_n, provided it is so for those duo subalgebras which are represented indecomposably on K^n and, when $n > 3$, only the latter can have dimension $g(n)$. (ii) If R is a duo ring which has a faithful indecomposable module satisfying the ascending and descending chain conditions,then R is a local ring with nilpotent radical. The theorem establishing $g(n)$ as an upper bound for the dimensions of duo subalgebras R of K_n is theorem 3 in section 2, which also states that for $n > 3$, R can be $g(n)$-dimensional only when $R/(\text{rad } R)$ is isomorphic to K. A supporting theorem, theorem 2, states that for $n \leqslant 3$ each maximal duo subalgebra of K_n is commutative with dimension $n = g(n)$.

Definition. Let $n > 1$. For $n^* \in \{1,..., (n-1)\}$, let $P(n^*) = P(n^*, n, K) \subseteq K_n$ be the zero algebra generated over K by

$$\{E_{ij} \mid 1 \leqslant i \leqslant n^*, (n^* + 1) \leqslant j \leqslant n\}.$$

$A(n^*) = A(n^*, n, K)$ denotes the algebra $KI_n + P(n^*)$.

Definition. Let $n > 1$. $A_n = A_n(K)$ denotes the class of all subalgebras R of K_n which are similar to $A(n^*, n, K)$ for one $n^* \in \{1,..., (n-1)\}$.

Definition. Let $n > 1$. $J_n = J_n(K)$ denotes the class of all subalgebras of K_n which are similar to

$A(n/2, n, K)$ when n is even;

$A((n-1)/2, n, K)$ or $A((n+1)/2, n, K)$ when n is odd.

For $n > 3$, Jacobson proved that, with one possible minor exception, J_n is the set of all maximal commutative subalgebras of K_n with dimension $g(n)$ [8, p. 436]. Proof that the exception does not occur was made by Gustafson [5, p. 560].

In section 3, theorem 6, we complete the proof that, for all n, a duo subalgebra of K_n having dimension $g(n)$ is commutative. This involves proving that, for $n > 3$, R is a duo subalgebra with dimension $g(n)$ if and only if $R \in J_n$; for $n \leqslant 3$, theorem 2 is quoted. A consequence of theorem 6 is that a noncommutative duo subalgebra of K_n with dimension $[n^2/4]$ is a maximal duo subalgebra.

Theorem 11 in section 4 states that duo subalgebras of K_n having dimension $g(n)$ are maximal right duo subalgebras and maximal left duo subalgebras of End K^n, $n = 1, 2,....$ This theorem depends heavily on theorem 10 which establishes these maximal duo-ness properties for the class A_n (and thus for its subclass J_n), of commutative algebras, $n = 2, 3,\ldots$. The effectiveness of "maximality in the endomorphism ring of K^n" is illustrated by analysis of one subalgebra $R \in A_n(K)$, where K is a finite dimensional extension of its prime field F. R, considered as a ring of linear transformations on F^{jn}, where $j = (K:F)$, is maximal right duo and maximal left duo in $F_{jn} = \text{End } K^n$. It is evident from the F-dimension of R in our example that these maximality properties are not provable by direct application to the F-linear situation of any theorem known to us.

Also in Section 4 (Theorem 7) we show that, when M is a faithful cyclic right R-module, no proper overring of R contained in End M can be a right duo ring. A dual theorem, concerning uniform representations, is shown to hold when $R \subseteq K_n$ (Theorem 8). Essentiality of the finiteness condition in theorem 8 is shown by a uniform R-module M which has infinite K-dimension, such that R is commutative but not maximal commutative in $\text{End}_K M$. In Section 5 non-commutative examples of maximal duo rings are presented, arising from Theorems 7 and 8. There is an example of a subalgebra of K_{12} which is not right duo and which has no right duo overring in End K^{12}. Section 6 is a list of open questions.

2. The Upper Bound Theorem

For a ring R with unity element every right ideal is an ideal if, and only if, $Rt \subseteq tR$ for every $t \in R$.

Definition. A ring is called a right duo ring if, and only if, every right ideal is an ideal. It is a duo ring if, and only if, it is a right duo ring and a left duo ring.

Evidently R is a duo ring if, and only if, $tR = Rt$ for every $t \in R$, since we consider only rings with unity element.

We prove some preliminary theorems which reduce the investigation of upper bounds to indecomposable representations and thus to local rings. The following lemma is proved in [8, p. 434]:

Lemma. *For positive integers n_1 and n_2, $g(n_1 + n_2) \geqslant g(n_1) + g(n_2)$. Equality holds only when $n_1 + n_2 = 2$ or 3.*

Theorem P1. *For $n = 1, 2,\ldots$, $\dim R \leqslant g(n)$ for all duo subalgebras R of K_n if and only if the inequality holds when K^n is an indecomposable R-module for the duo ring R. For $n > 3$, equality can hold only if K^n is indecomposable.*

Proof. If $e = e^2 \in \text{End}_R K^n$, $eR = \{er \mid r \in R\}$ is a duo ring. When the duo ring R is a maximal duo subalgebra of K_n, $e \in R$ can be deduced, since the direct sum of duo rings is a duo ring. Let K^n have a nontrivial direct decomposition: $K^n = M_1 \oplus M_2$, where M_1 and M_2 are R-modules. Let e be the corresponding projection on M_1; then $e \in R$. M_1 is a faithful (eR)-module and M_2 is a faithful $(I_n - e)$ R-module. eR and $(I_n - e)R$ are duo rings. Let $n_i = \dim M_i$, $i = 1, 2$. After applying an inner automorphism to K_n R assumes a diagonal block form with components in K_{n_1} and K_{n_2}, one of them similar to eR, the other to $(I_n - e)R$. The lemma, applied to n_1 and n_2, proves both statements of the theorem.

THEOREM P2. *Let M be an indecomposable right R-module satisfying the ascending and descending chain conditions. Let S be a duo ring such that M is an S-module and $R \subseteq S$. Then, for each $f \in S$, f is an automorphism of the additive group of M, or f is nilpotent modulo* $(0{:}\ M)_S$.

See, for example, the proof in [2, p. 138] of Fitting's Lemma. The lemma is concerned with R-endomorphisms but the proof is adaptable to theorem P2. One needs only to verify that the kernel and image of the group endomorphism f are R-submodules. Thus: If m is in the kernel of f, $mrf \in mfS = 0$ for each $r \in R$; $Sf \subseteq fS$ has been used to prove that the kernel is an R-submodule. Proof can be made that the image is an R-submodule, using the left duo property, $fS \subseteq Sf$.

THEOREM P3. *Let R be a duo ring which is represehted faithfully on an indecomposable module M satisfying the ascending and descending chain conditions. Then the set P of nilpotent elements is the radical of R, P is nilpotent, and R/P is a division ring.*

Proof. Let $x, y \in P$. Let $m > 0$ be such that $x^m = y^m = 0$. Since R is a right duo ring, $(x + y)^{2m} \in x^mR + y^mR = 0$, and $(xR)^m \subseteq x^mR = 0$. Thus P is a nil ideal. By Theorem P2, P is the unique maximal ideal of R and R/P is a division ring. Nakayama's Lemma [2, p. 169], applied to MP^i, yields $MP^i = 0$ or $MP^i \neq MP^{i+1}$ for each positive integer i. Evidently, $MP^j = 0$ for some j, whence $P^j = 0$.

NOTATION. We denote by $c(N)$ the composition length of an R-module N.

THEOREM 1. *Let R be a right duo local ring and let M_R be a faithful right R-module having composition length n. Then the composition length of R does not exceed $g(n)$.*

Proof. Let $\alpha = c(M/MP)$, where P is the radical of R. Let $\{x_1 + MP,..., x_\alpha + MP\}$ be an (R/P)-basis of M/MP. Then by Nakayama's Lemma

$$M = \sum_1^\alpha x_i R.$$

Let $s \in R$ be an annihilator of x_i. Since R is a right duo ring, $x_iRs \subseteq x_isR = 0$. We have $(0\colon x_i) = (0\colon x_iR)$, $i = 1,..., \alpha$, whence

$$(0\colon M) = \cap(0\colon x_iR) = \cap(0\colon x_i).$$

Let $\beta = n - \alpha$ denote $c(MP)$. Assume, if possible, that for some positive integer σ, $c(P) = \sigma + \alpha\beta$. Since R is a local ring,

$$(0\colon x_i) \subseteq P \qquad i = 1,..., \alpha.$$

Thus $c(P) = c((0\colon x_1)) + c(x_1P)$, whence $c((0\colon x_1)) \geqslant \sigma + (\alpha - 1)\beta$, since $c(x_1P) \leqslant \beta$. If, for any positive integer $j < \alpha$,

$$c\left(\bigcap^{j} (0 : x_i)\right) \geqslant \sigma + (\alpha - j)\beta$$

is assumed, then

$$c\left(x_{j+1}\left(\bigcap^{j} (0 : x_i)\right)\right) + c\left(\bigcap^{j+1} (0 : x_i)\right) = c\left(\bigcap^{j} (0 : x_i)\right)$$

and $c(x_{j+1}(\bigcap^j (0\colon x_i))) \leqslant c(MP) = \beta$, imply that

$$c\left(\bigcap^{j+1} (0 : x_i)\right) \geqslant \sigma + (\alpha - j - 1)\beta.$$

Thus we can obtain $c((0\colon M)) = c(\bigcap (0\colon x_i)) \geqslant \sigma > 0$, contrary to hypothesis. We have $c(P) \leqslant \alpha\beta$; $c(M) = n = \alpha + \beta$. It has been observed [5, p. 558] and is easily proved that the maximum of $\alpha\beta$ is $n^2/4$, occurring when $\alpha = \beta = n/2$. Since $c(P)$ is an integer, $c(P) \leqslant [n^2/4]$; $c(R) \leqslant g(n)$.

Remark. The obvious bijection proves that the dimension of a faithful cyclic module over a commutative K-algebra R equals that of R. This traditional result extends to right duo K-algebras: If $M = xR$, then for any annihilator s of x, $xRs \subseteq xsR = 0$, whence $s = 0$; $r \to xr$, $r \in R$, is an injection of R onto M.

THEOREM 2. *For $n = 1, 2, 3$, each maximal duo subalgebra R of K_n is commutative and has dimension $n = g(n)$. Either the R-module K^n is cyclic, or R is similar to the subalgebra R_1 of K_3 which is generated over K by I_3, E_{21} and E_{31}.*

Proof. Dim $R = n$ is implied by the second assertion of the theorem and the remark above. If K^n is directly decomposable, the pertinent projections belong to R. The module is cyclic and the ring is commutative if these properties hold for the direct summands. Thus we assume the indecomposability of K^n. By Theorem P3, R/P is a division ring, where P is the radical of R.

When R/P is not isomorphic to K and $n = 2$ or 3, the dimension of the simple module R/P divides n whence $\dim (R/P) = n$. Since R/P has dimension less than 4 over its center, it is a field. K^n is a simple, therefore cyclic, module. We have $g(c(K^n)) = g(1) = 1$. By Theorem 1, $c(R) = 1$, so that $P = 0$ and R is the field R/P.

Let $R/P \cong K$. We ignore the case $n = 1$. No maximal duo subalgebra of K_2 exists such that $P^2 \neq 0$, since in that case $\dim R \geqslant 3 > g(2) = 2$, contrary to Theorem 1. No maximal duo subalgebra of K_3 exists such that $P^3 \neq 0$ or that $P^2 \neq 0$ and $\dim (P/P^2) > 1$, since in these cases $c(R) = \dim R \geqslant 4 > g(3)$. Three cases remain: (i) $n = 2$ and $P^2 = 0$. By Theorem 1, $\dim R \leqslant 2$. Since each matrix in K_2 commutes with the identity, maximal duo-ness implies that $P \neq 0$. $R = KI_2 + P$ is commutative. Since $\dim R = 2$, $K^2 = xR$, if x is chosen so that $xP \neq 0$. (ii) $n = 3$ and $P^2 = 0$. Clearly, $R = KI_3 + P$ is commutative. By Theorem 1, $\dim R \leqslant 3$. KI_3 is not a maximal duo subalgebra of K_3, whence $P \neq 0$. If $M = K^3$ is cyclic, $\dim R = 3$. If M is not cyclic, we see from $P \neq 0$ that MP is one-dimensional and there are two generators, x_1 and x_2. x_1P and x_2P are nonzero, since M is indecomposable. If P were one-dimensional we would have (from $x_1P = x_2P = MP$) that R is similar to the algebra generated by I_3 and $E_{21} + E_{31}$ which is contained properly in the commutative subalgebra R_1 mentioned in the theorem. Thus $\dim P = 2$ and R is similar to R_1. (iii). $n = 3$, $P^2 \neq 0$, $P^3 = 0$ and $\dim (P/P^2) = 1$. Taking a nilpotent element $p \notin P^2$ we have $p^3 = 0$, $p^2 \neq 0$. Since $\dim R \leqslant g(3) = 3$, R equals its 3-dimensional commutative subalgebra generated by I_3, p and p^2. If x is chosen so that $xp^2 \neq 0$, $K^3 = xR$.

THEOREM 3. *Let R be a duo subalgebra of K_n. Then $\dim R \leqslant g(n)$. For $n > 3$, equality holds only when $R/(\text{rad } R) \cong K$.*

Proof. Considering Theorem 2, we may assume $n > 3$. According to Theorem P1 we prove the inequality if we prove it when K^n is indecomposable and equality is possible only under this assumption. By Theorem P3, R/P is a division ring, where $P = (\text{rad } R)$. Let β denote $\dim (R/P)$; let $m = c(K^n)$. Then $n = \beta m$. By Theorem 1 $c(R) \leqslant g(m)$, which is the required conclusion if $\beta = 1$. When $m = 1$, $c(R) = 1$, so that R ($\cong K^n$) has dimension $n < g(n)$, as $n > 3$.

Finally we have $\beta \geqslant 2$, $m \geqslant 2$. From

$$[n^2/4] = [(\beta m)^2/4] \geqslant \beta^2[m^2/4] \geqslant \beta + \beta[m^2/4] = \beta g(m) \geqslant \beta c(R),$$

we have $g(n) > \beta c(R)$, as required. When $\beta \neq 1$ strict inequality has been obtained, proving the second assertion.

3. Duo Subalgebras with Dimension $g(n)$

We prove now that duo subalgebras of K_n with dimension $g(n)$ are commutative.

For $i = 1,..., n$ let v_i be the n-tuple with 1 as the i-th entry and zeros elsewhere, so that $M = K^n$ has the basis $\{v_1, ..., v_n\}$. $v_\sigma E_{ij} = \delta_{\sigma i} v_j$ $[E_{ij} v_\sigma = \delta_{j\sigma} v_i]$, $1 \leqslant \sigma, i, j \leqslant n$, suffice to make K^n a right [left] R-module for $R \subseteq K_n$. It is helpful to consider an abstract background: Let M be a K-space with basis $\{u_1, ..., u_n\}$. The dual space $M^* = \mathrm{Hom}_K(M, K)$ has the basis $\{u_1^*, ..., u_n^*\}$ dual to that of M. To each $r \in R \subseteq \mathrm{End}\, M$, we define a linear transformation r^* on M^* by $(r^* t^*)(m) = t^*(mr)$, for all $m \in M$, all $t^* \in M^*$. The set R^* of linear transformations so obtained is a ring (the opposite of the ring of linear adjoints associated with R) isomorphic to R. Thus its radical, P^*, is the set of elements obtained from rad $R = P$. M^* is a left R^*-module. Our main interest is the equality

$$(0: (0: P)_M)_{M^*} = P^* M^*$$

[3, p. 190], which shows that $\{u_1^*, ..., u_{n-\beta}^*\}$ is a basis of P^*M^*, if $\{u_{n-\beta+1}, ..., u_n\}$ is a basis of $(0: P)_M$. Then, by Nakayama's Lemma, $M^* = R^* u_{n-\beta+1}^* + \cdots + R^* u_n^*$. To justify dropping asterisks in theorems to follow, we mention that $(\tilde{E}_{ij})^* u_\sigma^* = \delta_{\sigma j} u_i^*$, if $\{\tilde{E}_{ij}\}$ is the set of matrix units with respect to $\{u_1, ..., u_n\}$. We have proved the following proposition, except for the statement on bounds:

Proposition 4. *Let R be a duo subalgebra of K_n with $R/P \cong K$. Let $\{u_1, ..., u_n\}$ be a basis of K^n such that $\{u_{\alpha+1}, ..., u_n\}$ is a basis of MP and $\{u_{n-\beta+1}, ..., u_n\}$ is a basis of $(0: P)_M$. Then K^n is generated minimally by $\{u_1, ..., u_\alpha\}$ as a right R-module and by $\{u_{n-\beta+1}, ..., u_n\}$ as a left R-module. $\alpha(n - \alpha)$ and $\beta(n - \beta)$ are upper bounds of the dimension of P.*

Proof. We have $M = K^n = \sum_1^\alpha u_i R$. $N = \sum_1^\alpha K u_i$ is a vector space complement of MP in M. We claim for any nonzero $r \in P$ that the action of r on N is a nonzero linear transformation $\bar{r}$ of N into MP. For some element $m = \sum u_i s_i$, $s_i \in R$, $mr \neq 0$. Since R is a right duo ring, we have for $i = 1, ..., \alpha$

$$s_i r = r t_i \qquad t_i \in R.$$

$0 \neq mr = \sum u_i s_i r = \sum u_i r t_i$ proves that for at least one generator u_i, $u_i r \neq 0$. The injection $r \to \bar{r}$ is K-linear from P into the space of linear transformations of N into MP, whence $\dim P \leqslant \alpha(n - \alpha)$. Since the left R-module K^n is minimally generated by β elements and R is a left duo ring, $\dim P \leqslant \beta(n - \beta)$.

The procedure above involving induced linear transformations was introduced into upper bound investigations by Gustafson [5, p. 558].

Proposition 5. *Let $n = 2\alpha + 1$, $\alpha \geqslant 2$. K_n contains no duo subalgebra R achieving dimension $g(n)$ and satisfying*

$$\dim(M/MP) = \alpha = \dim((0: P)_M) \qquad M = K^n.$$

Proof. We assume the existence of such a subalgebra R: Dim $R = g(2\alpha+1) = \alpha^2 + \alpha + 1$. By Theorems P1 and 3 M is indecomposable and $R/P \simeq K$. Proposition 4 is applicable and its notation is adopted. For P to have $[n^2/4] = \alpha(\alpha + 1)$ dimensions, P must provide the full space of linear transformations of $N = \sum_1^\alpha Ku_i$ into MP. Since M is indecomposable, $MP \supseteq (0{:}\ P)$. Thus P contains the subspace with basis of α^2 elements

$$\{E_{ij} \mid 1 \leqslant i \leqslant \alpha, (\alpha + 2) \leqslant j \leqslant n\}.$$

With the help of these E_{ij} we may choose the remaining basis elements $\{p_1, \ldots, p_\alpha\}$ of P so that

$$u_j p_i = \delta_{ij} u_{\alpha+1} \qquad 1 \leqslant i, j \leqslant \alpha.$$

For some i, $u_{\alpha+1} p_i \neq 0$, since $u_{\alpha+1} \notin (0{:}\ P)$. Then for $j \neq i$

$$0 \neq u_{\alpha+1} p_i = u_j p_j p_i \in u_j p_i R$$

contradicts $u_j p_i = 0$, completing the proof that no such algebra R can exist.

Recall that the class $J_n(K)$ was defined in the introduction as consisting of all subalgebras R of K_n which are similar to $A(n^*, n, K)$, where $n^* = [n/2]$ or $[(n + 1)/2]$. $A(n^*, n, K) = KI_n + P(n^*, n, K)$, where $P(n^*, n, K)$ is the K-space generated by $\{E_{ij} \mid 1 \leqslant i \leqslant n^*, (n^* + 1) \leqslant j \leqslant n\}$. When $n > 3$, J_n is the class of all commutative subalgebras of K_n with dimension $g(n)$ [8] and [5, p. 560].

THEOREM 6. *Let R be a duo subalgebra of K_n, where K is an arbitrary field, and let* dim $R = g(n)$. *Then* (i) *for $n > 3$, $R \in J_n$* ; (ii) *for all n, R is commutative. Thus for each positive integer n R is a maximal duo subalgebra of K_n having dimension $g(n)$ if, and only if, R is a maximal commutative subalgebra of K_n having dimension $g(n)$.*

Proof. For $n = 1, 2, 3$, commutativity was proved in theorem 2, so that only (i) needs proof. Let $n > 3$. K^n is indecomposable and $R/P \simeq K$ by theorems P1, P3 and 3. Invoking proposition 4 and its notation, $\alpha = \beta = n/2$ when n is even, since $\alpha(n - \alpha)$ and $\beta(n - \beta)$ are upper bounds of the $(n^2/4)$-dimensional algebra P. To achieve the dimension, $(n^2 - 1)/4$, of P when n is odd, we must have $\alpha = (n - 1)/2$, $\beta = (n + 1)/2$ or the reverse choice, since, by Proposition 5, $\alpha = \beta = (n - 1)/2$ is impossible. Thus R is similar to $A(n^*)$, where $n^* = n/2$ $[(n - 1)/2$ or $(n + 1)/2]$, if n is even [odd]. For $n > 3$, $R \in J_n(K)$ has been proved.

COROLLARY. *If R is a noncommutative duo subalgebra of K_n such that* dim $R = [n^2/4]$, *then R is a maximal duo subalgebra of K_n.*

4. Maximality in the Endomorphism Ring of K^n

THEOREM 7. *Let M be a faithful cyclic right R-module for an arbitrary ring R. Let S be a right duo ring which contains R and is contained in* End M. *Then* $S = R$.

Proof. For some $x \in M$, $M = xR$. If $f \in S$, $xf = xr$ for some $r \in R$. Let $\alpha \in R$. Then $\alpha(f - r) \in (f - r)S$, so that $x\alpha(f - r) \in x(f - r)S = 0$. Since $M(f - r) = 0$ and M is a faithful S-module, $f = r$; $S = R$.

DEFINITION. An R-module M is called uniform, if $M_1 \cap M_2 \neq 0$ for all choices of nonzero submodules M_1 and M_2 of M. Eviedntly, when the descending chain condition holds on submodules of M, M is uniform if, and only if, M has a unique minimal submodule.

In Section 3 we denoted by M^* the dual of the n-dimensional K-space M. Recall that M^* is a faithful left module for a ring $R^* \cong R$, if M is a faithful right R-module. If V is a subspace of M [of M^*], the annihilators $(0\colon V)$ of V in M^* [in M] form a subspace with dimension $n - \dim V$. Thus the function $N \to (0\colon N)_{M^*}$ is a strictly order reversing mapping from the set $\{N\}$ of subspaces of M onto the set of subspaces of M^*.

We note that the set of annihilators in M^* [M] of a submodule of M [M^*] is a submodule, and conclude that M^* has a unique maximal submodule if, and only if, M has a unique minimal submodule. In the application to follow asterisks are avoided.

THEOREM 8. *Let a subalgebra R of K_n be such that K^n is a uniform right R-module. Let T be a left duo ring which contains R and is contained in* End K^n. *Then* $T = R$.

Proof. Since the right R-module K^n has a unique minimal submodule, the left R-module K^n has a unique maximal submodule N. Thus $K^n = Ry$ for every $y \notin N$. Since the left module K^n is cyclic, $T = R$ by Theorem 7.

Remark. Let the K-space M have countably infinite dimension; let $\{v_1, \ldots, v_j, \ldots\}$ be a basis of M. If R is the subalgebra of $\mathrm{End}_K M$ generated over K by the identity, $E_{21}, \ldots, E_{j1}, \ldots$, M is a uniform right R-module with $v_1 R = Kv_1$ as its unique minimal submodule. R is commutative, and is not a maximal commutative subalgebra of $\mathrm{End}_K M$, since the linear transformation $g = E_{21} + \cdots + E_{j1} + \cdots$ commutes with every element of R. This example demonstrates the essentiality of finiteness conditions in Theorem 8.

DEFINITION. Let $x_1, \ldots, x_m$, $y_1, \ldots, y_k$ be elements of an R-module M and let $r_{ij} \in R$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$. We say that $(r_{ij}\}$ is dense for the pair $((x_1, \ldots, x_m), (y_1, \ldots, y_k))$ if

$$x_\sigma r_{ij} = \delta_{\sigma i} y_j \qquad 1 \leqslant \sigma, i \leqslant m, 1 \leqslant j \leqslant k.$$

THEOREM 9. *Let R be a commutative ring and let M be a faithful right R-module generated by $\{x_1, \ldots, x_m\}$. Let Y be a proper submodule of M generated by $\{y_1, \ldots, y_k\}$. Let R have mk elements r_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, such that $\{r_{ij}\}$ is dense for the pair $((x_1, \ldots, x_m), (y_1, \ldots, y_k))$. Assume that*

$$(0\colon Y)_R \subseteq (Y\colon M)_R .$$

Then, if W is a right duo subring of End M *such that $R \subseteq W$ and $YW \subseteq Y$, $W = R$.*

Proof. Let $f \in W$. We fix elements $t_{ij} \in R$ such that

$$x_i f = \sum_j x_j t_{ij} \qquad i = 1, \ldots, m.$$

Let $i \neq j$. Since W is a right duo ring , $x_i f r_{j\alpha} \in x_i r_{j\alpha} W = 0$. Thus, when $i \neq j$, we have for $\alpha = 1, \ldots, k$

$$0 = x_i f r_{j\alpha} = \sum_q x_q t_{iq} r_{j\alpha} = \sum_q x_q r_{j\alpha} t_{iq} = x_j r_{j\alpha} t_{ij} = y_\alpha t_{ij} .$$

We have proved that

$$t_{ij} \in (0\colon Y)_R \subseteq (Y\colon M)_R \qquad i \neq j \tag{A}$$

Let $\beta \in \{1, \ldots, k\}$. An element $g_\beta \in W$ exists such that

$$f \sum_i r_{i\beta} = \sum_i r_{i\beta} g_\beta .$$

For $\alpha \in \{1, \ldots, m\}$, $y_\beta g_\beta = x_\alpha r_{\alpha\beta} g_\beta = x_\alpha(\sum_i r_{i\beta}) g_\beta = x_\alpha f(\sum_i r_{i\beta}) = \sum_{i,j} x_j t_{\alpha j} r_{i\beta} = y_\beta(\sum_i t_{\alpha i})$. But by assertion (A) $Y t_{\alpha i} = 0$, when $i \neq \alpha$, so that we have obtained

$$y_\beta g_\beta = y_\beta t_{\alpha\alpha} \qquad 1 \leqslant \alpha \leqslant m.$$

Thus for each α and for each generator y_β of Y, $y_\beta(t_{11} - t_{\alpha\alpha}) = 0$. Since R is commutative,

$$t_{11} - t_{\alpha\alpha} \in (0\colon Y)_R \subseteq (Y\colon M)_R \quad \alpha = 1, \ldots, m. \tag{B}$$

Let S_α denote the complement of $\{\alpha\}$ in $\{1, \ldots, m\}$. From assertions (A) and (B),

$$x_\alpha(f - t_{11}) = \sum_{S_\alpha} x_j t_{\alpha j} + x_\alpha(t_{\alpha\alpha} - t_{11}) \in Y \qquad 1 \leqslant \alpha \leqslant m.$$

Let $f^* = f - t_{11}$. For each $s \in R$, $x_\alpha s f^* \in x_\alpha f^* W \subseteq YW \subseteq Y$, $1 \leqslant \alpha \leqslant m$. Thus we have

$$Mf^* \subseteq Y, \qquad f^* = f - t_{11} .$$

For $i = 1,..., m$, let $x_i f^* = \sum_j y_j c_{ij}$, $c_{ij} \in R$. Then $x_i f^* = \sum_j x_i r_{ij} c_{ij}$. Let $p = \sum_{i,j} r_{ij} c_{ij}$. Since $x_\alpha(\sum_j r_{ij} c_{ij}) = 0$, when $\alpha \neq i$, we have

$$x_i f^* = x_i p \qquad i = 1,..., m.$$

For $i = 1,..., m$, $x_i R(f^* - p) \subseteq x_i(f^* - p)W = 0$. Since $M = \sum x_i R$ is faithful for W, $f^* = p$. We have proved that $f = t_{11} + p \in R$; $W = R$.

In the introduction the class $A_n(K)$ was defined as the union of similarity classes determined by $\{A(n^*, n, K) \mid n^* = 1,..., (n-1)\}$, where $A(n^*, n, K) = A(n^*)$ is the subalgebra $KI_n + P(n^*)$ of K_n, where $P(n^*)$ is the subalgebra generated over K by

$$\{E_{ij} \mid 1 \leqslant i \leqslant n^*, (n^* + 1) \leqslant j \leqslant n\}.$$

It was proved in [1, p. 44] that subalgebras in this form are maximal commutative subalgebras of K_n.

THEOREM 10. *Let $R \in A_n$, $n = 2, 3,...$. Then the commutative algebra R is a maximal right duo and maximal left duo subalgebra of* End K^n.

Proof. Let $R = A(n^*)$, $n^* \in \{1,..., (n-1)\}$. Let $P = (\text{rad } R)$. In terms of the canonical basis $\{v_1,..., v_n\}$ of $M = K^n$, let $Y = MP = Kv_{n^*+1} + \cdots + Kv_n = (0: P)_M$. We note that $M = \sum_1^{n^*} v_i R$, and that $\{E_{ij} \mid 1 \leqslant i \leqslant n^*, (n^* + 1) \leqslant j \leqslant n\}$ is dense for the pair $((v_1,..., v_{n^*}), (v_{n^*+1},..., v_n))$. The hypotheses of Theorem 9 are satisfied, if we show that $YW \subseteq Y$ for each right duo ring W which contains R and is contained in End K^n. Given such a ring W, we suppose that $f \in W$ satisfies

$$d = yf \notin Y$$

for some $y \in Y$. For some integer $h \leqslant n^*$, the h-th entry, d_h, of the n-tuple d is nonzero, then. Since W is a right duo ring, $fE_{hn} = E_{hn}w$ for some $w \in W$. Since $Y = (0: P)_M$, $yE_{hn}w = 0$. But this contradicts $yfE_{hn} = dE_{hn} = (0,..., 0, d_h)$, a nonzero n-tuple. We have $YW \subseteq Y$. By Theorem 9, R is a maximal right duo subalgebra of End K^n.

For the subalgebra R which is similar to $A(n^*)$, the algebra formed by taking transposes of the matrices, in R is similar to $A(n - n^*)$ and a maximal right duo subalgebra of End K^n, whence R is a maximal left duo subalgebra of End K^n.

Remark. In order to illustrate the effectiveness of theorems which imply that a subalgebra of K_n is a maximal duo subalgebra of End K^n, we make an example. Let K be an extension of its prime field F such that $(K: F) = j < \infty$. Then End $K = F_j$ and End $K^n = F_{jn}$. In our example $n = 6$ and K is a cubic extension field of its prime field F. Let $R = A(2, 6, K)$. The K-dimension of R is 9. R, considered as a ring of operators on F^{18}, is a commutative subalgebra of F_{18} having F-dimension 27. By Theorem 10, R is a maximal right duo and a maximal left duo subalgebra of F_{18}. One can verify that the F-dimension of S is either 18 or exceeds 32, when $S \in A_{18}(F)$. Thus our example is not an illustration of applying any of our results directly to the F-linear situation. Theorem 7, however, will not produce additional examples by this sort of indirect application, for the algebra would again be represented cyclically on F^{jn}. A similar comment holds for uniform representations.

THEOREM 11. *If R is a duo subalgebra of K_n and dim $R = g(n)$, R is a maximal right duo and a maximal left duo subring of* End K^n.

Proof. It is sufficient to prove that R is a maximal right duo subring, since the algebra formed by taking transposes of the matrices in R also satisfies the hypothesis. When $n \leqslant 3$, either R is similar to $A(2, 3, K)$ or the representation is cyclic by Theorem 2. When $n > 3$, $R \in A_n(K)$ by Theorem 6. Theorems 7 and 10 imply that R is a maximal right duo subring of End K^n.

5. EXAMPLES

Noncommutative maximal duo subalgebras of K_n will be presented. These, of course, have dimension less than $g(n)$. The third example has dimension greater than n.

The canonical basis of the n-tuple space will be denoted by $\{v_1, \dots, v_n\}$.

EXAMPLE A. Both cyclic and uniform representation are illustrated by the division ring Q of real quaternions imbedded in K_4, where K is the field of real numbers. Thus Q is a maximal right duo and a maximal left duo subring of End K^4. The following imbedding appears in [7, p. 8]:

$$\left\{ \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \middle| \; a, b, c, d \in K \right\}$$

EXAMPLE B. A noncommutative subalgebra of K_{12} which is not a right duo ring and is not contained in a right duo subring of End K^{12}. Let K be the field of real numbers. Let R be the tensor product, $Q \otimes S$, of the algebra Q of example A and the following subalgebra of K_3: $S = KI_3 + KE_{12} + KE_{13}$. Since $K^4 = (1, 0, 0, 0)Q$ and $K^3 = (1, 0, 0)S$, $K^{12} = vR$, where $v = (1, 0, 0, 0) \otimes (1, 0, 0)$. By Theorem 7 a proper overring of R contained in End K^{12} can not be a right duo ring. We show that R is not a right duo ring. For $t \in Q$, let tI denote $t \otimes I_3$, let $\bar{t}$ denote $t \otimes E_{12}$, and let t^* denote $t \otimes E_{13}$. Clearly, rad $R = \{\bar{t}_1 + t_2^* \mid$

$t_i \in Q\}$ and $(\text{rad } R)^2 = 0$. Let e be the unity element of Q. Choose noncommuting elements f and g in Q. Suppose that w exists in R such that

$$(gI)(\bar{f} + e^*) = (\overline{gf}) + g^* = (\bar{f} + e^*)w.$$

Then w must have the form $(f^{-1}\, gf)I + w_1$, where $w_1 \in \text{rad } R$. Since $(\bar{f} + e^*)w_1 = 0$, $(\bar{f} + e^*)w) = (\overline{gf}) + (f^{-1}\, gf)^*$. But by our assumption, $f^{-1}\, gf \neq g$. We have proved that R is not a right duo ring.

PROPOSITION. *Let R be a K-algebra with identity I such that R is the direct sum as a K-space of KI and the radical P of R. Assume that P is anticommutative. Then R is a duo ring.*

Proof. For $k \in K$ and $w, v \in P$, we have

$$(kI + v)w = kw + vw = w(kI - v) \in wR;$$
$$w(kI + v) = kw - vw = (kI - v)w \in Rw.$$

Since $yR = R = Ry$, when y is an invertible element of R, the proof is complete.

EXAMPLE C. A noncommutative duo ring R which is a 7-dimensional maximal right duo subalgebra of K_6. Let K be a field whose characteristic is not two. We claim that the algebra R defined below satisfies the hypothesis of the proposition above:

$$R = \left\{ \begin{pmatrix} a & b & c & d & e & 0 \\ 0 & a & 0 & -c & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & p & q & a \end{pmatrix} \middle| \; a, b, c, d, e, p, q \in K \right\}$$

We claim that, when $a = 0$, the radical P is obtained, that P is anticommutative and that $P^3 = 0$. Let $a = 0$, b, c, d, e, p and q define a matrix $T \in R$ and let $a = 0$, b', c', d', e', p' and q' define a matrix $T' \in R$. Then $TT' = (-bc' + cb')$ $E_{14} = -T'T$. The product of three such matrices is zero, then. By the proposition, R is a duo ring.

Let M' be the subspace of K^6 generated over K by $\{v_1, ..., v_5\}$. M' is the R-submodule v_1R of K^6. Thus, by Theorem 7, $R/(0{:}\, M')_R$ is a maximal right duo subring of End M'. Suppose, if possible, that R is properly contained in a right duo ring S, $S \subseteq K_6$. Let $\sigma \in S$, $\sigma \notin R$. Since σ's restriction to M' belongs to R, we are interested in an element

$$\sigma = \sum k_{ij}E_{ij}$$

where the summation is taken over the eleven pairs (i, j) such that i or j equals 6. Since E_{64} and $E_{65} \in S$, we stipulate $k_{64} = k_{65} = 0$. We proceed to prove that $\sigma = 0$. Let $\bar{b}$ denote the matrix in the definition of R which is obtained by setting $b = 1$, while the remaining parameters vanish. Let similar definitions hold for $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{p}$ and $\bar{q}$. The vanishing of k_{61}, k_{62}, and k_{63} is proved by the following steps in sequence: $v_6\sigma\bar{d} = k_{61}v_4 \in v_6\bar{d}S = 0$; $v_6\sigma\bar{c} = -k_{62}v_4 \in v_6\bar{c}S = 0$; $v_6\sigma\bar{b} = k_{63}v_4 \in v_6\bar{b}S = 0$. We have proved that $\sigma = \sum_i k_{i6}E_{i6}$.

For $i = 1,\dots, 5$, $v_i\sigma E_{64} = k_{i6}v_4 \in v_iE_{64}S = 0$. Thus k_{i6} vanishes, $1 \leqslant i \leqslant 5$, and we have proved that $\sigma = k_{66}E_{66}$.

$v_6E_{64}(k_{66}I_6 - \sigma) = k_{66}v_4 \in v_6(k_{66}I_6 - \sigma)S = 0$ proves that $k_{66} = 0$. $\sigma = 0$ has been proved, whence $S = R$. R is a duo, maximal right duo subalgebra of K_6. We mention that maximal left duoness is also provable. In proving this it is helpful that a quotient module of K^6, $K^6/(v_5R)$, is uniform.

6. Open Question

(a) Does a maximal commutative subalgebra of K_n exist which is not a maximal duo subalgebra of K_n?

(b) When K^n is a faithful cyclic right R-module, can R have a proper overring $S \subsetneq \operatorname{End} K^n$ which is a left duo ring?

(c) Section 3 establishes $[n^2/4]$ as an upper bound for dimensions of noncommutative duo subalgebras of K_n. For $n > 4$, is $[n^2/4]$ the maximum? For $n = 4$, see example A.

(d) We conjecture that maximal duo subalgebras of K_n exist with dimension less than n. In case they exist, find a lower bound of the dimensions of these subalgebras. In the case of maximal commutative subalgebras a lower bound, $n^{2/3}$, has been proved [5, p. 561].

(e) Assuming that conjecture (d) is true, is zero the greatest lower bound of the ratios $\{(\dim R)/n \mid R$ a maximal duo subalgebra of $K_n\}$? For maximal commutative subalgebras, this greatest lower bound property has been established [4, p. 9].

References

1. H. S. Allen, Commutative rings of linear transformations and infinite matrices, *Quart. J. Math. Oxford Ser. (2)* **8** (1957), 39–53.
2. F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules, Springer-Verlag, New York, 1974.
3. R. C. Courter, Maximal commutative algebras of linear transformations, *Trans. Amer. Math. Soc.* **101** (1961), 177–199.

4. R. C. COURTER, Maximal commutative subalgebras of K_n at exponent three, *Linear Algebra and Appl.* **6** (1973), 1-11.
5. W. H. GUSTAFSON, On maximal commutative algebras of linear transformations, *J. Algebra* **42** (1976), 557–563.
6. P. R. HALMOS, "Finite Dimensional Vector Spaces," 2d ed., Van Nostrand, New York, 1958.
7. I. N. HERSTEIN, "Noncommutative Rings," Carus Mathematical Monographs, No. 15, Math. Assoc. Amer., Washington, D.C., 1968.
8. N. JACOBSON, Schur's theorem on commutative matrices, *Bull. Amer. Math. Soc.* **50** (1944), 431–436.
9. I. SCHUR, Zur Theorie vertauschbaren Matrizen, *J. Reine Angew. Math.* **30** (1905), 66–76.