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# Equality of Higher Numerical Ranges of Matrices and a Conjecture of Kippenhahn on Hermitian Pencils

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ABSTRACT

Let $M_n$ be the algebra of all $n \times n$ complex matrices. For $1 \leqslant k \leqslant n$, the $k$th numerical range of $A \in M_n$ is defined by $W_k(A) = \{(1/k)\sum_{j=1}^{k} x_j^* A x_j : \{x_1, \dots, x_k\}$ is an orthonormal set in $\mathbb{C}^n\}$. It is known that $\{\mathrm{tr}\, A/n\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \cdots \subseteq W_1(A)$. We study the condition on $A$ under which $W_m(A) = W_k(A)$ for some given $1 \leqslant m < k \leqslant n$. It turns out that this study is closely related to a conjecture of Kippenhahn on Hermitian pencils. A new class of counterexamples to the conjecture is constructed, based on the theory of the numerical range. © 1998 Elsevier Science Inc.

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## 1. INTRODUCTION

Let $M_n$ be the algebra of all $n \times n$ complex matrices. For $1 \leqslant k \leqslant n$, the $k$th *numerical range* of $A \in M_n$ is defined by

$$W_k(A) = \left\{ \frac{1}{k} \sum_{j=1}^{k} x_j^* A x_j : \{x_1, \dots, x_k\} \text{ is an orthonormal set in } \mathbb{C}^n \right\}.$$

When $k = 1$, $W_k(A)$ reduces to the *classical numerical range* of $A$, which has been studied extensively (e.g., see [6]). The $k$th numerical range is very useful in studying matrices and operators (e.g., see [5, 13]), and many interesting properties of it have been obtained. We list some of them in the following:

(a) $W_k(A)$ is compact.
(b) $W_k(A)$ is convex [1].
(c) $W_k(U^*AU) = W_k(A)$ for any unitary $U$.
(d) $W_k(aA + bI) = aW_k(A) + b$ for any $a, b \in \mathbb{C}$.
(e) $kW_k(A) = \operatorname{tr} A - (n - k)W_{n-k}(A)$.
(f) Let $H$ and $K$ be Hermitian matrices. Suppose $A = H + iK$ and $\tilde{A} = aH + ibK$ for some nonzero real numbers $a$ and $b$. Then $x + iy \in W_k(A)$ if and only if $ax + iby \in W_k(\tilde{A})$.
(g) $\{\operatorname{tr} A/n\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \cdots \subseteq W_1(A)$ [4].

The purpose of this paper is to study the condition on $A$ so that $W_m(A) = W_k(A)$ for some given $1 \leqslant m < k \leqslant n$. It turns out that this study is related to a conjecture of Kippenhahn on Hermitian pencils. As a by-product of our study, we obtain a new class of $6 \times 6$ counterexamples to the conjecture of Kippenhahn (cf. Theorem 3.2).

## 2. CHARACTERIZATIONS AND A SUFFICIENT CONDITION

THEOREM 2.1. *Let $1 < k \leqslant n$, and let $A = H + iK \in M_n$ with $H = H^*$ and $K = K^*$. The following conditions are equivalent*:

(a) *There exists $m$ with $1 \leqslant m < k$ such that $W_m(A) = W_k(A)$.*
(b) *$W_r(A) = W_s(A)$ for all $1 \leqslant r < s \leqslant k$.*
(c) *The largest eigenvalue of $uH + vK$ has multiplicity at least $k$ for all $u, v \in \mathbb{R}$.*
(d) *The largest eigenvalue of* $\cos\theta\, H + \sin\theta\, K$ *has multiplicity at least $k$ for all $\theta \in [0, 2\pi)$.*

*Proof.* Let $\lambda_1(\theta) \geqslant \cdots \geqslant \lambda_n(\theta)$ be the eigenvalues of $\mathrm{Re}(e^{i\theta}A) = \cos\theta\, H - \sin\theta\, K$. Then

$$L(r,\theta) = \left\{ z \in \mathbb{C} : \mathrm{Re}\, z = \frac{1}{r}\sum_{j=1}^{r} \lambda_j(\theta) \right\}$$

is the right support line of the convex set $W_r(e^{i\theta}A) = e^{i\theta}W_r(A)$ (e.g., see [13]). As a result, $W_r(A) = W_s(A)$ if and only if $L(r,\theta) = L(s,\theta)$ for all $\theta \in [0, 2\pi)$.

Suppose (a) holds. By the arguments in the preceding paragraph, we have $L(m,\theta) = L(k,\theta)$ for all $\theta \in [0,2\pi)$. It is clear that $(1/m)\sum_{j=1}^{m}\lambda_j(\theta) = (1/k)\sum_{j=1}^{k}\lambda_j(\theta)$ if and only if $\lambda_1(\theta) = \cdots = \lambda_k(\theta)$. Thus, condition (d) holds.

Suppose (d) holds. Then for all $\theta \in [0,2\pi)$ and $1 \leqslant r < s \leqslant k$, $L(r,\theta) = L(s,\theta)$. Hence, (d) $\Rightarrow$ (b). The implications (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (d) are trivial, and (d) $\Rightarrow$ (c) because the multiplicity of the largest eigenvalue is invariant under multiplication with positive scalars. ■

By Theorem 2.1, we can focus our study on those matrices $A$ for which $W_1(A) = W_k(A)$ for a given $k > 1$. In some sense, a complete description of such matrices is already given by this theorem, but we are interested in more straightforward characterizations. A very simple one is available if $n < 2k$.

THEOREM 2.2. *Suppose $n < 2k$ and $A \in M_n$. Then $W_1(A) = W_k(A)$ if and only if $A = \lambda I$ for some $\lambda \in \mathbb{C}$.*

*Proof.* The $\Leftarrow$ direction is clear. To prove the converse, we let $A = H + iK$ with $H = H^*$ and $K = K^*$. Let $\mu_1 \geqslant \cdots \geqslant \mu_n$ be the eigenvalues of $H$. By condition (c) of Theorem 2.1, the largest eigenvalue of $H$ has multiplicity $k$, and the largest eigenvalue of $-H$ has multiplicity $k$. Thus $\mu_1 = \mu_k$ and $-\mu_n = -\mu_{n-k+1}$. Since $n < 2k$, it follows that $\mu_1 = \mu_n$, and hence $H$ is a scalar matrix. Similarly, $K$ is a scalar matrix, and so is $A$. ■

For general $n$ ($\geqslant k$) the following sufficiency result holds.

THEOREM 2.3. *Suppose $A \in M_n$ is unitarily similar to $A_1 \oplus \cdots \oplus A_k$ such that $W_1(A_j)$ are the same for all $1 \leqslant j \leqslant k$. Then $W_1(A) = W_k(A)$.*

*Proof.* By the hypothesis, the largest eigenvalue of $\mathrm{Re}(e^{i\theta}A_j)$ are all equal to the largest eigenvalue of $\mathrm{Re}(e^{i\theta}A)$ for any $\theta \in [0, 2\pi)$. Thus condition Theorem 2.1(c) holds, and the result follows. ∎

Analogous to the situation when $A \in M_n$ has a multiple eigenvalue, we say that *the numerical range of a matrix $A$ has multiplicity $k$* if $A$ is unitarily similar to $A_1 \oplus \cdots \oplus A_k$ such that $W_1(A_j)$ are the same for all $1 \leqslant j \leqslant k$. With this definition, Theorem 2.3 can be restated as follows:

*If the numerical range of $A \in M_n$ has multiplicity $k$, then $W_1(A) = W_k(A)$.*

We are interested in the converse of this statement, namely,

$$\text{if } A \in M_n \text{ satisfies } W_1(A) = W_k(A)\text{, then the numerical range of } A \text{ has multiplicity } k. \tag{2.1}$$

If $n < 2k$, then (2.1) follows from Theorem 2.2. Indeed, in this case $A = \lambda I_n$, and one may set $A_1 = \cdots = A_{k-1} = \lambda$, $A_k = \lambda I_{n-k+1}$.

The situation is more complicated when $2k \leqslant n$. We shall consider this problem in the next few sections. As will be seen, the study is closely related to a conjecture of Kippenhahn on Hermitian pencils.

## 3. KIPPENHAHN'S CONJECTURE AND NEW COUNTEREXAMPLES

Let $A = H + iK \in M_n$ with $H = (A + A^*)/2$. Consider the homogeneous polynomial

$$L_A(u, v, w) = \det(uH + vK + wI).$$

Every homogeneous polynomial $f$ in three variables defines an algebraic curve $\Gamma_f$ in the following sense: a line $ux + vy + w = 0$ is tangent to $\Gamma_f$ if and only if $f(u, v, w) = 0$. The latter equation is called a *line equation* of $\Gamma_f$. In particular, there exists a curve defined in such a way by the polynomial $L_A$; we will denote it by $C(A)$ and call it, as in [11], the *associated curve* of $A$. It is well known (see [10, 16]) that the numerical range of $A$ coincides with the convex hull of $C(A)$: $W_1(A) = \mathrm{Co}\, C(A)$.

According to Theorem 2.1, $W_1(A) = W_k(A)$ if and only if for all fixed $u, v \in \mathbb{R}$ the largest root $w$ of $L_A$ has multiplicity at least $k$. In its turn, the conclusion of (2.1) means in particular that $A$ is unitarily reducible, that is, unitarily similar to an orthogonal sum of matrices of lower orders. It is interesting to compare (2.1) with Kippenhahn's conjecture [10]

> Let $A \in M_n$ and $L_A = f^s g$, where $f, g$ are polynomials in $u, v, w$ and $s > 1$. Then $A$ is unitarily reducible. (3.1)

Since $f$ in (3.1) does not necessarily determine the largest eigenvalue of $uH + vK$, the hypothesis in (3.1) is weaker than that of (2.1). Obviously, the conclusion of (3.1) is also weaker than that of (2.1). Nevertheless, we will see in this section that the two statements are closely related.

Kippenhahn himself showed in [10] that (3.1) holds when the minimal polynomial $m(u, v, w)$ of $uH + vK$ has degree at most 2. Later, H. Shapiro proved it in the cases $\deg f = 1$, $s > n/3$ [18] and $\deg m \leqslant 3$ [17]. Combining the latter two cases, she observed in [17] the following

THEOREM 3.1. *Kippenhahn's conjecture* (3.1) *holds if* $n \leqslant 5$.

However, statement (3.1) is not true in general; counterexamples were first constructed by T. Laffey [12] and W. Waterhouse [20]. In the following, we describe those counterexamples and discuss their relations with the statement (2.1).

EXAMPLE 1 [12]. Let $B = \begin{bmatrix} P & X \\ -X^T & Q \end{bmatrix}$, $C = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$, where

$$P = \begin{bmatrix} 0 & -1 & 3 & -6 \\ 1 & 0 & -6 & -3 \\ -3 & 6 & 0 & 1 \\ 6 & 3 & -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -2 & -2 & 6 \\ 2 & 0 & 6 & 2 \\ 2 & -6 & 0 & 2 \\ -6 & -2 & -2 & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} -1 & -1 & 5 & -7 \\ -1 & 1 & -7 & -5 \\ -1 & 13 & 1 & -1 \\ 13 & 1 & -1 & -1 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

Set $H = B^2$ and $K = BC + CB$. Then (see [12]) $A = H + iK$ is unitarily irreducible and $L_A$ is of the form $f^2$, so that all the eigenvalues of $uH + vK$ have even multiplicities.

Due to Theorem 2.1, we have $W_1(A) = W_2(A)$. In other words, the statement (2.1) fails for this matrix $A \in M_8$.

EXAMPLE 2 [20]. Let $S, T$ be $r \times r$ positive definite Hermitian matrices with no nontrivial common invariant subspace (such matrices exist for all $r > 1$; see [20, Lemma]). Then for

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & S & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 0 & T \\ 0 & 0 & 0 \\ T & 0 & 0 \end{bmatrix},$$

the matrix $A = H + iK$ is unitarily irreducible, though $L_A(u, v, w)$ contains a multiple $w^r$.

Choosing $r = 2$, we see that the upper bound for $n$ in Theorem 3.1 is sharp. However, nonzero eigenvalues of $uH + vK$ in this case equal $\pm 1$ times the singular values of $u^2 S^2 + v^2 T^2$, and are therefore simple for all but finitely many directions $(u, v)$. Hence, [20] does not lead to an example of a $6 \times 6$ unitarily irreducible matrix $A$ with $W_1(A) = W_2(A)$.

In the following, we present such a class of matrices. They constitute counterexamples to (2.1) and, of course, are at the same time (new) counterexamples to (3.1).

THEOREM 3.2. *Let*

$$A = \begin{bmatrix} 0 & x & 0 & cy & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -cx & 0 & \sqrt{1 - c^2}\,\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.2}$$

*where*

$$x, y, \xi, \eta, c > 0, \qquad x^2 + y^2 = \xi^2 + \eta^2 = 4, \qquad c < 1. \tag{3.3}$$

*Then $A$ is unitarily irreducible such that $W_1(A) = W_2(A)$ is the unit circular disk $\overline{\mathbb{D}}$ centered at the origin. Furthermore, matrices of the form* (3.2) *obtained by different choices of positive vectors $(x, y, \xi, \eta, c)$ are not unitarily similar.*

*Proof.* Let $A = H + iK$ with $H = (A + A^*)/2$. One can easily check (by hand calculation or symbolic software such as Maple or Mathematica) that $\det\{(\cos t)H + (\sin t)K - zI\} = (z^2 - c^2\eta^2/4)(z^2 - 1)^2$, and hence $W_1(A) = W_2(A) = \overline{\mathbb{D}}$ by Theorem 2.1.

We claim that $A$ is unitarily irreducible. To prove our claim, note that $A$ is a rank 4 nilpotent matrix such that $A^3 \neq 0 = A^4$. Thus, $A$ is similar to a direct sum of a $2 \times 2$ and a $4 \times 4$ Jordan block. In particular, if $A$ is unitarily reducible, then it is unitarily similar to $\tilde{A} = A_1 \oplus A_2$, where

$$A_1 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

for some singular value $a$ of $A$. Suppose $A_1 = H_1 + iK_1$ with $H_1 = (A_1 + A_1^*)/2$. Then $\det\{(\cos t)H_1 + (\sin t)K_1 - zI\} = z^2 - a^2/4$ is a factor of $\det\{(\cos t)H + (\sin t)K - zI\}$. Thus $a^2 = c^2\eta^2$ or 4.

First, we show that $a^2$ cannot be 4. Since

$$A^*A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & cxy & 0 & 0 \\ 0 & 0 & y^2 + c^2x^2 & 0 & -c\sqrt{1-c^2}\,x\xi & 0 \\ 0 & cxy & 0 & c^2y^2 & 0 & 0 \\ 0 & 0 & -c\sqrt{1-c^2}\,x\xi & 0 & (1-c^2)\xi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta^2 \end{bmatrix},$$

the eigenvalues of $A^*A$ are $0, 0, \eta^2, x^2 + c^2y^2, \lambda_1$, and $\lambda_2$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the matrix

$$A_0 = \begin{bmatrix} y^2 + c^2x^2 & -c\sqrt{1-c^2}\,x\xi \\ -c\sqrt{1-c^2}\,x\xi & (1-c^2)\xi^2 \end{bmatrix}.$$

By the fact that $4 = x^2 + y^2 = \xi^2 + \eta^2$, we see that $\det(4I - A_0) = (1 - c^2)x^2\eta^2 > 0$, and hence $4I - A_0$ is positive definite. So both roots of $A_0$ are strictly less than 4. Obviously, $\eta^2 < 4$ and $x^2 + c^2y^2 < x^2 + y^2 = 4$ as well. Hence, all the singular values of $A$ are strictly less than 2.

Now suppose $a^2 = c^2\eta^2$. If $A_2 = H_2 + iK_2$ with $H_2 = (A_2 + A_2^*)/2$, then both $H_2$ and $K_2$ have eigenvalues $1, 1, -1, -1$, and hence $H_2^2 + K_2^2 = 2I$. Thus the eigenvalues of $H^2 + K^2$ are the same as those of $(H_1^2 + K_1^2) \oplus (H_2^2 + K_2^2)$, and equal $c^2\eta^2/2, c^2\eta^2/2, 2, 2, 2, 2$. On the other hand, a direct computation shows that $(AA^* + A^*A)/2 = H^2 + K^2$ is an orthogonal sum of a matrix in $M_5$ and the $1 \times 1$ matrix $[\eta^2/2]$. So $\eta^2/2$ is an eigenvalue of $H^2 + K^2$ that is neither $c^2\eta^2/2$ nor 2, which is a contradiction. Thus $A$ is unitarily irreducible.

Note that unitarily similar matrices of the form (3.2) have the same value of $\operatorname{tr}(A^*A) = 4(2 + c^2) - c^2\xi^2$ and have the same set of eigenvalues for $\operatorname{Re} A$: $1, 1, -1, -1, c\eta/2, -c\eta/2$. By the fact that $c, \xi, \eta$ are positive numbers satisfying $\xi^2 + \eta^2 = 4$, we see that these three values are uniquely determined. Also observe that the only nonzero entry of $A^3$ is $c\sqrt{1 - c^2}\,\xi\eta y$, in the $(1, 5)$ position. Since $x$ and $y$ are positive numbers satisfying $x^2 + y^2 = 4$, we see that $x$ and $y$ are also uniquely determined. Hence, unitarily similar matrices of the form (3.2) have to coincide. ■

We remark that the proof of the above theorem can be simplified using the result in [10] or the results in the next two sections. Our proof is elementary and self-contained.

## 4. MATRICES WITH MULTIPLE NUMERICAL RANGES

Although the statement (2.1) is false in general, there are several nontrivial (different from $n < 2k$) situations in which it holds. They are discussed in this section.

Note that if $A$ is Hermitian, then (2.1) follows from Theorem 2.1. In fact, one can extend this result to *convex matrices* (also known as *convexoids*), i.e., those matrices $A$ for which the boundary of $W(A)$ is a convex polygon. It is well known (e.g., see [6] and [13]) that normal matrices are convex matrices. We first establish the following lemma.

LEMMA 4.1. *Let $1 \leqslant k \leqslant n$ and let $A \in M_n$. If $z$ is a corner of $W_1(A)$ and $W_k(A)$, then $A$ is unitarily similar to $zI_m \oplus A_2$ with $m \geqslant k$ and $z \notin W_1(A_2)$.*

*Proof.* Since $z$ is a corner of $W_1(A)$, $A$ is unitarily similar to $zI_m \oplus A_2$ such that $z \notin \sigma(A_2)$ (e.g., see [2]). We may assume that $z = 0$ and the spectrum of $A_2$ is in the right open half plane of $\mathbb{C}$; otherwise replace $A$ by $\mu A + \eta I$ for some suitable $\mu, \eta \in \mathbb{C}$. Since $z = 0$ is a corner of $W_k(A)$, $z$ must be the sum of $k$ eigenvalues of $A$ (see [14]). Clearly, none of the eigenvalues of $A_2$ can be included; otherwise the resulting sum will have positive real part. Thus all the $k$ eigenvalues must be chosen from $zI_m$, and hence $m \geqslant k$. Finally, if $z \in W_1(A_2)$, then it is a corner point of it, and therefore $z \in \sigma(A_2)$, which is false. ∎

We are now ready to prove the result on convex matrices.

THEOREM 4.2. *If $A \in M_n$ is a convex matrix, then the statement* (2.1) *holds.*

*Proof.* Suppose the boundary of $W_1(A) = W_k(A)$ has vertices $z_1, \dots, z_m$. By Lemma 4.1, $A$ is unitarily similar to $z_1 I_k \oplus \cdots z_m I_k \oplus B$ for some $B \in M_{n-km}$. Let $A_j = \operatorname{diag}(z_1, \dots, z_m)$ for $j = 1, \dots, k-1$, and $A_k = A_1 \oplus B$. Then $W_1(A_1) = \cdots = W_1(A_k) = W_1(A)$. ∎

In the proof of the above theorem, we see that if $A$ satisfies $W_1(A) = W_k(A)$ and can be reduced to an orthogonal sum of matrices of lower orders, one may be able to arrange the summands into $k$ groups to form $A_1, \dots, A_k$, so that the conclusion of statement (2.1) holds. In fact, we have the following theorem.

THEOREM 4.3. *Let $A$ be unitarily similar to $B_1 \oplus \cdots \oplus B_m$ so that of all the blocks $B_i$ for which $W_1(B_i)$ and $W_1(A)$ have common extreme points, at most $k$ are in $M_4$ or $M_5$, and all others are of the size $3 \times 3$ or smaller. Then* (2.1) *holds.*

*Proof.* If some of the blocks $B_j$ are unitarily reducible, we may replace them with orthogonal sums of unitarily irreducible blocks of lower orders. Since this procedure does not increase the number of $4 \times 4$ and $5 \times 5$ blocks having common extreme points of the numerical range with $A$, we may without loss of generality suppose that the $B_j$ themselves are unitarily irreducible.

Hermitian matrices $\operatorname{Re}(e^{i\theta}B_j)$ depend analytically on $\theta \in \mathbb{R}$. Hence [9], their eigenvalues $\lambda_{jk}(\theta)$ can be numbered in such a way that each of them is an analytic function of $\theta \in \mathbb{R}$. Being analytic, any two of the functions $\lambda_{jk}$ are either identical or coincide only on an isolated subset of $\mathbb{R}$. The former possibility does not occur for $\lambda_{jk}$ corresponding to the same block $B_j$ due to Theorem 3.1. In fact, for low ($\leqslant 3$) dimensions a stronger result holds and can be established by elementary methods. Namely, suppose that for a certain $\theta_0$ two of the eigenvalues $\lambda_{jk}(\theta_0)$ coincide. If $B_j \in M_2$, that means that $\operatorname{Re}(e^{i\theta_0}B_j)$ is a scalar matrix, so that $e^{i\theta_0}B_j$ (and therefore $B_j$ itself) is normal, which contradicts our assumption of unitary irreducibility. So for $2 \times 2$ blocks the corresponding eigenvalue functions are different everywhere, and therefore one of them is bigger than the other everywhere on $\mathbb{R}$. Let us agree to denote the bigger one by $\lambda_{j1}$.

For $B_j \in M_3$, the point $\theta_0$ can indeed exist. Of course, all the points $\theta_0 + \pi\mathbb{Z}$ then have the same property. Suppose now that there are points $\theta$ different from those found above for which we also get a multiple eigenvalue. Then we have two linear combinations $H_k = c_k \operatorname{Re} B_j + d_k \operatorname{Im} B_j$, $k = 1, 2$, with noncollinear tuples $(c_k, d_k)$ of coefficients, each of which has a two dimensional invariant subspace. In a three dimensional space, these invariant subspaces have a nontrivial intersection, and this intersection is invariant under both $H_1$ and $H_2$. Therefore, it is invariant under $\operatorname{Re} B_j$ and $\operatorname{Im} B_j$, which again contradicts the unitary irreducibility of $B_j$. Hence, the multiple eigenvalues, if any, occur only on the grid $\theta_0 + \pi\mathbb{Z}$. This allows us for each $3 \times 3$ block $B_j$ to choose an interval $I_j$ of length $2\pi$ such that $\max_k \lambda_{jk}(\theta)$ coincides with one of the analytic functions $\lambda_{jk}$ for $\theta \in I_j$. Relabeling them if necessary, we may always assume that this branch is $\lambda_{j1}$.

Let now $I_j = [0, 2\pi)$ for all the blocks $B_j$ of sizes different from $3 \times 3$, and

$$\mathbb{T}_{jl} = \left\{ e^{i\theta} : \lambda_{jl}(\theta) = \max \operatorname{Re}(e^{i\theta}A),\ \theta \in I_j \right\}.$$

Of course, the sets $\mathbb{T}_{jl}$ coincide if the corresponding functions $\lambda_{jl}$ coincide; otherwise they have only finitely many points in common. Some of these sets may be empty [in particular, if $W_1(B_j)$ and $W_1(A)$ have disjoint boundaries, or $B_j \in M_2$ and $l = 2$] or consist of finitely many points (in particular, if $B_j \in M_3$ and $l = 2, 3$); others are unions of finitely many closed arcs. Denote by $\Omega_1, \dots, \Omega_N$ all different sets $\mathbb{T}_{jl}$ of positive measure, and by $\mathscr{B}_s'$ ($\mathscr{B}_s''$) the set of all blocks $B_j \in M_r$ for which $r \leqslant 3$ (respectively, $r = 4, 5$) and $\Omega_s = \mathbb{T}_{jl}$ for some $l$. According to our previous discussion, the only possibility for $\mathscr{B}_s'$ is $l = 1$, and therefore the sets $\mathscr{B}_s'$, $s = 1, \dots, N$, are mutually disjoint.

Now recall that $W_1(A) = W_k(A)$. By Theorem 2.1, the union $\mathscr{B}_s' \cup \mathscr{B}_s''$ contains at least $k$ matrices for each $s$. We construct matrices $A_1, \dots, A_k$ by combining the blocks $B_l$ as follows.

1. Suppose $B_{j_1}, \dots, B_{j_r}$, $r \leqslant k$, are all the $4 \times 4$ or $5 \times 5$ blocks that have common extreme points of the numerical range with $A$. Assign them to $A_1, \dots, A_r$.
2. For each $s = 1, \dots, N$, assign a block from $\mathscr{B}_s'$ to each of those $A_i$ that has not been assigned a block in $\mathscr{B}_s''$ by step 1.
3. If there are blocks $B_l$ left after steps 1 and 2 are done, assign them to $A_k$.

Step 1 can be performed because the total number of the specified $4 \times 4$ and $5 \times 5$ blocks does not exceed $k$. Step 2 can be performed because $|\mathscr{B}_s'| + |\mathscr{B}_s''| \geqslant k$.

For every $A_i$ constructed in such a manner, the maximal eigenvalues of $\operatorname{Re}(e^{i\theta}A_i)$ and $\operatorname{Re}(e^{i\theta}A)$ coincide for all $e^{i\theta} \in \bigcup_{s=1}^{N}\Omega_s = \mathbb{T}$. Thus, all the $W_1(A_i)$ are the same as $W_1(A)$. ■

Theorem 3.2 shows the size restriction in Theorem 4.3 is sharp. To show that the number of $4 \times 4$ and $5 \times 5$ blocks cannot be increased either, we need the following results.

LEMMA 4.4. *Let* $H = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$, $K = \begin{bmatrix} cP & M \\ M & cP \end{bmatrix}$, *with*

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \textit{and} \quad P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

*where* $0 < m_2 < 1 < m_1$, $m_1^2 + m_2^2 = 2$, *and* $c = m_1 + m_2$. *Then* $A = H + iK$ *is a* $4 \times 4$ *unitarily irreducible matrix with* $W_1(A)$ *equal to the convex hull of two circles*: $\mathbb{T}$ *and* $\mathbb{T} + ic$.

*Proof.* Direct computations show that

$$L_A(u, v, w) = (u^2 + v^2 - w^2)\{u^2 + v^2 - (w - cv)^2\}.$$

Therefore, the associated curve $C(A)$ is the union of two curves, the line equations of which are $u^2 + v^2 - w^2 = 0$ and $u^2 + v^2 - (w - cv)^2 = 0$. The first of them is obviously $\mathbb{T}$, and the second is obtained from the first by adding $ic$ to it. Hence, $W_1(A) = \operatorname{Co} C(A) = \operatorname{Co}(\mathbb{T}, ic + \mathbb{T})$.

To prove the unitary irreducibility of $A$, consider a subspace $\mathfrak{L}$ ($\neq \{0\}$) invariant simultaneously under $H$ and $K$. Then it is also invariant under $Q_1 = \frac{1}{2}(I + H)$, $Q_2 = I - Q_1$,

$$X_1 = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = Q_1 K Q_2, \qquad X_2 = \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = Q_2 K Q_1,$$

and, finally,

$$Z_1 = \begin{bmatrix} M^2 & 0 \\ 0 & 0 \end{bmatrix} = X_1 X_2, \qquad Z_2 = \begin{bmatrix} 0 & 0 \\ 0 & M^2 \end{bmatrix} = X_2 X_1.$$

For an arbitrary $(0 \neq)\ x \in \mathfrak{L}$, vectors $x_j = Q_j x$ lie in $\mathfrak{L}$ $(j = 1, 2)$, and at least one of them differs from 0. If $x_1 \neq 0$, then $x_1$, $Z_1 x_1$ form a basis for $\operatorname{Im} Q_1$, and $X_2 x_1, Z_2 X_2 x_1$ form a basis for $\operatorname{Im} Q_2$, so that $\dim \mathfrak{L} = 4$. The case $x_2 \neq 0$ can be dealt with similarly, and we conclude that $\mathfrak{L}$ is the whole space. ∎

LEMMA 4.5. *If a matrix $A$ is unitarily similar to an orthogonal sum $A_1 \oplus \cdots \oplus A_m$ of unitarily irreducible blocks $A_i$, then the sizes of these blocks (up to their order) and the number $m$ are defined by $A$ uniquely.*

This statement is well known (see [19, Section 8]), and is in fact (as was pointed out to us by the referee) a special case of the Krull-Schmidt theorem (e.g., see [7]). A simple direct proof for the matrix case can be found in [15].

From property (d) of numerical ranges (Section 1) it follows that for any $\mu_1, \mu_2 \in \mathbb{C}$ and $r > 0$ the matrix $\mu_1 I + \{(\mu_2 - \mu_1/ic\}A$, where $A$ is as in Lemma 4.4 and $c = |\mu_2 - \mu_1|/r$, has as its numerical range the convex hull of two circles with radii $r$, centered at $\mu_1$ and $\mu_2$. We will denote such a matrix by $A_{\mu_1, \mu_2, r}$.

EXAMPLE 3. Let $\mu_1, \mu_2, \mu_3$ be three distinct points in $\mathbb{C}$, $r > 0$. Consider a matrix $A$ unitarily similar to the orthogonal sum of $k - 1$ copies of $A_{\mu_1, \mu_2, r}$, one matrix $A_{\mu_1, \mu_3, r}$, one matrix $A_{\mu_2, \mu_3, r}$, and $k - 2$ copies of

$$\begin{bmatrix} \mu_3 & 2 \\ 0 & \mu_3 \end{bmatrix}.$$

Then $W_1(A)$ coincides with the convex hull of the numerical ranges of its blocks, and is therefore equal to $\mathrm{Co}(\Gamma_1, \Gamma_2, \Gamma_3)$, where $\Gamma_j$ is a circle of radius $r$ centered at $\mu_j$: $\Gamma_j = \mu_j + r\mathbb{T}$. Every support line to $W_1(A)$ is a support line to one of these three circles, and at the same time a support line to exactly $k$ of the $2k - 1$ diagonal blocks of $A$. According to Theorem 2.3, $W_k(A) = W_1(A)$, If, however, we try to unitarily reduce $A$ to an orthogonal sum of $k$ summands with the same numerical range $W_1(A)$, the associated curve of these summands should be $\bigcup_{j=1}^{3} \Gamma_j$. The line equation of this union has degree 6, and therefore the size of each summand is at least 6. On the other hand, the size of $A$ itself is $4(k+1)+2(k-2)=6k$, so that each summand has to be from $M_6$. If $A$ is unitarily similar to the orthogonal sum of $k$ $6\times 6$ matrices, $A$ cannot have $k+1$ $4\times 4$ unitarily irreducible blocks, by Lemma 4.5. The contradiction obtained shows that (2.1) is not true for our matrix $A$.

In some cases, one may apply Theorem 4.3 when reducibility of $A$ is not given *a priori* as shown in the following results.

THEOREM 4.6. *Let $A \in M_n$. Then the statement* (2.1) *holds if* $W_1(A)$ *have $m \geqslant (n-1)/k-2$ vertices.*

*Proof.* Let $W_1(A)=W_k(A)$ have $m \geqslant 0$ vertices $z_1,\dots,z_m$. By Lemma 4.1, $A$ is unitarily similar to $z_1 I_k \oplus \cdots z_m I_k \oplus B$ for some $B \in M_{n-km}$.

We may suppose that $W_1(A)$ is not a polygon (otherwise, Theorem 4.2 applies), and therefore that the boundary of $W_1(B)$ contains a nonflat arc $\gamma$ of an algebraic curve. For a support line of $W_1(B)$ passing through a point on $\gamma$ and forming an angle $\alpha$ with the positive $x$-axis, the largest eigenvalue of $\operatorname{Re}(e^{i(\pi/2-\alpha)}B)$ has multiplicity at least $k$. Since such support lines exist for infinitely many values of $\alpha$, the latter is possible only when $L_B$ can be factored as $L_B=f^k g$, and $\gamma$ lies on an algebraic curve with a line equation $f=0$. This implies that $\deg f \geqslant 2$ (otherwise $\gamma$ would have been a point). Note that the inequality $\deg L_B = k\deg f+\deg g = n-mk \leqslant 2k+1$ can only hold when $\deg f=2$ and $\deg g \leqslant 1$. The minimal polynomial $fg$ of the matrix $u \operatorname{Re} B + v \operatorname{Im} B$ then has degree at most 3, and the result in [17] implies that $B$ is unitarily reducible to an orthogonal sum of blocks having sizes $3\times 3$ and smaller. But then $A$ itself has the same kind of representation, and the result follows from Theorem 4.3. ∎

In the following result, we show that statement (2.1) holds if $n=2k$ or $n=2k+1$. Moreover, we determine all the possible shapes of the numerical ranges of such matrices (see Theorem 2.2).

COROLLARY 4.7. *Suppose $A \in M_n$ satisfies $W_1(A)=W_k(A)$, where $n = 2k$ or $2k+1$.*

(a) *If $n=2k$, then $A$ is unitarily similar to $A_1 \otimes I_k$, where $A_1 \in M_2$.*

(b) *If $n=2k+1$, then $A$ is unitarily similar to $A_0 \oplus A_1 \otimes I_{k-1}$, where $A_0 \in M_3$ and $A_1 \in M_2$ have the same numerical range.*

*In both cases,* $W_1(A) = W_k(A)$ *is an elliptical disk, that can degenerate to a line segment or a point.*

*Proof.* If $W_1(A)$ is a singleton, then (e.g., see [6]) $A$ is a scalar matrix, and the result holds.

If the boundary of $W_1(A)$ is a convex polygon, then the result follows from Theorem 4.2 (and its proof).

If the boundary of $W_1(A)$ contains a nonflat arc, one can use arguments in the proof of Theorem 4.6 to show that $L_A = f^k g$ with $\deg f = 2$ and $\deg g = 0$ or $1$ depending on whether $n = 2k$ or $2k + 1$. One can then apply Theorem 4.3 to conclude that $A$ is unitarily similar to $B_1 \oplus \cdots \oplus B_k$ such that $W_1(A) = W_1(B_i)$ is an elliptical disk for all $i$. We may assume that $B_2, \ldots, B_k \in M_2$, and $B_1$ belongs to $M_2$ (if $n = 2k$) or $M_3$ (if $n = 2k + 1$). Since $2 \times 2$ matrices with coinciding numerical ranges are unitarily similar, we may also assume that all $2 \times 2$ blocks $B_i$ are the same. ■

The next result gives us the largest possible $n$ so that statement (2.1) holds for a given $k$.

COROLLARY 4.8. *Let* $k > 1$. *Then* $n = 2k + 1$ *is the maximum integer* $n$ *such that the property* (2.1) *holds for all* $A \in M_n$.

*Proof.* If $n \leqslant 2k + 1$, then statement (2.1) is valid by Theorem 2.2 and Corollary 4.7.

Suppose $n > 2k + 1$. One can consider $A = B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1} \oplus 0_{n-2k-2}$, where $B_1 \in M_6$ is a matrix of type (3.2), and

$$B_2 = \cdots = B_{k-1} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then $W_1(A) = W_k(A) = \overline{\mathbb{D}}$. If, on the other hand, (2.1) holds, then every matrix $A_i$ contains an unitarily irreducible block of dimension at least 2. This contradicts Lemma 4.5. ■

## 5. EQUALITY OF HIGHER NUMERICAL RANGES FOR $A \in M_6$

By Theorem 2.2 and Corollary 4.7, if $1 < k \leqslant n \leqslant 5$, the structure of those $A \in M_n$ satisfying $W_1(A) = W_k(A)$ is well understood. In this section, we focus on the case when $n = 6$. If $k \geqslant 3$, Theorem 2.2 and Corollary 4.7 are applicable. So we assume that $k = 2$.

Suppose $A \in M_6$ satisfies

$$W_1(A) = W_2(A). \tag{5.1}$$

If $A$ is unitarily reducible, then one can apply Theorem 4.3 to conclude that the numerical range of $A$ has multiplicity 2. We may assume that $A_1 \in M_m$ with $m \leqslant 3$. Therefore, if $W_1(A) = W_2(A)$, then $W_1(A)$ is the same as the numerical range of a certain compression of $A$ on a subspace of dimension not greater than 3. On the other hand, for any $3 \times 3$ matrix $A_0$ we can find a $6 \times 6$ matrix $A$ such that $W_1(A) = W_2(A) = W_1(A_0)$, simply by choosing $A = A_0 \oplus A_0$. Therefore, the collection of shapes of numerical ranges of all $6 \times 6$ unitarily reducible matrices $A$ with the property (5.1) is the same as that of the shapes of numerical ranges of all $3 \times 3$ matrices, the description of which can be found in [10] (see also [11]). Thus the structure of those reducible $A \in M_6$ satisfying $W_1(A) = W_2(A)$ is also quite well understood.

The next theorem describes all unitarily irreducible matrices $A \in M_6$ satisfying (5.1), and it shows, among other things, that there is only one possible shape for the numerical range of such matrices, namely, an elliptical disk.

THEOREM 5.1. *Let $B \in M_6$ be unitarily irreducible. Then $W_1(B) = W_2(B)$ if and only if*

$$B = \alpha I + \beta H + i\gamma K, \tag{5.2}$$

*where $\alpha, \beta, \gamma \in \mathbb{C}$, $\gamma/\beta > 0$, and Hermitian matrices $H$, $K$ are such that $A = H + iK$ is a unitarily irreducible matrix satisfying $W_1(A) = W_2(A) = \overline{\mathbb{D}}$. If these conditions are satisfied, then $W_1(B)$ is an ellipse centered at $\alpha$ and having axes $2|\beta|$, $2|\gamma|$.*

*Proof.* If $B$ is of the form (5.2), then its numerical range is obtained from $W_1(A) = \overline{\mathbb{D}}$ by dilation along the $y$-axis with the coefficient $\gamma/\beta$, multiplication of the resulting ellipse by $\beta$, and shifting by $\alpha$. The resulting set is, of course, the ellipse described in the statement, and the coincidence of the first and second numerical ranges is preserved at each step.

To prove the converse, suppose that $W_1(B) = W_2(B)$. Since $B$ is unitarily irreducible, the boundary of $W_1(B)$ contains nonlinear portions. Hence, for infinitely many values $\zeta \in \mathbb{T}$ the minimal polynomial of $\operatorname{Re}(\zeta A)$ has a multiple eigenvalue. In other words, the polynomial $L_B$ can be factored as $f^2 g$, where $\deg f \geqslant 2$. If $\deg f > 2$, then $\deg f = 3$ and $g$ is a constant. By the result in [17], $B$ is unitarily reducible. Hence, we may assume that

$\deg f = \deg g = 2$, and the associated curve $C(B)$ consists of an ellipse $E_1$ (counted twice) defined by $f$, and an ellipse (possibly degenerating to a pair of points) $E_2$ defined by $g$. If $E_2$ does not lie inside $E_1$, then $B$ is either unitarily reducible (in case of pair of points) or property $W_1(B) = W_2(B)$ fails. Therefore, $E_2$ lies inside $E_1$, and $W_1(B)$ is bounded by $E_1$. For any $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, the matrix $\tilde{B} = \beta^{-1}(B - \alpha I)$ still has the property that $W_1(\tilde{B}) = W_2(\tilde{B})$ is an ellipse. By a proper choice of $\alpha, \beta$ we can center $W_1(\tilde{B})$ at zero, make its axes lie on $\mathbb{R}$ and $i\mathbb{R}$, and adjust the length of the former one to be 2. The same features are inherited by all matrices $\operatorname{Re} \tilde{B} + i\epsilon \operatorname{Im} \tilde{B}$, $\epsilon > 0$, and an appropriate choice of $\epsilon$ allows us to transform the ellipse $W_1(\tilde{B})$ into the unit disk $\overline{\mathbb{D}}$. It remains only to denote the corresponding matrix $\operatorname{Re} \tilde{B} + i\epsilon \operatorname{Im} \tilde{B}$ by $A$. ∎

As follows from Theorem 5.1, to study unitarily irreducible matrices $A \in M_6$ satisfying $W_1(A) = W_2(A)$, one can focus on those $A \in M_6$ for which

$$W_1(A) = W_2(A) = \overline{\mathbb{D}}. \tag{5.3}$$

The following theorem gives a complete description of all (unitarily reducible or not) $6 \times 6$ matrices $A$ having the property (5.3). In what follows, we denote by $\mathscr{D}_n$ the set of all $n \times n$ matrices $Z$ such that $W_1(Z) = \overline{\mathbb{D}}$.

THEOREM 5.2. *Let $A \in M_6$. Then $W_1(A) = W_2(A) = \overline{\mathbb{D}}$ if and only if $A$ is unitarily similar to a matrix of the form*

$$\begin{bmatrix} A_0 & A_{01} \\ A_{10} & A_1 \end{bmatrix}$$

*belonging to one of the following classes*:

(1) $A_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $A_{01} = 0$, $A_{10} = 0$, *and* $A_1 \in \mathscr{D}_4$, *or*

(2) $A_0 = \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & \lambda \end{bmatrix}$, $A_{01} = c\begin{bmatrix} \bar{y}/\bar{z} & 0 & 0 \\ -1 & 0 & 0 \\ -\lambda/y & 0 & 0 \end{bmatrix}$,

$$A_{10} = \frac{c\lambda y}{|z|^2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$A_1 = DZD - \operatorname{diag}\left(\frac{c^2\lambda}{|z|^2}, 0, 0\right)$$

$$with \quad D = \operatorname{diag}\left(\sqrt{1 - \frac{c^2}{|z|^2}}, 1, 1\right), \quad Z \in \mathscr{D}_3,$$

*where* $xy\bar{z} = -\lambda(|y|^2 + |z|^2) \neq 0$, $|x|^2 + |y|^2 + |z|^2 = 4$, $|\lambda| \leqslant 1$, $0 \leqslant c < |z|$, *or*

$$(3) \quad A_0 = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} cy & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} 0 & 0 & -cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

*and*

$$A_1 = DZD, \qquad with \quad D = \operatorname{diag}\left(\sqrt{1 - c^2}, 1, 1\right), \quad Z \in \mathscr{D}_3,$$

*where* $x, y > 0$, $|x|^2 + |y|^2 = 4$, $0 \leqslant c < 1$.

*Proof.* Necessity: According to Theorem 2.1, the condition (5.3) implies that for every $\zeta \in \mathbb{T}$ the largest eigenvalue of $\operatorname{Re}(\zeta^{-1}A)$ equals 1 and has multiplicity at least 2. For all the corresponding unit eigenvectors $w$, $\operatorname{Re}(\zeta^{-1}Aw, w) = 1$. Since $W_1(\zeta^{-1}A) = \zeta^{-1}W_1(A) = \overline{\mathbb{D}}$, this implies that in fact $(Aw, w) = \zeta$.

Choosing four different values of $\zeta$ (say, $\zeta_1, \ldots, \zeta_4$), we may therefore produce at least eight corresponding eigenvectors $u_j, v_j$ such that $(Aw, w) = \zeta_j$ for all unit vectors $w \in \operatorname{span}\{u_j, v_j\}$, and each pair $\{u_j, v_j\}$ is orthonormal ($j = 1, \ldots, 4$). The set $\{u_j, v_j\}_{j=1}^4$ of eight vectors in $\mathbb{C}^6$ has to be linearly dependent. Relabeling these vectors if necessary, we may suppose that $u_4$ is a linear combination of $\{u_j, v_j\}_{j=1}^3$: $u_4 = \sum_{j=1}^3(\alpha_j u_j + \beta_j v_j)$. Put

$$w_j = \begin{cases} \dfrac{\alpha_j u_j + \beta_j v_j}{\|\alpha_j u_j + \beta_j v_j\|} & \text{if} \quad \alpha_j u_j + \beta_j v_j \neq 0, \\ u_j & \text{otherwise}. \end{cases} \qquad (j = 1, 2, 3),$$

and let $\mathfrak{L} = \operatorname{span}\{w_1, w_2, w_3\}$ have dimension $m$. Let, further, $V$ be a unitary transformation of $\mathbb{C}^6$ mapping the first $m$ elements of the standard basis into $\mathfrak{L}$. Then

$$A = V^* \begin{bmatrix} A_0 & X \\ Y & B \end{bmatrix} V,$$

where the block $A_0$ is a compression of $A$ onto $\mathfrak{L}$. Since $\mathfrak{L}$ contains $w_1, w_2, w_3$ and $u_4$, the numerical range of $A_0$ contains $(A_0 w_j, w_j) = (Aw_j, w_j) = \zeta_j$ $(j = 1, 2, 3)$ and $(A_0 u_4, u_4) = (Au_4, u_4) = \zeta_4$. On the other hand, $W_1(A_0) \subseteq W_1(A) = \overline{\mathbb{D}}$. Being contained in $\overline{\mathbb{D}}$, and containing 4 ($> m$) of its boundary points, $W_1(A_0)$ has to coincide with $\overline{\mathbb{D}}$ (e.g., see [3, Theorem 5.8]).

By a proper choice of an orthonormal basis in $\mathfrak{L}$ (in other words, the first $m$ rows of $V$) we may put $A_0$ in the form

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & \lambda \end{bmatrix}, \tag{5.4}$$

where

$$xy\bar{z} = -\lambda\left(|y|^2 + |z|^2\right), \qquad |x|^2 + |y|^2 + |z|^2 = 4, \qquad |\lambda| \leqslant 1, \tag{5.5}$$

depending on whether $m = 2$ or $m = 3$ (for $m = 2$ this is well known; for $m = 3$ see [11, Corollary 2.5]).

Consider now the matrix

$$VAV^* = \begin{bmatrix} A_0 & X \\ Y & B \end{bmatrix},$$

unitarily similar to $A$. If $A_0 \in M_2$, the condition $W_1(A_0) = W_1(A) = \overline{\mathbb{D}}$ implies that $X = 0$, $Y = 0$ [8]. From here and $W_2(A) = \overline{\mathbb{D}}$ it follows that $W_1(B) = \overline{\mathbb{D}}$. Hence, we are in class (1) of the theorem.

The same reasoning applies if $A_0 \in M_3$ is unitarily reducible. Indeed, it can then be put in the form

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus [\lambda], \qquad |\lambda| \leqslant 1,$$

and we may consider a $2 \times 2$ subblock

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

of $A_0$ as a new matrix $A_0$.

It remains to cover the case of a unitarily irreducible matrix $A_0$ of the form (5.4). Since $W_1(A_0) = \overline{\mathbb{D}}$, two eigenvalues of the matrix $\operatorname{Re}(\zeta A_0)$ for all $\zeta \in \mathbb{T}$ equal 1 and $-1$; the third one is therefore $\operatorname{Re}(\zeta\lambda)$. Denote by $U_0(\zeta)$ a $3 \times 3$ unitary matrix which reduces $\operatorname{Re}(\zeta A_0)$ to its diagonal form:

$$U_0^*(\zeta)\operatorname{Re}(\zeta A_0)\,U_0(\zeta) = \operatorname{diag}(1, -1, \operatorname{Re}(\lambda\zeta)).$$

Then

$$\operatorname{Re}(\zeta A) = \begin{bmatrix} U_0(\zeta) & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & -1 & 0 & K(\zeta) \\ 0 & 0 & \operatorname{Re}(\lambda\zeta) & \\ \hline & K^*(\zeta) & & \operatorname{Re}(\zeta B) \end{array}\right]$$
$$\times \begin{bmatrix} U_0^*(\zeta) & 0 \\ 0 & I \end{bmatrix}, \tag{5.6}$$

where $K(\zeta) = \frac{1}{2}U_0^*(\zeta)(\zeta X + \zeta^{-1}Y^*)$.

The condition (5.3) implies that 1 and $-1$ are the largest and smallest eigenvalues of $\operatorname{Re}(\zeta A)$, the first two rows of the block $K(\zeta)$ must vanish, i.e.,

$$U_0^*(\zeta)(\zeta X + \zeta^{-1}Y^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_1(\zeta) & t_2(\zeta) & t_3(\zeta) \end{bmatrix}. \tag{5.7}$$

It follows that $\operatorname{Re}(\zeta A)$ is unitarily similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus N(\zeta)$, where

$$N(\zeta) = \left[\begin{array}{c|ccc} \operatorname{Re}(\lambda\zeta) & \frac{1}{2}t_1(\zeta) & \frac{1}{2}t_2(\zeta) & \frac{1}{2}t_3(\zeta) \\ \hline \frac{1}{2}\overline{t_1(\zeta)} & & & \\ \frac{1}{2}\overline{t_2(\zeta)} & & \operatorname{Re}(\zeta B) & \\ \frac{1}{2}\overline{t_3(\zeta)} & & & \end{array}\right], \tag{5.8}$$

and

$$\{-1, 1\} \subset \sigma(N(\zeta)) \subset [-1, 1] \qquad \text{for all} \quad \zeta \in \mathbb{T}. \tag{5.9}$$

To find the blocks $X, Y$, rewrite (5.7) in the form

$$\zeta X + \zeta^{-1} Y^* = u_3(\zeta)\left[t_1(\zeta), t_2(\zeta), t_3(\zeta)\right], \tag{5.10}$$

where $u_3$ is the third column of $U_0$, in other words, the unit eigenvector of $\operatorname{Re}(\zeta A_0)$ corresponding to the eigenvalue $\operatorname{Re}(\lambda\zeta)$.

We now consider cases $\lambda = 0$ and $\lambda \neq 0$ separately.

*Case 1.* $\lambda \neq 0$. From (5.5) it follows that either $xyz \neq 0$ or $y = z = 0$. We do not have to consider the latter possibility, since it corresponds to the unitarily reducible matrix $A_0$. If $xyz \neq 0$, the direct computation shows that for all the values of $\zeta$ (except $\zeta = \pm\lambda^{-1}$ when $|\lambda| = 1$, but a finite number of exceptions do not change the following reasoning)

$$u_3(\zeta) = \phi(\zeta)\left(\frac{\bar{y}}{\bar{z}}, -1, -\frac{\lambda}{y} + \frac{\bar{y}}{|z|^2}\bar{\lambda}\zeta^{-2}\right)^T.$$

Here $\phi$ is a scalar function of $\zeta$ with the absolute value

$$|\phi(\zeta)| = \left\{\left|\frac{y}{z}\right|^2 + 1 + |\lambda|^2\left(\frac{1}{|y|^2} + \frac{|y|^2}{|z|^4}\right) - \frac{2}{|z|^2}\operatorname{Re}(\lambda^2\zeta^2)\right\}^{-1/2}$$

$$= \frac{|z|}{2}\left(1 - \{\operatorname{Re}(\lambda\zeta)\}^2\right)^{-1/2}. \tag{5.11}$$

From this and (5.10),

$$\zeta X + \zeta^{-1} Y^* = \left(\frac{\bar{y}}{\bar{z}}, -1, -\frac{\lambda}{y} + \frac{\bar{y}}{|z|^2}\bar{\lambda}\zeta^{-2}\right)^T (\phi t_1, \phi t_2, \phi t_3). \tag{5.12}$$

Comparing the second rows of the obtained equality, we see that

$$\phi(\zeta) t_j(\zeta) = c_j\zeta + d_j\zeta^{-1} \qquad (c_j, d_j \in \mathbb{C}, \quad j = 1, 2, 3).$$

Looking at the third rows, we then conclude that $d_j = 0$, that is,

$$t_j(\zeta) = c_j\zeta\phi^{-1}(\zeta) \qquad (j = 1, 2, 3). \tag{5.13}$$

If $c_1 = c_2 = c_3 = 0$, then (5.10) implies that $X = Y = 0$. The matrix $A$ is then unitarily similar to $A_0 \oplus B$. Due to property (e) in Section 1, $W_2(A_0) = \frac{1}{2}[\lambda - W_1(A_0)]$ is a disk of the radius $\frac{1}{2}$. Hence, it has at most one common boundary point with $W_1(A_0) = \overline{\mathbb{D}}$, and the condition (5.3) holds if and only if $W_1(B) = \overline{\mathbb{D}}$. This corresponds to the situation (2) with $c = 0$. If at least one of the constants $c_j$ differs from 0, put $c = (|c_1|^2 + |c_2|^2 + |c_3|^2)^{1/2}$, and introduce a unitary $3 \times 3$ matrix $U_1$ with the first column $c^{-1}(c_1, c_2, c_3)^*$. The matrix $A$ is unitarily similar to

$$\begin{bmatrix} I & 0 \\ 0 & U_1^* \end{bmatrix}\begin{bmatrix} A_0 & X \\ Y & B \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix} = \begin{bmatrix} A_0 & XU_1 \\ U_1^* Y & U_1^* B U_1 \end{bmatrix},$$

where, due to (5.12) and (5.13),

$$XU_1\zeta + Y^*U_1\zeta^{-1} = \zeta\begin{bmatrix} \bar{y}/\bar{z} \\ -1 \\ -\dfrac{\lambda}{y} + \dfrac{\bar{y}}{|z|^2}\bar{\lambda}\zeta^{-2} \end{bmatrix}[c_1, c_2, c_3]U_1$$

$$= \zeta\begin{bmatrix} \bar{y}/\bar{z} \\ -1 \\ -\dfrac{\lambda}{y} + \dfrac{\bar{y}}{|z|^2}\bar{\lambda}\zeta^{-2} \end{bmatrix}[c, 0, 0].$$

Hence,

$$XU_1 = c\begin{bmatrix} \bar{y}/\bar{z} & 0 & 0 \\ -1 & 0 & 0 \\ -\lambda/y & 0 & 0 \end{bmatrix}, \qquad U_1^* Y = c\begin{bmatrix} 0 & 0 & \lambda y/|z|^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.14}$$

The unitary similarity

$$\begin{bmatrix} \epsilon & 0 \\ 0 & U_1 \end{bmatrix}, \qquad \text{where} \quad \epsilon(\zeta) = \frac{\zeta|\phi(\zeta)|}{\phi(\zeta)},$$

transforms the matrix (5.8) into

$$\left[\begin{array}{c|ccc} \operatorname{Re}(\lambda\zeta) & \psi(\zeta) & 0 & 0 \\ \hline \psi(\zeta) & & & \\ 0 & & \operatorname{Re}(\zeta A_1) & \\ 0 & & & \end{array}\right],$$

$A_1 = U_1^* B U_1$, $\psi(\zeta) = \frac{1}{2}c/|\phi(\zeta)| = (c/|z|)\sqrt{1 - \{\operatorname{Re}(\lambda\zeta)\}^2}$. The property (5.9) is equivalent to

$$\left[\begin{array}{c|ccc} \operatorname{Re}(\lambda\zeta) - 1 & \psi(\zeta) & 0 & 0 \\ \hline \psi(\zeta) & & & \\ 0 & & \operatorname{Re}(\zeta A_1) - I & \\ 0 & & & \end{array}\right]$$

being nonpositive and singular for all $\zeta \in \mathbb{T}$. The latter matrix is congruent via

$$\begin{bmatrix} \{1 - \operatorname{Re}(\lambda\zeta)\}^{-1/2} & 0 \\ \{1 - \operatorname{Re}(\lambda\zeta)\}^{-1}\psi(\zeta) & 1 \end{bmatrix} \oplus I_2$$

to the orthogonal sum of $[-1]$ with

$$\operatorname{Re}\left\{\zeta A_1 + \zeta \operatorname{diag}\left(\frac{c^2\lambda}{|z|^2}, 0, 0\right)\right\} - \operatorname{diag}\left(1 - \frac{c^2}{|z|^2}, 1, 1\right). \qquad (5.15)$$

Therefore, (5.9) holds if and only if the matrix (5.15) is nonpositive and singular for all $\zeta \in \mathbb{T}$. Considering the (1, 1) entry of (5.15), we see that it means in particular that $1 - c^2/|z|^2 \geqslant 0$, which allows us to introduce a nonnegative matrix

$$D = \operatorname{diag}\left(\sqrt{1 - \frac{c^2}{|z|^2}}, 1, 1\right).$$

We have now to distinguish between the cases $c = |z|$ and $c < |z|$.

*Case 1-1.* $c < |z|$. In this situation $D$ is invertible, and the matrix (5.15) is, in turn, congruent to

$$\operatorname{Re}(\zeta Z) - I, \tag{5.16}$$

where

$$Z = D^{-1}\left\{A_1 + \operatorname{diag}\left(\frac{c^2\lambda}{|z|^2}, 0, 0\right)\right\}D^{-1}.$$

For the matrix (5.16) to be nonpositive and singular for all $\zeta \in \mathbb{T}$ it is necessary and sufficient that $W_1(Z) = \overline{\mathbb{D}}$. Hence, we are in situation (2).

*Case 1-2.* $c = |z|$. In this case nonpositivity of (5.15) implies that

$$\operatorname{Re}(\zeta\{A_1 + \operatorname{diag}(\lambda, 0, 0)\}) = [0] \oplus \begin{bmatrix} * & * \\ * & * \end{bmatrix} \quad \text{for all} \quad \zeta \in \mathbb{T}.$$

In other words, $A_1 = [-\lambda] \oplus Z_0$, $Z_0 \in M_2$. From here and (5.14) it follows that $A$ is unitarily similar to an orthogonal sum of $Z_0$ with a $4 \times 4$ matrix

$$Z_1 = \begin{bmatrix} 0 & x & z & c\bar{y}/\bar{z} \\ 0 & 0 & y & -c \\ 0 & 0 & \lambda & -c\lambda/y \\ 0 & 0 & c\lambda y/|z|^2 & -\lambda \end{bmatrix}.$$

The condition (5.3) implies that either $W_1(Z_0) = W_1(Z_1) = \overline{\mathbb{D}}$, or $W_1(Z_0) \subset \overline{\mathbb{D}}$, $W_1(Z_1) = W_2(Z_1) = \overline{\mathbb{D}}$. In the first case, we already are in situation (1) (with $A_0 = Z_0$, $A_1 = Z_1$). In the second case, the matrix $Z_1$ is unitarily similar to an orthogonal sum $Z_2 \oplus Z_3$ of two $2 \times 2$ matrices with $W_1(Z_2) = W_1(Z_3) = \overline{\mathbb{D}}$ due to Corollary 4.7. The matrix $A$ is again in class (1), with $A_0 = Z_2$ and $A_1 = Z_0 \oplus Z_3$.

*Case 2.* $\lambda = 0$. According to (5.5), $xyz = 0$ as well. We do not have to consider the cases $x = 0$ or $y = 0$, in which $A_0$ becomes unitarily reducible. Therefore, we are left with the only possibility $z = 0$. Applying an additional unitary similarity if necessary, we may suppose that $x, y > 0$. Direct computations show that $u_3(\zeta)$ can be chosen in the form

$$u_3(\zeta) = \tfrac{1}{2}[\zeta y, 0, -\zeta^{-1}x]^T.$$

From this and (5.10), we have

$$\zeta X + \zeta^{-1} Y^* = \frac{1}{2}\begin{bmatrix} \zeta t_1(\zeta) y & \zeta t_2(\zeta) y & \zeta t_3(\zeta) y \\ 0 & 0 & 0 \\ -\zeta^{-1} t_1(\zeta) x & -\zeta^{-1} t_2(\zeta) x & -\zeta^{-1} t_3(\zeta) x \end{bmatrix}.$$

The entries in the left hand side of the latter equation are linear combinations of $\zeta$ and $\zeta^{-1}$. Therefore, the entries in the right hand side have to be of the same form. Obviously, this is possible if and only if all the functions $t_j(\zeta)$ are in fact constant: $t_j(\zeta) = 2c_j$, $c_j \in \mathbb{C}$ $(j = 1, 2, 3)$. Then

$$\zeta X + \zeta^{-1} Y^* = \begin{bmatrix} \zeta y c_1 & \zeta y c_2 & \zeta y c_3 \\ 0 & 0 & 0 \\ -\zeta^{-1} x c_1 & -\zeta^{-1} x c_2 & -\zeta^{-1} x c_3 \end{bmatrix},$$

that is,

$$X = y\begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y = -x\begin{bmatrix} 0 & 0 & \overline{c_1} \\ 0 & 0 & \overline{c_2} \\ 0 & 0 & \overline{c_3} \end{bmatrix}.$$

Similarly to case 1, either $c_1 = c_2 = c_3 = 0$, which means that $X = Y = 0$ and $A$ is unitarily similar to $A_0 \oplus B$, $B \in \mathscr{D}_3$, or additional unitary transformation puts $A$ in the form

$$\tilde{A} = \left[\begin{array}{ccc|ccc} 0 & x & 0 & cy & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -cx & & & \\ 0 & 0 & 0 & & A_1 & \\ 0 & 0 & 0 & & & \end{array}\right].$$

The matrix (5.8), in turn, takes the form

$$\left[\begin{array}{c|ccc} 0 & c & 0 & 0 \\ \hline c & & & \\ 0 & & \mathrm{Re}(\zeta A_1) & \\ 0 & & & \end{array}\right], \tag{5.17}$$

and the condition (5.9) is now equivalent to

$$\left[\begin{array}{c|ccc} -1 & c & 0 & 0 \\ \hline c & & & \\ 0 & & \mathrm{Re}(\zeta A_1) - I & \\ 0 & & & \end{array}\right]$$

being nonpositive and singular for all $\zeta \in \mathbb{T}$. Rewriting the latter matrix as

$$\left(\begin{bmatrix} 1 & 0 \\ -c & 1 \end{bmatrix} \oplus I_2\right)\left([-1] \oplus \{\mathrm{Re}(\zeta A_1) + \mathrm{diag}(c^2, 0, 0) - I_3\}\right)$$
$$\times\left(\begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} \oplus I_2\right),$$

we see that the latter condition is satisfied if and only if

$$\mathrm{Re}(\zeta A_1) - \mathrm{diag}(1 - c^2, 1, 1) \tag{5.18}$$

is a nonpositive singular matrix for all $\zeta \in \mathbb{T}$. Considering the (1, 1) entry of (5.18), we conclude that $c \leqslant 1$. After we choose $D = \mathrm{diag}(\sqrt{1 - c^2}, 1, 1)$, the rest of the reasoning goes exactly as in case 1. Namely, if $c < 1$, then (5.18) can be rewritten as

$$D\{\mathrm{Re}(\zeta Z) - I\}D, \qquad Z = D^{-1}A_1 D^{-1},$$

and its nonpositiveness and singularity for all $\zeta \in \mathbb{T}$ mean exactly that $W_1(Z) = \overline{\mathbb{D}}$. In other words, $A$ is in class (3). If $c = 1$, then $A_1 = [0] \oplus Z_0$,

$Z_0 \in M_2$. Hence, $A$ is unitarily similar to $Z_0 \oplus Z_1$, where

$$Z_1 = \begin{bmatrix} 0 & x & 0 & cy \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -cx & 0 \end{bmatrix},$$

and therefore $A$ belongs to class (1).

Sufficiency: In all the cases considered above, $\operatorname{Re}(\zeta A)$ is unitarily similar to the orthogonal sum of $\operatorname{diag}(1, -1)$ with the matrix $N(\zeta)$ given by (5.8), and the latter matrix satisfies (5.9). In other words, 1 and $-1$ are eigenvalues of $\operatorname{Re}(\zeta A)$ with multiplicities at least 2, and remaining eigenvalues also lie in $[-1, 1]$. From this and Theorem 2.1, (5.3) follows. ■

Obviously, matrices in class (1) are unitarily reducible (into orthogonal sums of $2 \times 2$ and $4 \times 4$ blocks), as are matrices in classes (2), (3) with $c = 0$ (into sums of two $3 \times 3$ blocks). In fact, by the result in [8], a matrix $A \in M_6$ satisfying the condition (5.3) is in class (1) if and only if the spectral norm of $A$ equals 2. It would be nice to have a simple condition to determine when the matrices in classes (2) and (3) are irreducible. Note that all the (unitarily irreducible) matrices described by Theorem 3.2 are in class (3) and correspond to the choice

$$Z = \begin{bmatrix} 0 & \xi & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{bmatrix}.$$

Actually, it was Theorem 5.2 that led to the discovery of our counterexamples (3.2) to the statement (2.1).

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