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Statistical properties of an estimator for the mean function of a compound cyclic Poisson process in the presence of linear trend

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Abstract. The problem of estimating the mean function of a compound cyclic Poisson process with linear trend is considered. An estimator of this mean function is constructed and investigated. The cyclic component of intensity function of this process is not assumed to have any parametric form, but its period is assumed to be known. The slope of the linear trend is assumed to be positive, but its value is unknown. Moreover, we consider the case when there is only a single realization of the Poisson process is observed in a bounded interval. Asymptotic bias and variance of the proposed estimator are computed, when the size of interval indefinitely expands.

Keywords: Asymptotic bias and variance; Compound cyclic Poisson; Mean function; Linear trend

2010 Mathematics Subject Classification: 60E20; 60G20; 62M20

1. INTRODUCTION

Let $\{N(t), t \geq 0\}$ be a Poisson process with (unknown) locally integrable intensity function λ which is assumed to consist of two components, namely, a periodic or cyclic component with period $\tau > 0$ and a linear trend component. In other words, for any point

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$s \geq 0$, the intensity function λ can be written as

$$\lambda(s) = \lambda_c(s) + as,$$

where $\lambda_c(s)$ is a periodic function with (known) period τ and a denotes the slope of the linear trend which is assumed that $a > 0$. We do not assume any (parametric) form of $\lambda_c(s)$ except that it is periodic, that is, the equality

$$\lambda_c(s) = \lambda_c(s + k\tau)$$

holds for all $s \geq 0$ and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.

Let $\{Y(t), t \geq 0\}$ be a process with

$$Y(t) = \sum_{i=1}^{N(t)} X_i, \tag{1}$$

where $\{X_i,\ i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean $\mu < \infty$ and variance $\sigma^2 < \infty$, which is also independent of the process $\{N(t), t \geq 0\}$. The process $\{Y(t), t \geq 0\}$ is said to be a *compound cyclic Poisson process with linear trend*. The model presented in (1) is an extension of the model presented in [4]. We refer to [1,3,5,6] for some applications of the compound Poisson process.

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the process $\{N(t), t \geq 0\}$ defined on probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Furthermore, suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say ith data point, $i = 1, 2, \ldots, N([0, n])$, its corresponding random variable X_i is also observed.

The mean function (expected value) of $Y(t)$, denoted by $\psi(t)$, is given by:

$$\psi(t) = E[N(t)]\, E[X_1] = \Lambda(t)\mu$$

with $\Lambda(t) = \int_0^t \lambda(s)ds$. Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where for any real number x, $\lfloor x \rfloor$ denote the largest integer that less than or equal to x, and let also $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then, for any given real number t, we can write $t = k_{t,\tau}\tau + t_r$, with $0 \leq t_r < \tau$. Let $\theta = \frac{1}{\tau} \int_0^\tau \lambda_c(s)ds$ is the global intensity of the cyclic component of the Poisson process $\{N(t), t \geq 0\}$. We assume that $\theta > 0$. Then, for any given $t \geq 0$, we have

$$\Lambda(t) = k_{t,\tau}\tau\theta + \Lambda_c(t_r) + a\frac{t^2}{2}$$

which implies

$$\psi(t) = \left(k_{t,\tau}\tau\theta + \Lambda_c(t_r) + a\frac{t^2}{2} \right) \mu.$$

An estimator for the mean function $\psi(t)$ of the process $\{Y(t), t \geq 0\}$ using the observed realization have been constructed. Our goal in this paper is to compute asymptotic bias and variance of an estimator for the mean function $\psi(t)$ of the process $\{Y(t), t \geq 0\}$.

The rest of this paper is organized as follows. The estimator and main results are presented in Section 2, some technical lemmas are presented in Section 3, and the proofs of the main results are given in Section 4.

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2. The estimator and main results

Let $k_{n,\tau} = \left\lfloor \frac{n}{\tau} \right\rfloor$. The estimator of the mean function $\psi(t)$ using the available data set at hand is given by

$$\widehat{\psi}_n(t) = \left(k_{t,\tau}\tau\widehat{\theta}_n + \widehat{\Lambda}_{c,n}\left(t_r\right) + \hat{a}_n\frac{t^2}{2} \right)\widehat{\mu}_n,$$

where

$$\hat{a}_n = \frac{2N\left[0,n\right]}{n^2},$$

$$\widehat{\theta}_n = \frac{1}{\ln\left(k_{n,\tau}\right)\tau}\sum_{k=1}^{k_{n,\tau}}\frac{N\left(\left[\left(k-1\right)\tau, k\tau\right]\right)}{k} - \hat{a}_n\left(\frac{k_{n,\tau}\tau}{\ln\left(k_{n,\tau}\right)} - \frac{\tau}{2}\right),$$

$$\widehat{\Lambda}_{c,n}\left(t_r\right) = \frac{1}{\ln\left(k_{n,\tau}\right)}\sum_{k=1}^{k_{n,\tau}}\frac{N\left(\left[\left(k-1\right)\tau, \left(k-1\right)\tau + t_r\right]\right)}{k} - \hat{a}_n\left(\frac{k_{n,\tau}\tau t_r}{\ln\left(k_{n,\tau}\right)} + \frac{\left(t_r^2 - 2t_r\tau\right)}{2}\right),$$

and

$$\widehat{\mu}_n = \frac{1}{N\left(\left[0,n\right]\right)}\sum_{i=1}^{N\left(\left[0,n\right]\right)} X_i,$$

with the understanding that $\widehat{\mu}_n = 0$ when $N\left(\left[0,n\right]\right) = 0$. Thus, $\widehat{\psi}_n(t) = 0$ when $N\left(\left[0,n\right]\right) = 0$.

Our main results are presented in the following theorem. The Theorem is about asymptotic approximation to the bias of $\widehat{\psi}_n(t)$ and about asymptotic approximation to the variance of $\widehat{\psi}_n(t)$.

Theorem. *Asymptotic approximation to the bias of* $\widehat{\psi}_n(t)$*:*

$$bias\left[\widehat{\psi}_n(t)\right] = \left(\frac{k_{t,\tau}\tau\left(2\theta\gamma - a\tau\gamma\right) + 2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)}{2\ln\left(k_{n,\tau}\right)}\right)\mu + o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right), \tag{2}$$

and asymptotic approximation to the variance of $\widehat{\psi}_n(t)$*:*

$$var\left[\widehat{\psi}_n(t)\right] = \frac{\mu^2}{\ln\left(k_{n,\tau}\right)}\Bigg(\left(k_{t,\tau}\tau\right)^2\left(a + 2\theta^2\gamma - a\tau\theta\gamma\right) + \left(a\tau t_r + 2\gamma\left(\Lambda_c\left(t_r\right)\right)^2 + a\gamma\Lambda_c\left(t_r\right)\left(t_r^2 - 2\tau t_r\right)\right) + 2k_{t,\tau}\tau\left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma - a\tau\gamma\right) + \theta\left(2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)\right) + 2at_r}{2}\right)$$

$$+k_{t,\tau}\tau t^2\left(\frac{2a\theta\gamma-a^2\tau\gamma}{2}\right)+t^2\left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma+a^2\gamma\left(t_r^2-2\tau t_r\right)\right)}{2}\right)$$
$$-\frac{\psi(t)}{\mu}\left(k_{t,\tau}\tau\left(2\theta\gamma-a\tau\gamma\right)+2\gamma\Lambda_c\left(t_r\right)+a\gamma\left(t_r^2-2\tau t_r\right)\right)\Bigg)+o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right) \quad (3)$$

as $n \to \infty$, where $\gamma = 0.577\ldots$ is an Euler's constant.

3. Some technical lemmas

In this section, we present some lemmas which are needed in the proofs of our theorem.

Lemma 1. *Asymptotic approximation to the bias of $\hat{a}_n$:*

$$E\left[\hat{a}_n\right]=a+\frac{2\theta}{n}+O\left(\frac{1}{n^2}\right) \quad (4)$$

and asymptotic approximation to the variance of $\hat{a}_n$:

$$Var\left[\hat{a}_n\right]=\frac{2a}{n^2}+O\left(\frac{1}{n^3}\right) \quad (5)$$

as $n \to \infty$.

Proof. We refer to [2].

Lemma 2. *Asymptotic approximation to the bias of $\widehat{\Lambda}_{c,n}\left(t_r\right)$*

$$E\left[\widehat{\Lambda}_{c,n}\left(t_r\right)\right]=\Lambda_c\left(t_r\right)+\frac{2\gamma\Lambda_c\left(t_r\right)+a\gamma\left(t_r^2-2\tau t_r\right)}{2\ln\left(k_{n,\tau}\right)}+o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right) \quad (6)$$

and asymptotic approximation to the variance of $\widehat{\Lambda}_{c,n}\left(t_r\right)$

$$Var\left[\widehat{\Lambda}_{c,n}\left(t_r\right)\right]=\frac{a\tau t_r}{\ln\left(k_{n,\tau}\right)}+\frac{2\pi^2\Lambda_c\left(t_r\right)+a\pi^2\left(t_r^2-2\tau t_r\right)+12a\tau t_r\gamma}{12\left(\ln\left(k_{n,\tau}\right)\right)^2}$$
$$+o\left(\frac{1}{\left(\ln\left(k_{n,\tau}\right)\right)^2}\right) \quad (7)$$

as $n \to \infty$, where $\gamma = 0.577\ldots$ is an Euler's constant.

Proof. The expected value of $\widehat{\Lambda}_{c,n}\left(t_r\right)$ can be computed as follows:

$$E\left[\widehat{\Lambda}_{c,n}\left(t_r\right)\right]=\frac{1}{\ln\left(k_{n,\tau}\right)}\sum_{k=1}^{k_{n,\tau}}\frac{E\left[N\left(\left[(k-1)\tau,(k-1)\tau+t_r\right]\right)\right]}{k}$$
$$-\left(\frac{k_{n,\tau}\tau t_r}{\ln\left(k_{n,\tau}\right)}+\frac{\left(t_r^2-2t_r\tau\right)}{2}\right)E\left[\hat{a}_n\right]. \quad (8)$$

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A simple calculation shows that

$$\sum_{k=1}^{k_{n,\tau}} \frac{E\left[N\left(\left[(k-1)\tau,(k-1)\tau+t_r\right]\right)\right]}{k} = \Lambda_c(t_r) + a\frac{t_r^2-2\tau t_r}{2} + \frac{\Lambda_c(t_r)\gamma + a\gamma\frac{t_r^2-2\tau t_r}{2} + ak_{n,\tau}\tau t_r}{\ln(k_{n,\tau})}. \quad (9)$$

By substituting (4) and (9) into the r.h.s. of (8), we obtain (6).

The variance of $\widehat{\Lambda}_{c,n}(t_r)$ can be computed as follows:

Let

$A_n = \frac{1}{\ln(k_{n,\tau})}\sum_{k=1}^{k_{n,\tau}} \frac{N([(k-1)\tau,(k-1)\tau+t_r])}{k}$ and $B_n = \hat{a}_n\left(\frac{k_{n,\tau}\tau t_r}{\ln(k_{n,\tau})} + \frac{(t_r^2-2t_r\tau)}{2}\right)$.

So

$$Var\left[\widehat{\Lambda}_{c,n}(t_r)\right] = Var[A_n] + Var[B_n] - 2Cov(A_n,B_n). \quad (10)$$

First, we compute

$$Var[A_n] = \frac{1}{(\ln(k_{n,\tau}))^2}\sum_{k=1}^{k_{n,\tau}}\frac{1}{k^2}Var\left[N\left(\left[(k-1)\tau,(k-1)\tau+t_r\right]\right)\right] = \frac{1}{(\ln(k_{n,\tau}))^2}\sum_{k=1}^{k_{n,\tau}}\frac{1}{k^2}E\left[N\left(\left[(k-1)\tau,(k-1)\tau+t_r\right]\right)\right]. \quad (11)$$

A simple calculation shows that

$$\sum_{k=1}^{k_{n,\tau}}\frac{1}{k^2}E\left[N\left(\left[(k-1)\tau,(k-1)\tau+t_r\right]\right)\right] = \left(\Lambda_c(t_r) + a\frac{t_r^2-2\tau t_r}{2}\right)\frac{\pi^2}{6} + a\tau t_r\ln(k_{n,\tau}) + a\tau t_r\gamma + o(1) \quad (12)$$

as $n\to\infty$. By substituting (12) into the r.h.s of (11), we have

$$Var[A_n] = \frac{a\tau t_r}{\ln(k_{n,\tau})} + \frac{\left(\Lambda_c(t_r) + a\frac{t_r^2-2\tau t_r}{2}\right)\frac{\pi^2}{6} + a\tau t_r\gamma}{(\ln(k_{n,\tau}))^2} + o\left(\frac{1}{(\ln(k_{n,\tau}))^2}\right), \quad (13)$$

as $n\to\infty$.

Next, we compute

$$Var[B_n] = \left(\frac{k_{n,\tau}\tau t_r}{\ln(k_{n,\tau})} + \frac{(t_r^2-2t_r\tau)}{2}\right)^2 Var[\hat{a}_n] = \left(\frac{(k_{n,\tau}\tau t_r)^2}{(\ln(k_{n,\tau}))^2} + \frac{k_{n,\tau}\tau t_r(t_r^2-2t_r\tau)}{\ln(k_{n,\tau})} + \frac{(t_r^2-2t_r\tau)^2}{4}\right)Var[\hat{a}_n]. \quad (14)$$

By substituting (5) into the r.h.s of (14), we have

$$Var\left[B_n\right] = \frac{2a\left(k_{n,\tau}\tau t_r\right)^2}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{4\theta\left(k_{n,\tau}\tau t_r\right)^2}{n^3\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{2ak_{n,\tau}\tau t_r\left(t_r^2 - 2t_r\tau\right)}{n^2\ln\left(k_{n,\tau}\right)} + \frac{4\theta k_{n,\tau}\tau t_r\left(t_r^2 - 2t_r\tau\right)}{n^3\ln\left(k_{n,\tau}\right)} + \frac{a\left(t_r^2 - 2t_r\tau\right)^2}{2n^2} + \frac{\theta\left(t_r^2 - 2t_r\tau\right)^2}{n^3} + O\left(\frac{1}{n^4}\right), \quad (15)$$

as $n \to \infty$.

Last, we compute

$$\begin{aligned} 2Cov\left(A_n, B_n\right) &= \left(\frac{2k_{n,\tau}\tau t_r}{\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{\left(t_r^2 - 2t_r\tau\right)}{\ln\left(k_{n,\tau}\right)}\right) \times Cov\left(\sum_{k=1}^{k_{n,\tau}} \frac{N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)}{k}, \hat{a}_n\right) \\ &= \left(\frac{4k_{n,\tau}\tau t_r}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{2\left(t_r^2 - 2t_r\tau\right)}{n^2\ln\left(k_{n,\tau}\right)}\right) \times Cov\left(\sum_{k=1}^{k_{n,\tau}} \frac{N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)}{k}, N\left[0,n\right]\right) \\ &= \left(\frac{4k_{n,\tau}\tau t_r}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{2\left(t_r^2 - 2t_r\tau\right)}{n^2\ln\left(k_{n,\tau}\right)}\right) \times \sum_{k=1}^{k_{n,\tau}} \frac{Var\left(N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)\right)}{k} \\ &= \left(\frac{4k_{n,\tau}\tau t_r}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{2\left(t_r^2 - 2t_r\tau\right)}{n^2\ln\left(k_{n,\tau}\right)}\right) \times \sum_{k=1}^{k_{n,\tau}} \frac{E\left(N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)\right)}{k}. \end{aligned} \quad (16)$$

By substituting (9) into the r.h.s. of (16), we have

$$2Cov\left(A_n, B_n\right) = \frac{4a\left(k_{n,\tau}\tau t_r\right)^2}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + \frac{2k_{n,\tau}\tau t_r\left(2\Lambda_c\left(t_r\right) + a\left(t_r^2 - 2t_r\tau\right)\right)}{n^2\ln\left(k_{n,\tau}\right)} + \frac{2k_{n,\tau}\tau t_r\left(2\Lambda_c\left(t_r\right)\gamma + a\gamma\left(t_r^2 - 2t_r\tau\right)\right)}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2} + o\left(\frac{4k_{n,\tau}\tau t_r}{n^2\left(\ln\left(k_{n,\tau}\right)\right)^2}\right), \quad (17)$$

as $n \to \infty$. By substituting (13), (15) and (17) into the r.h.s. of (10), we obtain (7). This completes the proof of Lemma 2.

Lemma 3. *Asymptotic approximation to the bias of* $\widehat{\theta}_n$

$$E\left[\widehat{\theta}_n\right] = \theta + \frac{2\theta\gamma - a\tau\gamma}{2\ln(k_{n,\tau})} + o\left(\frac{1}{\ln(k_{n,\tau})}\right) \tag{18}$$

and asymptotic approximation to the variance of $\widehat{\theta}_n$

$$Var\left[\widehat{\theta}_n\right] = \frac{a}{\ln(k_{n,\tau})} + \frac{2\theta\pi^2 + a\left(12\tau\gamma - \tau\pi^2\right)}{12\tau\left(\ln(k_{n,\tau})\right)^2} + o\left(\frac{1}{\left(\ln(k_{n,\tau})\right)^2}\right)$$

as $n \to \infty$*, where* $\gamma = 0.577\ldots$ *is an Euler's constant.*

Proof. To prove Lemma 3, note that,

$$\widehat{\theta}_n = \frac{1}{\tau}\widehat{\Lambda}_{c,n}(\tau).$$

Since $\widehat{\theta}_n$ is (almost) special case of $\widehat{\Lambda}_{c,n}(t_r)$ with $t_r = \tau$, the proof of Lemma 3 is similar (and simpler) than the proof of Lemma 2. Hence, it is omitted.

Lemma 4. *Asymptotic approximation to the bias of* $\widehat{\theta}_n\hat{a}_n$

$$E\left(\widehat{\theta}_n\hat{a}_n\right) = a\theta + \frac{2a\theta\gamma - a^2\tau\gamma - 12a\theta}{2\ln(k_{n,\tau})} + o\left(\frac{1}{\ln(k_{n,\tau})}\right) \tag{19}$$

and asymptotic approximation to the bias of $\widehat{\Lambda}_{c,n}(t_r)\hat{a}_n$

$$E\left(\widehat{\Lambda}_{c,n}(t_r)\hat{a}_n\right) = a\Lambda_c(t_r) + \frac{\left(2a\Lambda_c(t_r)\gamma + a^2\gamma\left(t_r^2 - 2\tau t_r\right)\right)}{2\ln(k_{n,\tau})} + o\left(\frac{1}{\ln(k_{n,\tau})}\right) \tag{20}$$

as $n \to \infty$*, where* $\gamma = 0.577\ldots$ *is an Euler's constant.*

Proof. The value of $E\left(\widehat{\theta}_n\hat{a}_n\right)$ can be computed as follows:

$$\begin{aligned} E\left(\widehat{\theta}_n\hat{a}_n\right) &= \frac{2}{n^2\ln(k_{n,\tau})\tau}E\left(\sum_{k=1}^{k_{n,\tau}}\frac{N([(k-1)\tau,k\tau])}{k}N[0,n]\right) \\ &\quad - \left(\frac{k_{n,\tau}\tau}{\ln(k_{n,\tau})} - \frac{\tau}{2}\right)E\left(\hat{a}_n^2\right). \end{aligned} \tag{21}$$

A simple calculation shows that

$$\begin{aligned} &E\left(\sum_{k=1}^{k_{n,\tau}}\frac{N([(k-1)\tau,k\tau])}{k}N[0,n]\right) \\ &= \frac{a^2n^2k_{n,\tau}\tau^2}{2} + \left(\theta\tau - \frac{a\tau^2}{2}\right)\frac{an^2\ln(k_{n,\tau})}{2} + \frac{an^2\left(\theta\tau - \frac{a\tau^2}{2}\right)\gamma}{2} \end{aligned}$$

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$$+ a\theta n k_{n,\tau}\tau^2 + \left(\theta\tau - \frac{a\tau^2}{2}\right)\theta n \ln\left(k_{n,\tau}\right) + \left(\theta\tau - \frac{a\tau^2}{2}\right)\theta n\gamma$$
$$+ a k_{n,\tau}\tau^2 + \left(\theta\tau - \frac{a\tau^2}{2}\right)\ln\left(k_{n,\tau}\right) + \left(\theta\tau - \frac{a\tau^2}{2}\right)\gamma + o(1), \tag{22}$$

as $n \to \infty$.

Now note that, by Lemma 1, we have

$$E\left((\hat{a}_n)^2\right) = a^2 + \frac{4a\theta}{n} + O\left(\frac{1}{n^2}\right), \tag{23}$$

as $n \to \infty$. By substituting (22) and (23) into the r.h.s of (21), we obtain (19).

The value of $E\left(\widehat{\Lambda}_{c,n}(t_r)\,\hat{a}_n\right)$ can be computed as follows:

$$E\left(\widehat{\Lambda}_{c,n}(t_r)\,\hat{a}_n\right) = \frac{2}{n^2\ln\left(k_{n,\tau}\right)} E\left(\sum_{k=1}^{k_{n,\tau}} \frac{N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)}{k} N[0,n]\right)$$
$$- \left(\frac{k_{n,\tau}\tau t_r}{\ln\left(k_{n,\tau}\right)} + \frac{\left(t_r^2 - 2t_r\tau\right)}{2}\right) E\left(\hat{a}_n^2\right). \tag{24}$$

A simple calculation shows that

$$E\left(\sum_{k=1}^{k_{n,\tau}} \frac{N\left(\left[(k-1)\tau, (k-1)\tau + t_r\right]\right)}{k} N[0,n]\right)$$
$$= \frac{a^2 n^2 k_{n,\tau}\tau t_r}{2} + \left(\Lambda_c(t_r) + \frac{a\left(t_r^2 - 2\tau t_r\right)}{2}\right)\frac{a n^2 \ln\left(k_{n,\tau}\right)}{2}$$
$$+ \left(\Lambda_c(t_r)\gamma + \frac{a\gamma\left(t_r^2 - 2\tau t_r\right)}{2}\right)\frac{a n^2}{2}$$
$$+ a\theta n k_{n,\tau}\tau t_r + \left(\Lambda_c(t_r) + \frac{a\left(t_r^2 - 2\tau t_r\right)}{2}\right)\theta n \ln\left(k_{n,\tau}\right)$$
$$+ \left(\Lambda_c(t_r)\gamma + \frac{a\gamma\left(t_r^2 - 2\tau t_r\right)}{2}\right)\theta n$$
$$+ a k_{n,\tau}\tau t_r + \left(\Lambda_c(t_r) + \frac{a\left(t_r^2 - 2\tau t_r\right)}{2}\right)\ln\left(k_{n,\tau}\right)$$
$$+ \left(\Lambda_c(t_r)\gamma + \frac{a\gamma\left(t_r^2 - 2\tau t_r\right)}{2}\right) + o(1), \tag{25}$$

as $n \to \infty$. By substituting (23) and (25) into the r.h.s of (24), we obtain (20). This completes the proof of Lemma 4.

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Lemma 5. *Asymptotic approximation to the bias of* $\widehat{\theta}_n \widehat{\Lambda}_{c,n}(t_r)$

$$E\left(\widehat{\theta}_n \widehat{\Lambda}_{c,n}(t_r)\right) = \theta \Lambda_c(t_r) + \frac{\Lambda_c(t_r)(2\theta\gamma - a\tau\gamma) + \theta\left(2\gamma\Lambda_c(t_r) + a\gamma\left(t_r^2 - 2\tau t_r\right)\right) + 2at_r}{2\ln(k_{n,\tau})} + o\left(\frac{1}{\ln(k_{n,\tau})}\right) \tag{26}$$

as $n \to \infty$, *where* $\gamma = 0.577\ldots$ *is an Euler's constant.*

Proof. To compute $E\left(\widehat{\theta}_n \widehat{\Lambda}_{c,n}(t_r)\right)$ we argue as follows.

Let

$$\widehat{\Lambda}_{c,n}(t_r)^C = \frac{1}{\ln(k_{n,\tau})} \sum_{k=1}^{k_{n,\tau}} \frac{N([(k-1)\tau + t_r,\ k\tau])}{k} - \hat{a}_n\left(\frac{k_{n,\tau}\tau(\tau - t_r)}{\ln(k_{n,\tau})} - \frac{(\tau - t_r)^2}{2}\right).$$

So

$$\widehat{\theta}_n = \frac{1}{\tau}\left(\widehat{\Lambda}_{c,n}(t_r) + \widehat{\Lambda}_{c,n}(t_r)^C\right).$$

Note that $\widehat{\Lambda}_{c,n}(t_r)$ and $\widehat{\Lambda}_{c,n}(t_r)^C$ are independent random variables.

Hence,

$$\begin{aligned} E\left(\widehat{\theta}_n \widehat{\Lambda}_{c,n}(t_r)\right) &= Cov\left(\widehat{\theta}_n, \widehat{\Lambda}_{c,n}(t_r)\right) + E\left(\widehat{\theta}_n\right) E\left(\widehat{\Lambda}_{c,n}(t_r)\right) \\ &= \frac{1}{\tau} Cov\left(\widehat{\Lambda}_{c,n}(t_r), \widehat{\Lambda}_{c,n}(t_r)\right) + \frac{1}{\tau} Cov\left(\widehat{\Lambda}_{c,n}(t_r)^C, \widehat{\Lambda}_{c,n}(t_r)\right) \\ &\quad + E\left(\widehat{\theta}_n\right) E\left(\widehat{\Lambda}_{c,n}(t_r)\right) \\ &= \frac{1}{\tau} Var\left(\widehat{\Lambda}_{c,n}(t_r)\right) + E\left(\widehat{\theta}_n\right) E\left(\widehat{\Lambda}_{c,n}(t_r)\right). \end{aligned} \tag{27}$$

By substituting (6), (7) and (18) into the r.h.s of (27), we obtain (26). This completes the proof of Lemma 5.

4. Proof of theorem

In this section, we present the proofs of our theorem. Asymptotic approximation to the bias of $\widehat{\psi}_n(t)$ can be computed as follows:

First, we compute the expected value of $\widehat{\psi}_n(t)$ as follows:

$$\begin{aligned} E\left[\widehat{\psi}_n(t)\right] &= E\left[E\left[\widehat{\psi}_n(t) | N([0,n])\right]\right] \\ &= \sum_{m=1}^{\infty} E\left[\widehat{\psi}_n(t) | N([0,n]) = m\right] P(N([0,n]) = m) \end{aligned}$$

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$$= \sum_{m=1}^{\infty} E\left(k_{t,\tau}\tau\widehat{\theta}_n + \widehat{\Lambda}_{c,n}\left(t_r\right) + \hat{a}_n\frac{t^2}{2}\right)$$
$$\times E\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right) P\left(N\left([0,n]\right) = m\right)$$
$$= \left(k_{t,\tau}\tau E\left(\widehat{\theta}_n\right) + E\left(\widehat{\Lambda}_{c,n}\left(t_r\right)\right) + E\left(\hat{a}_n\right)\frac{t^2}{2}\right)$$
$$\times \mu \sum_{m=1}^{\infty} P\left(N\left([0,n]\right) = m\right). \tag{28}$$

By substituting (4) of Lemma 1, (6) of Lemma 2 and (18) of Lemma 3 into the r.h.s. of (28), and after some algebras, we obtain that

$$E\left[\widehat{\psi}_n(t)\right] = \left(\psi(t) + \left(\frac{k_{t,\tau}\tau\left(2\theta\gamma - a\tau\gamma\right) + 2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)}{2\ln\left(k_{n,\tau}\right)}\right.\right.$$
$$\left.\left. + \ o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right)\right)\mu\right)\left(1 - e^{-\Lambda(n)}\right). \tag{29}$$

A simple calculation shows that

$$\Lambda\left(n\right) = E\left[N\left(0,n\right)\right] = \theta n + \frac{an^2}{2} + O\left(1\right), \tag{30}$$

as $n \to \infty$. By substituting (30) into the r.h.s. of (29) and after some simplification, we obtain

$$E\left[\widehat{\psi}_n(t)\right] = \psi(t) + \left(\frac{k_{t,\tau}\tau\left(2\theta\gamma - a\tau\gamma\right) + 2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)}{2\ln\left(k_{n,\tau}\right)}\right)\mu$$
$$+ o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right), \tag{31}$$

as $n \to \infty$. By (31), we obtain (2). Asymptotic approximation to the variance of $\widehat{\psi}_n(t)$ can be computed as follows:

First we compute $E\left[\left(\widehat{\psi}_n(t)\right)^2\right]$ as follows:

$$E\left[\left(\widehat{\psi}_n(t)\right)^2\right] = E\left[E\left[\left(\widehat{\psi}_n(t)\right)^2 | N\left([0,n]\right)\right]\right]$$
$$= \sum_{m=1}^{\infty} E\left[\left(k_{t,\tau}\tau\widehat{\theta}_n + \widehat{\Lambda}_{c,n}\left(t_r\right) + \hat{a}_n\frac{t^2}{2}\right)^2\right]$$
$$\times E\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)^2 P\left(N\left([0,n]\right) = m\right)$$
$$= \left(k_{t,\tau}\tau E\left(\left(\widehat{\theta}_n\right)^2\right) + E\left(\left(\widehat{\Lambda}_{c,n}\left(t_r\right)\right)^2\right) + \frac{t^4}{4}E\left(\left(\hat{a}_n\right)^2\right)\right.$$
$$+ 2k_{t,\tau}\tau E\left(\widehat{\theta}_n\widehat{\Lambda}_{c,n}\left(t_r\right)\right) + k_{t,\tau}\tau t^2 E\left(\widehat{\theta}_n\hat{a}_n\right)$$

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$$+\ t^2E\left(\widehat{\Lambda}_{c,n}\left(t_r\right)\hat{a}_n\right)\Big)$$

$$\times\sum_{m=1}^{\infty}E\left(\frac{1}{m}\sum_{i=1}^{m}X_i\right)^2P\left(N\left([0,n]\right)=m\right). \tag{32}$$

Since $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean μ and variance σ^2, a simple calculation shows that

$$E\left(\frac{1}{m}\sum_{i=1}^{m}X_i\right)^2=\mu^2+\frac{\sigma^2}{m}. \tag{33}$$

Now note that, by Lemma 2 we have

$$E\left(\left(\widehat{\Lambda}_{c,n}\left(t_r\right)\right)^2\right)=\left(\Lambda_c\left(t_r\right)\right)^2$$
$$+\frac{a\tau t_r+2\gamma\left(\Lambda_c\left(t_r\right)\right)^2+a\gamma\Lambda_c\left(t_r\right)\left(t_r^2-2\tau t_r\right)}{\ln\left(k_{n,\tau}\right)}+o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right) \tag{34}$$

and by Lemma 3 we have

$$E\left(\left(\widehat{\theta}_n\right)^2\right)=\theta^2+\frac{a+2\theta^2\gamma-a\tau\theta\gamma}{\ln\left(k_{n,\tau}\right)}+o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right) \tag{35}$$

as $n\to\infty$. By substituting (23), (33), (34), (35), Lemma 4 and Lemma 5 into the r.h.s. of (32), after some simplification, we have

$$E\left[\left(\widehat{\psi}_n(t)\right)^2\right]=\left(\left(\left(\left(k_{t,\tau}\tau\theta+\Lambda_c\left(t_r\right)+a\frac{t^2}{2}\right)\mu\right)^2\right.\right.$$
$$+\frac{\mu^2}{\ln\left(k_{n,\tau}\right)}\left(\left(k_{t,\tau}\tau\right)^2\left(a+2\theta^2\gamma-a\tau\theta\gamma\right)\right.$$
$$+\left(a\tau t_r+2\gamma\left(\Lambda_c\left(t_r\right)\right)^2+a\gamma\Lambda_c\left(t_r\right)\left(t_r^2-2\tau t_r\right)\right)$$
$$+2k_{t,\tau}\tau\left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma-a\tau\gamma\right)+\theta\left(2\gamma\Lambda_c\left(t_r\right)+a\gamma\left(t_r^2-2\tau t_r\right)\right)+2at_r}{2}\right)$$
$$\left.\left.\left.+\ k_{t,\tau}\tau t^2\left(\frac{2a\theta\gamma-a^2\tau\gamma}{2}\right)+t^2\left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma+a^2\gamma\left(t_r^2-2\tau t_r\right)\right)}{2}\right)\right)\right)\right)$$
$$\times\left(\sum_{m=\ 1}^{\infty}P\left(N\left([0,n]\right)=\ m\right)\right)+\left(\left(k_{t,\tau}\tau\theta+\Lambda_c\left(t_r\right)+a\frac{t^2}{2}\right)^2\right.$$
$$+\frac{1}{\ln\left(k_{n,\tau}\right)}\left(\left(k_{t,\tau}\tau\right)^2\left(a+2\theta^2\gamma-a\tau\theta\gamma\right)\right.$$
$$+\left(a\tau t_r+2\gamma\left(\Lambda_c\left(t_r\right)\right)^2+a\gamma\Lambda_c\left(t_r\right)\left(t_r^2-2\tau t_r\right)\right)$$
$$+2k_{t,\tau}\tau\left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma-a\tau\gamma\right)+\theta\left(2\gamma\Lambda_c\left(t_r\right)+a\gamma\left(t_r^2-2\tau t_r\right)\right)+2at_r}{2}\right)$$

$$+ \ k_{t,\tau}\tau t^2 \left(\frac{2a\theta\gamma - a^2\tau\gamma}{2}\right) + t^2 \left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma + a^2\gamma\left(t_r^2 - 2\tau t_r\right)\right)}{2}\right)\Bigg)\Bigg)$$

$$\times \frac{\sigma^2}{m}\left(\sum_{m=1}^{\infty}\frac{1}{m}P\left(N\left([0,n]\right) = m\right)\right) + o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right)$$

$$\times \left(\sum_{m=1}^{\infty} P\left(N\left([0,n]\right) = m\right) + \sum_{m=1}^{\infty}\frac{1}{m}P\left(N\left([0,n]\right) = m\right)\right). \tag{36}$$

The first term on the r.h.s. of (36) is equal to

$$(\psi(t))^2 + \frac{\mu^2}{\ln\left(k_{n,\tau}\right)}\Big(\left(k_{t,\tau}\tau\right)^2\left(a + 2\theta^2\gamma - a\tau\theta\gamma\right)$$

$$+ \left(a\tau t_r + 2\gamma\left(\Lambda_c\left(t_r\right)\right)^2 + a\gamma\Lambda_c\left(t_r\right)\left(t_r^2 - 2\tau t_r\right)\right)$$

$$+ 2k_{t,\tau}\tau\left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma - a\tau\gamma\right) + \theta\left(2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)\right) + 2at_r}{2}\right)$$

$$+ \ k_{t,\tau}\tau t^2\left(\frac{2a\theta\gamma - a^2\tau\gamma}{2}\right) + t^2\left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma + a^2\gamma\left(t_r^2 - 2\tau t_r\right)\right)}{2}\right)\Bigg)$$

$$+ O\left(e^{-n}\right) \tag{37}$$

as $n \to \infty$, while its second term can be simplified as

$$\Bigg((\psi(t))^2 + \frac{1}{\ln\left(k_{n,\tau}\right)}\Big(\left(k_{t,\tau}\tau\right)^2\left(a + 2\theta^2\gamma - a\tau\theta\gamma\right)$$

$$+ \left(a\tau t_r + 2\gamma\left(\Lambda_c\left(t_r\right)\right)^2 + a\gamma\Lambda_c\left(t_r\right)\left(t_r^2 - 2\tau t_r\right)\right)$$

$$+ 2k_{t,\tau}\tau\left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma - a\tau\gamma\right) + \theta\left(2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)\right) + 2at_r}{2}\right)$$

$$+ \ k_{t,\tau}\tau t^2\left(\frac{2a\theta\gamma - a^2\tau\gamma}{2}\right) + t^2\left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma + a^2\gamma\left(t_r^2 - 2\tau t_r\right)\right)}{2}\right)\Bigg)\Bigg)$$

$$\times \sigma^2\left(\frac{1}{\theta n + \frac{an^2}{2}} + O\left(\frac{1}{n^2}\right)\right)$$

$$+ o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right)\left(1 + O\left(e^{-n}\right) + \frac{1}{\theta n + \frac{an^2}{2}} + O\left(\frac{1}{n^2}\right)\right)$$

$$= o\left(\frac{1}{ln\left(n/\tau\right)}\right), \tag{38}$$

as $n \to \infty$. By substituting (37) and (38) into the r.h.s. of (36), then we have

$$E\left[\left(\widehat{\psi}_n(t)\right)^2\right] = (\psi(t))^2 + \frac{\mu^2}{\ln\left(k_{n,\tau}\right)}\Big(\left(k_{t,\tau}\tau\right)^2\left(a + 2\theta^2\gamma - a\tau\theta\gamma\right)$$

$$+ \left(a\tau t_r + 2\gamma\left(\Lambda_c\left(t_r\right)\right)^2 + a\gamma\Lambda_c\left(t_r\right)\left(t_r^2 - 2\tau t_r\right)\right)$$

$$+ 2k_{t,\tau}\tau \left(\frac{\Lambda_c\left(t_r\right)\left(2\theta\gamma - a\tau\gamma\right) + \theta\left(2\gamma\Lambda_c\left(t_r\right) + a\gamma\left(t_r^2 - 2\tau t_r\right)\right) + 2at_r}{2} \right)$$
$$+ \, k_{t,\tau}\tau t^2 \left(\frac{2a\theta\gamma - a^2\tau\gamma}{2}\right) + t^2 \left(\frac{\left(2a\Lambda_c\left(t_r\right)\gamma + a^2\gamma\left(t_r^2 - 2\tau t_r\right)\right)}{2} \right)\Bigg)$$
$$+ \, o\left(\frac{1}{\ln\left(k_{n,\tau}\right)}\right), \tag{39}$$

as $n \to \infty$. By (31) and (39) we obtain (3). This completes the Theorem.

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