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The Connection Matrix Theory for Semiflows on (Not Necessarily Locally Compact) Metric Spaces

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The index theory of Rybakowski for isolated invariant sets and attractor–repeller pairs in the setting of a semiflow on a not necessarily locally compact metric space is extended to include a connection matrix theory for Morse decompositions. Partially ordered Morse decompositions and attractor semifiltrations of invariant sets are defined and shown to be equivalent. The definition and proof of existence of index filtrations for an ordered Morse decomposition is provided. Via the index filtration, the homology index braid and the connection matrices of the Morse decomposition are defined. © 1988 Academic Press, Inc.

INTRODUCTION

In a series of papers the Conley index theory for locally compact local (semi)flows in a flow on a Hausdorff space is developed. Conley [1] defines the index for an isolated invariant set. The index is defined via the index pair, a pair of compact sets which act, roughly, as an isolating neighborhood of the invariant set and an exit set for the isolating neighborhood. Conley [1] and Kurland [7, 8] extend the theory to include an index sequence for attractor–repeller pairs in an isolated invariant set. The index sequence is defined via index triples for the

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attractor–repeller pair. Conley and Zehnder [2] generalize the index triple to an index filtration for a totally ordered Morse decomposition of an isolated invariant set, and using the index filtration develop Morse inequalities for such Morse decompositions. The first author in [4–6] further develops the index theory for Morse decompositions. The homology index braid of a Morse decomposition of an isolated invariant set is introduced in [4]. The homology index braid is defined via index filtrations for partially ordered Morse decompositions. The connection matrix theory for Morse decompositions is developed in [5]. The collection of connection matrices is an algebraic invariant of the homology index braid.

Rybakowski [10, 11] and Rybakowski and Zehnder [12] develop a corresponding index theory for the setting of a local semiflow on a (not necessarily locally compact) metric space. In this setting the compactness condition on the index pair (and therefore index triple and filtration) is weakened to a condition called admissibility. The index of an isolated (by an admissible neighborhood) invariant set and the index sequence of an attractor–repeller pair in such an invariant set are developed in [10] and [11], respectively. Morse inequalities for Morse decompositions are established in [12]. In this paper we further develop this index theory to include a connection matrix theory. It is via index filtrations for partially ordered Morse decompositions that the connection matrices are ultimately defined, so it is the main intent of this paper to establish the existence of index filtrations.

The paper begins in Section 1 with basic background definitions. In Section 2 we define ordered Morse decompositions and attractor (semi)filtrations and establish the relationship between them. In Section 3 we define and prove the existence of index filtrations for partially ordered Morse decompositions. Finally, in Section 4 the homology index braid and the connection matrices of a Morse decomposition are studied.

1. Definitions

Throughout this paper P denotes a finite indexing set with p elements. A *partial order* on P is a relation, $<$, on the elements of P satisfying

(1) $\pi < \pi$ never holds for $\pi \in P$,

(2) $\pi < \pi'$ and $\pi' < \pi''$ imply $\pi < \pi''$.

A *total order* on P also satisfies

(3) for every $\pi, \pi' \in P$ either $\pi < \pi'$ or $\pi' < \pi$.

Assume throughout that $<$ is a partial order on P.

An *extension* of $<$ is a partial order $<_*$ on for which $\pi < \pi'$ implies $\pi <_* \pi'$. If $<_*$ is a total order, then it is called a *linear extension* of $<$.

An *interval* in $<$ is a subset $I \subset P$ for which $\pi, \pi' \in I$ and $\pi < \pi'' < \pi'$ imply $\pi'' \in I$. The set of intervals in $<$ is denoted $I(<)$. $I \in I(<)$ is called an *attracting interval* if $\pi \in I$ and $\pi' < \pi$ imply $\pi' \in I$. The set of attracting intervals in $<$ is denoted $A(<)$. Theorem 2.4 provides the justification for the use of the term "attracting." It should be noted that $\phi \in A(<)$.

An *adjacent n-tuple of intervals in* $<$ is an ordered collection $(I_1, ..., I_n)$ of mutually disjoint intervals of $<$ satisfying

(1) $\bigcup_{i=1}^{n} I_i \in I(<)$,

(2) $\pi \in I_j$, $\pi' \in I_k$, $j < k$ imply $\pi' \not< \pi$.

The collection of adjacent n-tuples in $<$ is denoted $I_n(<)$. Note that $I(<) = I_1(<)$. If (I, J) is an adjacent pair (i.e., 2-tuple) of intervals, then we set $IJ = I \cup J$. If $(I_1, ..., I_n) \in I_n(<)$ and $\bigcup_{i=1}^{n} I_i = I$, then $(I_1, ..., I_n)$ is called a *decomposition* of I. If (I, J), $(J, I) \in I_2(<)$, then I and J are said to be *noncomparable*. If $\{\pi, \pi'\} \in I(<)$, then π and π' are said to be *adjacent* in the partial order.

See [4] for more details on partial orders.

The dynamics in which we are interested lie in a metric space X with metric $d(\cdot, \cdot)$. Specifically, let D be an open subset of $[0, \infty) \times X$ and $\psi: D \to X$ be continuous. Set $x \cdot t = \psi(t, x)$. ψ is called a *local semiflow* if the following properties are satisfied:

(1) For every $x \in X$ there exists ω_x, $0 < \omega_x \leqslant \infty$, such that $(t, x) \in D$ if and only if $0 \leqslant t < \omega_x$,

(2) $x \cdot 0 = x$,

(3) if (t, x), $(s, x \cdot t) \in D$, then $(t+s, x) \in D$ and $x \cdot (t+s) = (x \cdot t) \cdot s$.

Throughout the remainder of the paper assume that a local semiflow ψ is fixed.

If $Y \subset X$ and $Q \subset [0, \infty)$, then we define $Y \cdot Q = \{x \cdot t \mid x \in Y,\ t \in Q\}$.

If $Y \subset X$ is such that $\omega_x = \infty$ for each $x \in Y$, then we define the *ω-limit set* of Y to be the set $\omega(Y) = \bigcap_{t \geqslant 0} cl\{Y \cdot [t, \infty)\}$.

If $x \in X$, then a *solution* through x is a continuous map

$$\sigma_x : (a, \omega_x) \to X$$

where $a \in [-\infty, 0)$ and σ_x satisfies

(1) $\sigma_x(0) = x$,

(2) for $t \in (a, \omega_x)$ and $s > 0$ with $s + t < \omega_x$, it follows that $s < \omega_{\sigma_x(t)}$ and $\sigma_x(t) \cdot s = \sigma_x(t+s)$.

If $a = -\infty$, then σ_x is called a *full left solution* through x. If, in addition, $\omega_x = \infty$, then σ_x is called a *full solution* through x.

If σ_x is a full left solution through x, then we define the *ω^*-limit set* of σ_x to be the set $\omega^*(\sigma_x) = \bigcap_{t \geqslant 0} \text{cl}\{\sigma_x((-\infty, -t])\}$. Note that since σ_x is not uniquely determined for x, $\omega^*(\sigma_x)$ is dependent on the full left solution chosen, not just on x.

For $Y \subset X$ we set

$$A^+(Y) = \{x \in X \mid x \cdot [0, \omega_x) \subset Y\}$$

$$A^-(Y) = \{x \in X \mid \text{there is a full left solution } \sigma_x \text{ through } x \text{ such that } \sigma_x((-\infty, 0] \subset Y\}.$$

A set $S \subset X$ is called *invariant* if $S = A^+(S) = A^-(S)$. It is implicit in this definition that there is a full left solution through each point in an invariant set.

2. Morse Decompositions and Attractor Semifiltrations

Assume throughout the remainder of the paper that S is a compact invariant set and $\omega_x = \infty$ for each $x \in S$. Therefore there is a full solution through each $x \in S$.

A subset $A \subset S$ is called an *attractor* in S if there exists a neighborhood U of A such that $\omega(U \cap S) = A$. If A is an attractor in S, then $A^* := \{x \in S \mid \omega(x) \cap A = \phi\}$ is called the *repeller* dual to A in S. The pair (A, A^*) is called an attractor–repeller pair in S.

One is often interested in decomposing the invariant set S into finer invariant subsets. One way to do this is to consider sets of attractors in S. More specifically,

Definition 2.1. An *attractor filtration* of S is a finite collection, $\mathscr{A}$, of attractors in S satisfying

(1) $\phi, S \in \mathscr{A}$,

(2) if $A_1, A_2 \in \mathscr{A}$, then $A_1 \cap A_2$, $A_1 \cup A_2 \in \mathscr{A}$.

Another refinement can be obtained by considering mutually disjoint compact invariant subsets of S. Specifically,

Definition 2.2. A ($<$-ordered) *Morse decomposition* of S is a collection $M(S) = \{M(\pi)\}_{\pi \in P}$ of mutually disjoint compact invariant subsets of

S such that if $x \in S$ and $\sigma_x: \mathbb{R} \to S$ is a full solution through x, then one of the following holds:

(1) there exists $\pi \in P$ such that $\sigma_x(\mathbb{R}) \subset M(\pi)$,

(2) there exist π, $\pi' \in P$ such that $\pi < \pi'$, $\omega^*(\sigma_x) \subset M(\pi')$, and $\omega(x) \subset M(\pi)$.

We usually write M for $M(S)$; however, it is important to note that the definitions below do not depend only on the collection of sets M, but also on the invariant set S of which M is a Morse decomposition.

If S_1 and S_2 are compact invariant subsets of S, then we set

$$C(S_2, S_1) = \{x \in S \mid \omega(x) \subset S_1 \text{ and } \omega^*(\sigma_x) \subset S_2 \text{ for some full solution } \sigma_x: \mathbb{R} \to S\}.$$

Throughout the remainder of the paper assume $M = \{M(\pi)\}_{\pi \in P}$ is a Morse decomposition of S. Note that $C(M(\pi), M(\pi)) = M(\pi)$ for each $\pi \in P$. Furthermore note that if $x \in S \backslash \bigcup_{\pi \in P} M(\pi)$, then there exists $\pi < \pi'$ such that $x \in C(M(\pi'), M(\pi))$; however, because of the nonuniqueness of full solutions through x, there does not necessarily exist a unique $\pi' \in P$ such that $x \in C(M(\pi'), M(\pi))$.

The partial order $<$ on P induces an obvious partial order $<$ on M, called an *admissible ordering* of M. M may have many admissible orderings, but there is an "extremal" admissible ordering on M, called the *flow ordering* of M and denoted $<_F$, which is such that $\pi <_F \pi'$ if and only if there exists a sequence of distinct elements of P: $\pi = \pi_0, ..., \pi_n = \pi'$, such that $C(M(\pi_j), M(\pi_{j-1})) \neq \phi$ for each $j = 1, ..., n$. It is not difficult to see that every admissible ordering of M is an extension of the flow ordering of M.

Conley [1] and Franzosa [3] show the equivalence of attractor filtrations and ordered Morse decompositions for the setting of a flow on a locally compact space. In the setting that we consider here Rybakowski and Zehnder [12] show the equivalence holds if the admissible ordering of the Morse decomposition is a total order. In this section we consider the case where the admissible ordering is a partial order. We need a broader type of filtration of attractors to establish the desired equivalence.

DEFINITION 2.3. An *attractor semifiltration* of S is a finite collection $\mathscr{A}$ of attractors in S satisfying

(1) $\phi, S \in \mathscr{A}$,

(2) if $A_1, A_2 \in \mathscr{A}$, then $A_1 \cup A_2 \in \mathscr{A}$ and $\omega(A_1 \cap A_2) \in \mathscr{A}$.

If A is an attractor, then $\omega(A) = A$, hence an attractor filtration is an attractor semifiltration. We show that given an ordered Morse decomposition of S, there is a corresponding attractor semifiltration, and vice versa. The admissible ordering plays an important role in the correspondence.

For $I \in I(<)$ define

$$M(I) = \left(\bigcup_{\pi \in I} M(\pi) \right) \cup \left(\bigcup_{\pi, \pi' \in I} C(M(\pi), M(\pi')) \right)$$

The sets $M(I)$ are called the *Morse sets* of the admissible ordering of the Morse decomposition and the collection $\{M(I)\}_{I \in I(<)}$ is denoted $MS(<)$. If $<$ is the flow ordering of the Morse decomposition then we call $MS(<)$ the Morse sets of the Morse decomposition and denote it by $MS(M)$. It is easy to see that $MS(<) \subset MS(M)$ for each admissible ordering $<$ of M.

The Morse sets that correspond to attracting intervals in $<$ form an attractor semifiltration of S (Theorem 2.4 below), but not necessarily an attractor filtration (Fig. 2.1).

THEOREM 2.4. *The collection* $\mathscr{A} := \{M(I) \mid I \in A(<)\}$ *is an attractor semifiltration of* S.

Before proving Theorem 2.4 we need

LEMMA 2.5 (cf. [11, Corollary 3.1]). *If* (A, A^*) *is an attractor–repeller pair in* S, *and* C *is compact and such that* $A \subset C$ *and* $A^* \cap C = \phi$, *then* $\omega(C) = A$.

Proof. We prove that for each $\varepsilon > 0$ there is a t_ε such that $d(C \cdot t, A) < \varepsilon$ for all $t > t_\varepsilon$. From this the lemma easily follows.

Assume that the claim is not true, i.e., that there exists $\varepsilon > 0$ and sequences $\{z_n\} \subset C$, $t_n \to \infty$ such that $d(z_n \cdot t_n, A) \geq \varepsilon$. Set $B = \{x \in S \mid d(x, A) \geq \varepsilon\}$. Thus $z_n \cdot t_n \in B$ for all n. Let ε' be such that if $d(z, A^*) < \varepsilon'$ then $z \notin C$, and set $B' = \{x \in S \mid d(x, A^*) < \varepsilon'\}$. B is closed and disjoint from A; therefore by [11, Lemma 3.1] it follows that for sufficiently large n, $z_n \in B'$. But $B' \cap C = \phi$ and $z_n \in C$; contradiction. ∎

Proof of Theorem 2.4. We begin by showing that $M(I)$ is an attractor for each $I \in A(<)$. If $I \in A(<)$, then it is easy to see that there exists a linear extension $<_*$ of $<$ such that $I \in A(<_*)$. M is a Morse decom-

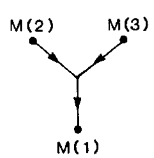


FIG. 2.1. The attracting Morse sets do not form an attractor filtration since $M(12) \cap M(13) \neq M(1)$.

position of S with total admissible ordering $<_*$. By Rybakowski and Zehnder's definition of Morse decompositions in terms of nested attractor filtrations, along with their propositions 2.3, 4 [12], it follows that $M(I)$ is an attractor in S.

$\phi = M(\phi)$ and $S = M(P)$; therefore ϕ, $S \in \mathscr{A}$. If I, $J \in A(<)$, then $I \cup J \in A(<)$; therefore $M(I \cup J)$ is an attractor. We claim that $M(I) \cup M(J) = M(I \cup J)$, and therefore $M(I) \cup M(J) \in \mathscr{A}$. Clearly $M(I) \cup M(J) \subset M(I \cup J)$. Let $x \in M(I \cup J)$; we show that $x \in M(I) \cup M(J)$. If $\sigma_x : \mathbb{R} \to S$ is a full solution through x, then there exists π', $\pi \in I \cup J$ such that $\pi \leqslant \pi'$, $\omega^*(\sigma_x) \subset M(\pi')$, and $\omega(x) \subset M(\pi)$. If $\pi' \in I$, then $\pi \in I$, therefore since $x \in C(M(\pi'), M(\pi))$, it follows that $x \in M(I) \subset M(I) \cup M(J)$. Similarly, if $\pi' \in J$, then $x \in M(I) \cup M(J)$. Thus $M(I \cup J) = M(I) \cup M(J)$.

If I, $J \in A(<)$, then $I \cap J \in A(<)$; thus $M(I \cap J)$ is an attractor in S. We claim that $\omega(M(I) \cap M(J)) = M(I \cap J)$, and therefore $\omega(M(I) \cap M(J)) \in \mathscr{A}$. Clearly $M(I) \cap M(J)$ is compact and contains the attractor $M(I \cap J)$. It is easy to see that if $x \in M(I) \cap M(J)$, then $\omega(x) \subset M(I \cap J)$; therefore $M(I) \cap M(J)$ is disjoint from the repeller complementary to $M(I \cap J)$. Lemma 2.5 then implies that $\omega(M(I) \cap M(J)) = M(I \cap J)$. ▌

Thus an admissible ordering of a Morse decomposition determines an attractor semifiltration; we call this the *attractor semifiltration defined by the admissible ordering* $<$ *of* M. The converse is proved in Theorem 2.6.

In the proof of Theorem 2.4 it is shown that if I is an attracting interval then $M(I)$ is an attractor in S. It is not difficult to see that $M(P \setminus I)$ is the complementary repeller, and, more generally, if $(I, J) \in I_2(<)$ then $(M(I), M(J))$ is an attractor–repeller pair in $M(IJ)$.

THEOREM 2.6. *If $\mathscr{A}$ is an attractor semifiltration of S, then there exists a Morse decomposition M of S with an admissible ordering such that $\mathscr{A}$ is the attractor semifiltration defined by the admissible ordering.*

Proof. Order the attractors in $\mathscr{A}$: $A_0, ..., A_k$, so that $A_i \subset A_j$ implies that $i < j$. Let $\bar{A}_i = \bigcup_{j=1}^{i} A_j$; $\{\bar{A}_i\}$ is an increasing sequence of attractors in S, $\bar{A}_0 = \phi$, and $\bar{A}_k = S$. By Rybakowski–Zehnder [12, Definition 2.2 and Propositions 2.3, 4] the collection of $\bar{A}_i$s determines a Morse decomposition $M = \{M(i)\}_{i=1,...,k}$ of S where $M(i) = \bar{A}_i \cap \bar{A}_{i-1}^*$ for each $i = 1, ..., k$.

We need to define an admissible ordering $<_*$ on M so that $\mathscr{A}$ is the attractor semifiltration defined by $<_*$. First we claim that for each $M(i) \in M$ and $A \in \mathscr{A}$, if $M(i) \cap A \neq \phi$ then $M(i) \subset A$. To see this, first note that $M(i) \subset A_i$; therefore $\omega(A_i \cap A) \cap M(i) \neq \phi$. Thus by the construction of the $\bar{A}_i$'s and the definition of M it follows that $\omega(A_i \cap A) \not\subset A_j$ for $j = 1, ..., i-1$. Therefore, since $\omega(A_i \cap A) \in \mathscr{A}$ and $\omega(A_i \cap A) \subset A_i$, it follows by the ordering of the attractors in $\mathscr{A}$ that $\omega(A_i \cap A) = A_i$. But $\omega(A_i \cap A) \subset A$; so $M(i) \subset A_i = \omega(A_i \cap A) \subset A$.

Now define a partial order $<_*$ on $\{1, ..., k\}$ $i <_* j$ if there exits $A \in \mathscr{A}$ containing $M(i)$ but not $M(j)$ and if every $A \in \mathscr{A}$ containing $M(j)$ also contains $M(i)$. It is easy to see that $<_*$ is a partial order; we claim that M is a $<_*$-ordered Morse decomposition. We must show that if $x \in S$ and $\sigma_x: \mathbb{R} \to S$ is a full solution through x then either $\sigma_x(\mathbb{R}) \subset M(i)$ for some i, or there exists $i <_* j$ such that $\omega(x) \subset M(i)$ and $\omega^*(\sigma_x) \subset M(j)$. Suppose the former does not hold; then since M is a $<$-ordered Morse decomposition where $<$ is the usual order on the integers, it follows that there exists $i < j$ such that $\omega(x) \subset M(i)$ and $\omega^*(\sigma_x) \subset M(j)$. We claim that $i <_* j$. It is easy to see that $\bar{A}_i \in \mathscr{A}$ contains $M(i)$ but not $M(j)$. We need to show that every $A \in \mathscr{A}$ containing $M(j)$ contains $M(i)$. Thus assume $M(j) \subset A \in \mathscr{A}$; then since $\omega^*(\sigma_x) \subset M(j) \subset A$, it follows (by [12, Proposition 2.1] that $\sigma_x(\mathbb{R}) \subset A$. Therefore $\omega(x) \subset A$, and since $\omega(x) \subset M(i)$, it follows that $M(i) \cap A \neq \phi$, implying $M(i) \subset A$.

It remains to prove that $\mathscr{A}$ is the attractor semifiltration defined by the admissible ordering $<_*$ of M. We need to show that if $A \in \mathscr{A}$ then there exists $I \in A(<_*)$ such that $A = M(I)$, and furthermore, if $I \in A(<_*)$ then $M(I) \in \mathscr{A}$. Define $I_A = \{i \mid M(i) \subset A\}$ for each $A \in \mathscr{A}$. We first show that $A = M(I_A)$. By definition of I_A, $M(i) \subset A$ for each $i \in I_A$; Proposition 2.1 [12] then implies that $C(M(j), M(i)) \subset A$ for each $i, j \in I_A$. Therefore $M(I_A) \subset A$. Consider the converse containment. Suppose $x \in A$; there is a full solution $\sigma_x: \mathbb{R} \to S$ through x such that $\sigma_x(\mathbb{R}) \subset A$. Now either $\sigma_x(\mathbb{R}) \subset M(i)$ for some i, or there exists $i <_* j$ such that $\omega(x) \subset M(i)$ and $\omega^*(\sigma_x) \subset M(j)$. In the former case $i \in I_A$ and $x \in M(i) \subset M(I_A)$. In the latter case $i, j \in I_A$ and $x \in C(M(j), M(i)) \subset M(I_A)$. Therefore $A \subset M(I_A)$, and we have shown that if $A \in \mathscr{A}$ then $A = M(I)$ for some $I \in A(<_*)$.

Now suppose $I \in A(<_*)$; we show that $M(I) \in \mathscr{A}$. Let A be a minimal element of $\mathscr{A}$ containing $M(I)$. By above $A = M(I_A)$; we prove that $M(I) = M(I_A)$ (but not necessarily $I = I_A$), and therefore $M(I) \in \mathscr{A}$. We do this by showing that if $i \in I_A \backslash I$ then $M(i) = \phi$, implying $M(I) = M(I_A)$. Suppose not; i.e., assume A is a minimal element of $\mathscr{A}$ containing $M(I)$, $M(i)$ is nonempty and contained in A, and $i \notin I$. Since $i \notin I$, it follows that if $j \in I$ then $i \not<_* j$. Therefore either there exists an attractor $B_j \in \mathscr{A}$ such that B_j contains $M(j)$ but not $M(i)$, or every A in $\mathscr{A}$ contains either both $M(i)$ and $M(j)$ or neither $M(i)$ nor $M(j)$. In the latter case because $M(i)$ is nonempty it follows that $M(j) = \phi$. Let B be the union of the B_j's over all j for which the former case holds. $B \in \mathscr{A}$, and clearly $M(i) \cap B = \phi$. Furthermore note that B contains each nonempty $M(j)$ in $M(I)$; therefore $M(I) \subset B$. Now $\omega(B \cap A)$ is in $\mathscr{A}$, contains $M(I)$, and is properly contained in A. The containment in A is proper because $\omega(B \cap A)$ does not contain $M(i)$. Thus A is not a minimal attractor in $\mathscr{A}$ containing $M(I)$; contradiction. Thus $M(I) \in \mathscr{A}$, and the proof of Theorem 2.5 is complete. ∎

3. Index Filtrations

If S is the maximal invariant set in a closed neighborhood N of itself in X, then S is called an *isolated invariant set* and N is called an *isolating neighborhood* of S. Assume throughout the remainder of the paper that S is an isolated invariant set.

A closed subset $N \subset X$ is called *admissible* provided that

(1) if $\{x_n\} \subset N$ and $\{t_n\} \subset [0, \infty)$ are sequences satisfying $x_n \cdot [0, t_n] \subset N$ and $t_n \to \infty$, then the sequence $\{x_n \cdot t_n\}$ is precompact,

(2) if $x \in N$ and $\omega_x < \infty$, then $x \cdot [0, \omega_x) \not\subset N$.

Clearly a closed subset of an admissible set is admissible.

If $N_0 \subset N_1$ are closed sets in X, then N_0 is said to be *positively invariant* relative to N_1 if $x \in N_0$, $0 < t < \omega_x$ and $x \cdot [0, t] \subset N_1$ imply $x \cdot [0, t] \subset N_0$.

Definition 3.1. If $N_0 \subset N_1$ are closed, then the pair (N_1, N_0) is called an *index pair* for S if

(1) $S \subset \text{int}(N_1 \backslash N_0)$, and S is the maximal invariant set in $\text{cl}(N_1 \backslash N_0)$,

(2) N_0 is positively invariant relative to N_1,

(3) if $x \in N_1$, $0 < t < \omega_x$ and $x \cdot t \notin N_1$, then there exists $t' < t$ such that $x \cdot [0, t'] \subset N_1$ and $x \cdot t' \in N_0$.

The index pair is called admissible if N_1 (and therefore N_0) is admissible.

If (N_1, N_0) is an index pair for S, then we call the pointed quotient space N_1/N_0 an *index space* for X.

Rybakowski [10] proves the existence of index pairs. Furthermore, he proves that if (N_1, N_0) and (N_1', N_0') are admissible index pairs for S, then there is a flow-defined homotopy equivalence between the corresponding index spaces. Thus, associated to an invariant set S with an admissible isolating neighborhood there is a homotopy type of a pointed space, $h(S)$, and if (N_1, N_0) is an admissible index pair for S, then $h(S)$ equals the homotopy type of the index space N_1/N_0. We call $h(S)$ the *Conley index* of S.

Note that our definition of an index pair differs from that of Rybakowski because we do not require N_1 to be an isolating neighborhood of S. However, it can easily be seen that if (N_1, N_0) is an index pair for S (as in Definition 3.1 above) and N is an isolating neighborhood of S containing $\text{cl}(N_1 \backslash N_0)$, then $(N \cap N_1, N \cap N_0)$ is an index pair for S in the sense of Rybakowski. It follows that the index theory is unaffected by the difference in the definitions.

Assume that (A, A^*) is an attractor–repeller pair in S.

LEMMA 3.2 (cf. [4, Lemma 3.2]). *If N is an isolating neighborhood of S, and N' is a closed neighborhood of A disjoint from A^* and contained in N, then N' is an isolating neighborhood of A.*

Proof. Let T be the maximal invariant set in N'. We need to show that $A = T$. Clearly $A \subset T \subset S$. If $x \in S \backslash A$ and $\sigma_x : \mathbb{R} \to S$ is a full solution through x, then by [12, Proposition 2.1] $\omega^*(\sigma_x) \subset A^*$. Therefore since $A^* \cap N' = \phi$, it follows that $\omega^*(\sigma_x) \cap N' = \phi$. Thus $x \notin T$, implying $A = T$. ∎

Note that the roles of A^* and A can be reversed in Lemma 3.2, and therefore we have an analogous result for A^*. Also note that such sets N' can always be found, and thus A and A^* are isolated invariant sets.

The index pair for S is generalized to an index triple for (A, A^*) via the following

PROPOSITION 3.3 (cf. [4, Proposition 3.3]). *Assume $N_0 \subset N_1 \subset N_2$. If (N_1, N_0) is an index pair for A, and (N_2, N_0) is an index pair for S, then (N_2, N_1) is an index pair for A^*.*

We call such a triple (N_2, N_1, N_0) an *index triple* for the attractor–repeller pair (A, A^*) in S. The index triple is called admissible if N_2 (and therefore each N_i) is admissible. Rybakowski [11] proves the existence of index triples for an attractor–repeller pair (A, A^*) in S.

The proof of Proposition 3.3 is almost exactly the proof of Proposition 3.3 in [4] with the only difference being that Lemma 3.2 referred to therein must be replaced by Lemma 3.2 above. The details are left to the reader.

Recall that $M = \{M(\pi)\}_{\pi \in P}$ is a $<$-ordered Morse decomposition of S. The index triple for an attractor–repeller pair is generalized to the index filtration for an admissible ordering of a Morse decomposition in Definition 3.4 below. Specifically,

DEFINITION 3.4. An *index filtration* for the admissible ordering $<$ of M is a collection of closed sets $\mathcal{N} = \{N(I)\}_{I \in A(<)}$ satisfying

(1) for each $I \in A(<)$, $(N(I), N(\phi))$ is an index pair for $M(I)$,

(2) for each $I_1, I_2 \in A(<)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cap I_2) = N(I_1) \cup N(I_2)$.

An index filtration is called admissible if $N(P)$ (and therefore each $N(I)$) is admissible.

Now assume that $\mathcal{N} = \{N(I)\}_{I \in A(<)}$ is an index filtration for the admissible ordering $<$ of M. Property 1 in Definition 3.4 ensures that in $\mathcal{N}$ there is an index pair for $M(I)$ for each $I \in A(<)$, i.e., for every attractor in the attractor semifiltration defined by $<$. It is easy to see that if $J \in I(<)$, then there exist $I, K \in A(<)$ such that (I, J) is a decomposition of K. It follows that $(N(K), N(I), N(\phi))$ is an index triple for the attractor–repeller pair $(M(I), M(J))$ in $M(K)$, and therefore $(N(K), N(I))$ is an index pair for the Morse set $M(J)$. Thus the index filtration associates an index pair to every Morse set of the admissible ordering. Now if $(N(K_i), N(I_i))$, $i = 1, 2$, are index pairs in $\mathcal{N}$ for the Morse set $M(J)$, then it is not difficult to see that Property 2 in Definition 3.4 implies that $N(K_1) \backslash N(I_1) = N(K_2) \backslash N(I_2)$, and therefore the pointed quotient spaces $N(K_i)/N(I_i)$ are homeomorphic. The importance of this is brought out in Section 4.

The remainder of this section is devoted to the proof of the existence of an index filtration for the admissible ordering $<$ of M. This is done by a sequence of steps constructing an index filtration.

Let (N_1, N_0) be an index pair for S. Rybakowski [11, proof of Theorem 3.1] shows that for each $I \in A(<)$ there exists N_I such that (N_1, N_I, N_0) is an index triple for the attractor–repeller pair $(M(I), M(P \backslash I))$ in S. $\{N_I\}_{I \in A(<)}$ fails to be an index filtration only by (the crucial) property 2 in Definition 3.4. More work is needed to construct the desired index filtration.

Now, note that if $\pi \in I$, then $M(\pi) \subset \operatorname{int}(N_I \backslash N_0)$, and if $\pi \notin I$, then $M(\pi) \subset \operatorname{int}(N_1 \backslash N_I)$. For each $\pi \in P$ let D_π be the intersection of the sets $\operatorname{int}(N_I \backslash N_0)$ for which $\pi \in I$ and the sets $\operatorname{int}(N_1 \backslash N_I)$ for which $\pi \notin I$. Note that $M(\pi) \subset D_\pi$ for each $\pi \in P$. Further properties of the D_π and the sets E_π defined at the next stage are discussed in Propositions 3.7 and 3.8 below. We continue with the construction of the index filtration here.

Define E_π to be the set of all $x \in N_1$ such that there exists t with $x \cdot [0, t] \subset N_1$ and $x \cdot t \in D_\pi$. Now for each $I \in A(<)$ set

$$N(I) = N_1 \backslash \bigcup_{\pi \in P \backslash I} E_\pi$$

The existence of index filtrations is established by

THEOREM 3.5. *$\mathcal{N} = \{N(I)\}_{I \in A(<)}$ is an index filtration for the admissible ordering $<$ of M.*

Before proving Theorem 3.5 we must establish some properties of the sets constructed above. The first proposition states that it is necessary to have $\pi \leqslant \pi'$ to be able to flow from $D_{\pi'}$ to D_π in N_1.

PROPOSITION 3.6. *If $x \in D_{\pi'}$, $x \cdot [0, t] \subset N_1$, and $x \cdot t \in D_\pi$, then $\pi \leqslant \pi'$.*

Proof. Suppose not. Then there exists $I \in A(<)$ such that $\pi' \in I$ and $\pi \notin I$. So $D_{\pi'} \subset \operatorname{int}(N_I \backslash N_0)$ and $D_\pi \subset \operatorname{int}(N_1 \backslash N_I)$. Now $x \in D_{\pi'} \subset N_I$ and

$x \cdot [0, t] \subset N_1$; therefore by the positive invariance of N_I relative to N_1, it follows that $x \cdot [0, t] \subset N_I$. In particular, $x \cdot t \in N_I$. However, $x \cdot t \in D_\pi$ and $D_\pi \cap N_I = \phi$; contradiction. ▌

PROPOSITION 3.7. *The E_π have the following properties*:

(1) *E_π is open in N_1*,

(2) *if π and π' are noncomparable, then $E_\pi \cap E_{\pi'} = \phi$*,

(3) *if $I_1, I_2 \in A(<)$, then*

$$\left(\bigcup_{\pi \in P \setminus I_1} E_\pi\right) \cap \left(\bigcup_{\pi \in P \setminus I_2} E_\pi\right) = \left(\bigcup_{\pi \in P \setminus (I_1 \cup I_2)} E_\pi\right)$$

Proof. (1) Suppose $x \in E_\pi$; then there exists t so that $x \cdot [0, t] \subset N_1$ and $x \cdot t \in D_\pi$. Note that $D_\pi \cap N_0 = \phi$, and therefore by the positive invariance of N_0 relative to N_1 it follows that $x \cdot [0, t] \cap N_0 = \phi$. D_π is open; let V be an open neighborhood of $x \cdot t$ in D_π. By the continuity of the flow, $U := \{y \mid y \cdot t \in V\}$ is open. $U \cap N_1$ is an open neighborhood of x in N_1 such that if $y \in U \cap N_1$ then $y \cdot t \in D_\pi$. We claim that there is an open neighborhood U' of x in $U \cap N_1$ such that if $y \in U'$ then $y \cdot [0, t] \subset N_1$ (thus proving that E_π is open in N_1). Suppose not; then there exist sequences $\{y_n\} \subset U \cap N_1$ and $\{t_n\} \subset [0, t]$ such that $y_n \to x$ and $y_n \cdot t_n \notin N_1$ for each n. Since (N_1, N_0) is an index pair, there exists a sequence $\{t'_n\} \subset [0, t]$ such that $y_n \cdot t'_n \in N_0$ for each n. We may assume that $t'_n \to t' \in [0, t]$. Therefore $y_n \cdot t'_n \to x \cdot t'$, implying that $x \cdot t' \in N_0$. However, $x \cdot [0, t] \cap N_0 = \phi$; contradiction.

(2) We prove the contrapositive. Thus assume $x \in E_\pi \cap E_{\pi'}$; we show that either $\pi \leqslant \pi'$ or $\pi' \leqslant \pi$. There exists t, t' such that $x \cdot [0, t] \subset N_1$, $x \cdot t \in D_\pi$, $x \cdot [0, t'] \subset N_1$, and $x \cdot t' \in D'_\pi$. If $t' \leqslant t$, then it is easy to see that Proposition 3.6 implies that $\pi \leqslant \pi'$. Similarly $t \leqslant t'$ implies $\pi' \leqslant \pi$.

(3) Clearly the containment $\supset$ holds. Consider the reverse containment. We need to show that if $x \in E_{\pi_i} \in P \setminus I_i$, $i = 1, 2$, then there exists $\pi' \in P \setminus (I_1 \cup I_2)$ such that $x \in E_{\pi'}$. From (2) it follows that either $\pi_1 \leqslant \pi_2$ or $\pi_2 \leqslant \pi_1$. If $\pi_i \leqslant \pi_j$, then since I_i is an attracting interval and $\pi_i \notin I_i$, it follows that $\pi_j \notin I_i$. Therefore $\pi_j \notin I_1 \cup I_2$, and if we set $\pi' = \pi_j$, then $\pi' \in P \setminus (I_1 \cup I_2)$ and $x \in E_{\pi'}$. ▌

PROPOSITION 3.8. *For each $I \in A(<)$, $N(I)$ contains N_I and is positively invariant relative to N_1.*

Proof. Suppose $x \in N_1 \setminus N(I)$; then $x \in E_\pi$ for some $\pi \in P \setminus I$. So there exists t such that $x \cdot [0, t] \subset N_1$ and $x \cdot t \in D_\pi \subset \operatorname{int}(N_1 \setminus N_I)$. By the positive invariance of N_I relative to N_1 it then follows that $x \cdot [0, t] \cap N_I = \phi$. Therefore $x \in N_1 \setminus N_I$; i.e., $N_I \subset N(I)$.

Now suppose that $x \in N(I)$ and $x \cdot [0, t] \subset N_1$. Furthermore assume $x \cdot [0, t] \not\subset N(I)$; then there exist $\pi \in P \setminus I$ and $t' \in [0, t]$ such that $x \cdot t' \in E_\pi$. It is easy to see that this implies $x \in E_\pi$, contradicting $x \in N(I)$. Thus $x \cdot [0, t] \subset N(I)$. implying that $N(I)$ is positively invariant relative to N_1. ∎

Proof of Theorem 3.5. We begin by showing that $(N(I), N(\phi))$ is an index pair for $M(I)$. It follows from Proposition 3.7(1) that $N(I)$ and $N(\phi)$ are closed.

It is easy to see that $M(I) \cap N(\phi) = \phi$. $M(I) \subset \text{int}(N_I) \subset \text{int}(N(I))$ where the first containment holds because (N_I, N_0) is an index pair for $M(I)$ and the second holds by Proposition 3.8. It follows that $M(I) \subset \text{int}(N(I) \setminus N(\phi))$.

To see that $\text{cl}(N(I) \setminus N(\phi))$ is an isolating neighborhood of $M(I)$, first note that $M(P \setminus I) \subset \bigcup_{\pi \in P \setminus I} E_\pi$. Now since $M(I) \cap N(\phi) = \phi$, it then follows that $\text{cl}(N(I) \setminus N(\phi))$ is a closed neighborhood of $M(I)$ that is disjoint from $M(P \setminus I)$ and contained in the isolating neighborhood $\text{cl}(N_1 \setminus N_0)$ of S. Lemma 3.2 implies that $\text{cl}(N(I) \setminus N(\phi))$ is an isolating neighborhood of $M(I)$.

$N(\phi) \subset N(I) \subset N_1$, and by Proposition 3.8 $N(\phi)$ is positively invariant relative to N_1; therefore $N(\phi)$ is positively invariant relative to $N(I)$.

Now suppose that $x \in N(I)$, $0 < t < \omega_x$ and $x \cdot t \notin N(I)$. Then since $N(I)$ is positively invariant relative to N_1, there exists $t' \in (0, t]$ such that $x \cdot t' \notin N_1$. (N_1, N_0) is an index pair; therefore there exists $t'' \in [0, t']$ such that $x \cdot [0, t''] \subset N_1$ and $x \cdot t'' \in N_0$. By the positive invariance of $N(I)$ relative to N_1 and the fact that $N_0 \subset N(\phi)$, it follows, respectively, that $x \cdot [0, t''] \subset N(I)$ and $x \cdot t'' \in N(\phi)$. Therefore $(N(I), N(\phi))$ is an index pair for $M(I)$.

Finally note that if $I_1, I_2 \in A(<)$, then $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ follows trivially using DeMorgan's laws, and $N(I_1 \cup I_2) = N(N_1) \cup N(I_2)$ follows by DeMorgan's laws and Proposition 3.7(3). Therefore $\mathcal{N}$ is an index filtration for the admissible ordering $<$ of M. ∎

Note that if (N_1, N_0) is an admissible index pair for S, then $\mathcal{N}$ is an admissible index filtration, thus establishing the existence of admissible index filtrations. Furthermore note that if N_1 is positively invariant relative to some isolating neighborhood N of S (and therefore (N_1, N_0) is an "index pair in N" as in Rybakowski [10]), then each $N(I) \in \mathcal{N}$ is also positively invariant relative to N.

4. The Homology Index Braid and Connection Matrices

In this section we outline the derivation of the homology index braid omitting details. The steps in the derivation here follow those in [4] exactly, except that where Kurland's results are used in [4], the corresponding result of Rybakowski [11] must be used here. We point out that, in most of the cases considered herein, Rybakowski's result is exactly Kruland's (i.e., in such cases Kurland does not explicitly use compactness of index pairs). Once the homology index braid is defined, the connection matrices, being an algebraic property of the homology index braid, follow immediately (see [5]).

For the discussion that follows assume a coefficient module G (over a PID) is fixed, and let $C(\cdot)$ and $H_*(\cdot)$ denote the singular chain complex and the singular homology, respectively, with coefficients in G of the corresponding topological space.

If S is an isolated invariant set and (N_1, N_0) is an admissible index pair for S, then the homotopy type of the pointed space N_1/N_0 is the Conley index of S and is denoted $h(S)$. Define $H(S)$, the *homology index* of S, to be equal to the homology of the Conley index of S; i.e., $H(S) = H_*(h(S))$.

Now let (N_2, N_1, N_0) be an index triple for an attractor-repeller pair (A, A^*) in S. There exist inclusion induced maps on index spaces

$$N_1/N_0 \xrightarrow{i} N_2/N_0 \xrightarrow{p} N_2/N_1$$

and induced chain maps

$$C(N_1/N_0) \xrightarrow{i} C(N_2/N_0) \xrightarrow{p} C(N_2/N_1) \tag{4.1}$$

i is clearly injective. Note that in (4.1) $pi = 0$; therefore p defines a chain map $\rho\colon C(N_2/N_0, N_1/N_0) \to C(N_2/N_1)$. ρ induces an isomorphism on homology (see [4, Proposition 4.1]). These are exactly the requirements that the sequence be weakly exact, and therefore there is an exact homology sequence (see [5])

$$\cdots \longrightarrow H_*(N_1/N_0) \xrightarrow{i} H_*(N_2/N_0) \xrightarrow{p} H_*(N_2/N_1)$$
$$\xrightarrow{\partial} H_*(N_1/N_0) \longrightarrow \cdots$$

This sequence is independent of the admissible index triple, and therefore there is defined an exact sequence of homology indices and maps

$$\cdots \longrightarrow H(A) \xrightarrow{i} H(S) \xrightarrow{p} H(A^*) \xrightarrow{\partial} H(A) \longrightarrow \cdots$$

We call this the *homology index sequence* of the attractor–repeller pair. Note that this is basically the homology of Rybakowski's connection index of the attractor–repeller pair.

Now let $\mathcal{N} = \{N(I)\}_{I \in A(<)}$ be an admissible index filtration for the admissible ordering $<$ of the Morse decomposition $M = \{M(\pi)\}_{\pi \in P}$ of S. If $I \in I(<)$, then we denote the homology index of $M(I)$ by $H(I)$. There exists at least one index pair in $\mathcal{N}$ for the Morse set $M(I)$. If (N_1^i, N_0^i), $i = 1, 2$, are two such index pairs, then the index spaces N_1^i/N_0^i are homeomorphic and therefore the chain complexes $C(N_1^i/N_0^i)$ are isomorphic. It follows that a chain complex $C_{\mathcal{N}}(I)$ is defined for each $I \in I(<)$ and the homology of the chain complex is $H(I)$, the homology index of the Morse set $M(I)$.

If $(I, J) \in I_2(<)$, then $(M(I), M(J))$ is an attractor–repeller pair in $M(IJ)$ and there is defined a weakly exact sequence

$$C_{\mathcal{N}}(I) \xrightarrow{i(I,IJ)} C_{\mathcal{N}}(IJ) \xrightarrow{p(IJ,J)} C_{\mathcal{N}}(J) \tag{4.2}$$

Passing to homology we obtain

$$\cdots \longrightarrow H(I) \xrightarrow{i(I,IJ)} H(IJ) \xrightarrow{p(IJ,J)} H(J) \xrightarrow{\partial(J,I)} H(I) \longrightarrow \cdots$$

the homology index sequence of $(M(I), M(J))$.

The collection consisting of the chain complexes and chain maps as in sequence (4.2) is called the *chain complex braid* of the index filtration. It has the following properties:

(1) for each $I \in I(<)$ there is a chain complex $C_{\mathcal{N}}(I)$,

(2) for each $(I, J) \in I_2(<)$ there are chain maps

$$C_{\mathcal{N}}(I) \xrightarrow{i(I,IJ)} C_{\mathcal{N}}(IJ) \xrightarrow{p(IJ,J)} C_{\mathcal{N}}(J) \tag{4.3}$$

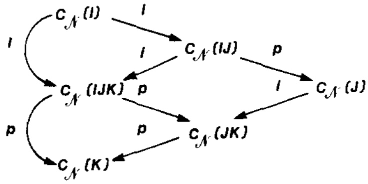
satisfying

(a) sequence (4.3) is weakly exact,

(b) the homology sequence associated to sequence (4.3) is the homology index sequence of the attractor–repeller pair $(M(I), M(J))$,

(c) if I and J are noncomparable, the $p(JI, I)\, i(I, IJ) = \mathrm{id}|C_{\mathcal{N}}(I)$,

(d) if $(I, J, K) \in I_3(<)$, then the following braid diagram commutes:



Passing to homology in the chain complex braid of the index filtration we obtain $\mathscr{H}(<)$, the *homology index braid of the admissible ordering* of the Morse decomposition.

$\mathscr{H}(<)$ has the following properties:

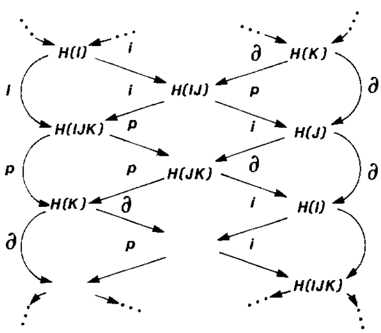
(1) for each $I \in I(<)$ there is a homology index $H(I) = H(M(I))$,

(2) for each $(I, J) \in I_2(<)$ there are maps between homology indices, $i(I, IJ)$: $H(I) \to H(IJ)$, $p(IJ, J)$: $H(IJ) \to H(J)$, and $\partial(J, I)$: $H(J) \to H(I)$, satisfying

(a) $\cdots \to H(i) \to^{i} H(IJ) \to^{p} H(J) \to^{\partial} H(I) \to \cdots$ is exact,

(b) if I and J are noncomparable, then $p(JI, I)\; i(I, IJ) = \mathrm{id} \mid H(I)$,

(c) if $(I, J, K) \in I_3(<)$, then the following braid diagram commutes:



$\mathscr{H}(<)$ is independent of the index filtration for the admissible ordering. Since every admissible ordering of M is an extension of the flow ordering $<_F$, it follows that $\mathscr{H}(<) \subset \mathscr{H}(<_F)$. Therefore we refer to $\mathscr{H}(<_F)$ as the *homology index braid of the Morse decomposition*, and denote it by $\mathscr{H}(M)$.

At this point the appropriate structures for a connection matrix theory are defined. The connection matrices are algebraic invariants of the algebraic structure $\mathscr{H}(<)$. We refer the reader to [5] for the appropriate definitions and theorems regarding the algebraic connection matrix theory.

$\mathscr{H}(<)$ is a chain complex generated graded module braid; therefore if $C = \{C\Delta(\pi)\}_{\pi \in P}$ is a collection of free chain complexes such that the homology of $C\Delta(\pi)$ is isomorphic to $H(\pi)$ for each π, then there exist upper triangular (with respect to $<$) boundary maps

$$\Delta \colon \bigoplus_{\pi \in P} C\Delta(\pi) \to \bigoplus_{\pi \in P} C\Delta(\pi)$$

that generate an isomorphic image of $\mathscr{H}(<)$ (see [5]).

The map Δ is called a *C-connection matrix of the admissible ordering* $<$, and the (nonempty!) collection of such maps is denoted $\mathscr{CM}(<; C)$. The collection $\mathscr{CM}(<_F; C)$ is also denoted $\mathscr{CM}(M; C)$, and each Δ in $\mathscr{CM}(M; C)$ is called a *C-connection matrix of the Morse decomposition.* Since $\mathscr{H}(M)$ contains $\mathscr{H}(<)$, the collection $\mathscr{CM}(M; C)$ is defined with more algebraic restrictions than $\mathscr{CM}(<; C)$, implying that $\mathscr{CM}(M; C) \subset \mathscr{CM}(<; C)$. If each chain complex $C\Delta(\pi)$ in C is equal to the graded module $H(\pi)$ with trivial boundary map, then the above collections are called the connection matrices of the admissible ordering and the connection matrices of the Morse decomposition and are, respectively, denoted $\mathscr{CM}(<)$ and $\mathscr{CM}(M)$. Note that this case occurs when the homology indices $H(\pi)$ are free for each π (e.g., when homology is computed using coefficients in a field).

Note that all of the algebraic indices defined in this section are dependent on the coefficient module G chosen. Therefore reference to G should be made in the indices; e.g., $H(S; G)$ = the homology index of S with coefficients in G, $\mathscr{CM}(M; C, G)$ = the C-connection matrices of M with coefficients in G, etc. We have left out reference to G here for simplicity.

To begin interpreting the information in the (C-)connection matrices we have the following proposition which is just proposition 5.3 in [5].

PROPOSITION 5.1. *If $\Delta \in \mathscr{CM}(M; C)$, π and π' are adjacent in the flow ordering, and $\Delta(\pi', \pi) \neq 0$ (where $\Delta(\pi', \pi)$ is the entry in the matrix Δ mapping $C\Delta(\pi')$ to $C\Delta(\pi)$), then $C(M(\pi'), M(\pi))$, the set of orbits connecting $M(\pi')$ to $M(\pi)$ in S, is nonempty.*

Proposition 5.1 describes a situation where information about the set of connecting orbits between elements of a Morse decomposition can be obtained via the connection matrices of the Morse decomposition. This is only an initial interpretation result; as is indicated in [5], it is evident that the connection matrices do contain deeper information about the structure of the invariant set and the Morse decomposition. The problem of a general interpretation theory for the connection matrices needs further investigation.

REFERENCES

1. C. CONLEY, "Isolated Invariant Sets and the Morse Index," CBMS Regional Conf. Ser. in Mathematics Vol. 38, Amer. Math. Soc., Providence, RI, 1980.
2. C. CONLEY AND E. ZEHNDER, Morse-type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.* **37** (2) (1984).
3. R. FRANZOSA, "Index Filtrations and Connection Matrices for Partially Ordered Morse Decompositions," Ph.D. dissertation, University of Wisconsin, Madison, 1984.
4. R. FRANZOSA, Index filtrations and the homology index braid for partially ordered Morse decompositions, *Trans. Amer. Math. Soc.* **298** (1986).
5. R. FRANZOSA, The connection matrix theory for Morse decompositions, *Trans. Amer. Math. Soc.*, submitted.

6. R. Franzosa, The continuation theory for connection matrices and Morse decompositions, in preparation.
7. H. Kurland, The Morse index of an isolated invariant set is a connected simple system, *J. Differential Equations* **42** (1981).
8. H. Kurland, Homotopy invariants of repeller–attractor pairs. I. The Puppe sequence of an *r–a* pair, *J. Differential Equations* **46** (1982).
9. H. Kurland, Homotopy invariants of repeller–attractor pairs. II. Continuation of *r–a* pairs, *J. Differential Equations* **49** (1983).
10. K. Rybakowski, On the homotopy index for infinite-dimensional semiflows, *Trans. Amer. Math. Soc.* **269** (1982).
11. K. Rybakowski, The Morse index, repeller–attractor pairs, and the connection index for semiflows on noncompact spaces, *J. Differential Equations* **47**, (1983).
12. K. Rybakowski and E. Zehnder, A Morse-equation in Conley's index theory for semiflows on metric spaces, *Ergodic Theory Dynamical Systems* **5** (1985).