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NOTE

Almost Periodic Passive Tracer Dispersion[1]

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The authors investigate the impact of external sources on the pattern formation of concentration profiles of passive tracers in a two-dimensional shear flow. By using the pullback attractor technique for the associated nonautonomous dynamical system, it is shown that a unique temporally, almost periodic concentration profile exists for the temporally almost periodic external source. © 2000 Academic Press

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1. INTRODUCTION

For the benefit of a better environment, it is important to understand the evolution of passive tracers such as pollutants, temperature, and salinity, in geophysical systems. Tracers are called passive when they do not dynamically affect the background fluid velocity field.

The Eulerian approach to studying passive tracer dispersion attempts to understand the evolution of the tracer concentration profile as a continuous field quantity [4, 17].

We consider two-dimensional passive tracer dispersion in a (bounded) shear flow $(u(y), 0)$ such as in a river or in an oceanic jet. The passive tracer concentration profile $C(x, y, t)$ then satisfies the advection–diffusion equation [4]

$$C_t + u(y)C_x = \kappa\left(C_{xx} + C_{yy}\right) + f(x, y, t), \tag{1}$$

where $\kappa > 0$ is the diffusivity constant and the source (or sink) term $f(x, y, t)$ accounts for effects of chemical reactions [4], external injections of pollutants, or heating and cooling [17, 13]. The source is generally dependent on time or even random in time, such as a random discharge of pollutants into a river or an oceanic jet.

There has been considerable research on the advection–diffusion equation *without source*; see, for example, [4, 15, 16, 19, 20].

In this paper, we study the impact of the external sources on the pattern formation of the concentration profile. We assume that the concentration profile satisfies the double-periodic boundary conditions,

$$C, C_x, C_y \text{ are double-periodic in } x \text{ and } y \text{ with period } 1 \tag{2}$$

and the appropriate initial condition

$$C(x, y, 0) = C_0(x, y). \tag{3}$$

We use the standard abbreviations $\dot{L}^2_{\text{per}} = \{u \in L^2(D), u \text{ is } D\text{-periodic and } \int_D u = 0\}$; $\dot{H}^1_{\text{per}} = \dot{H}^1_{\text{per}}(D) = \{u \in H^1(D), u, \nabla u \in \dot{L}^2_{\text{per}}\}$, with $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denoting the usual scalar product and norm, respectively, in L^2. We need the following properties and estimates:

The Poincaré Inequality [10].

$$\|g\|^2 = \int_D g^2(x, y)\, dx\, dy \le \frac{|D|}{\pi} \int_D |\nabla g|^2\, dx\, dy = \frac{|D|}{\pi} \|\nabla g\|^2 \tag{4}$$

for $g \in \dot{H}^1_{\text{per}}$.

Young's Inequality [10].

$$AB \le \frac{\epsilon}{2}A^2 + \frac{1}{2\epsilon}B^2, \tag{5}$$

where A, B are non-negative real numbers and $\epsilon > 0$.

In this paper, we prove the following main result.

THEOREM 1. *Assume that $C_0(x, y) \in \dot{L}^2_{\text{per}}$ and that $f(x, y, t)$ is temporally almost periodic with its $L^2(D)$-norm bounded uniformly in time $t \in \mathbb{R}$. Then the model for passive tracer dispersion* (1)–(3) *has a unique temporally almost periodic solution that exists for all time $t \in \mathbb{R}$.*

2. DISSIPATION AND CONTRACTION

In this section we consider the dissipation and contraction properties of the advection–diffusion equation with a temporally almost periodic source (1). These properties are crucial in the proof of Theorem 1 in the next section.

By integrating both sides of (1) with respect to x, y on the domain $D = [0, 1] \times [0, 1]$, we get

$$\frac{d}{dt}\iint C\,dx\,dy + \iint u(y)C_x\,dx\,dy$$
$$= \kappa\iint (C_{xx} + C_{yy})\,dx\,dy + \iint f(x, y, t)\,dx\,dy. \tag{6}$$

Note that

$$\iint u(y)C_x\,dx\,dy = 0$$

and

$$\iint (C_{xx} + C_{yy})\,dx\,dy = 0$$

due to the double-periodic boundary conditions (2). We thus have

$$\frac{d}{dt}\iint C\,dx\,dy = \iint f(x, y, t)\,dx\,dy. \tag{7}$$

Here and hereafter, all integrals are with respect to x, y over D. Thus, when there is no source, the spatial average or mean of the concentration

$C(x, y, t)$ does not change with time. When there is a source, the time-evolution of the spatial average of $C(x, y, t)$ is determined only by the source term. In order to understand the more delicate impact of the source on the evolution of $C(x, y, t)$ it is appropriate to assume that the source has a zero spatial average or mean:

$$\iint f(x, y, t)\, dx\, dy = 0. \tag{8}$$

With such a source, the mean of $C(x, y, t)$ is a constant. Without loss of generality or after removing the non-zero constant by a translation, we may assume that $C(x, y, t)$ has zero mean. So we study the dynamical behavior of $C(x, y, t)$ in zero-mean spaces.

Note that the linear operator $-\kappa(\partial_{xx} + \partial_{yy}) + u(y)\partial_x$ is sectorial [12, p. 19] in $\dot{L}^2_{\mathrm{per}}(D)$. Thus if $f(x, y, t)$ has a continuous derivative in time t the linear system (1)–(3) has a unique strong solution for every $C_0(x, y)$ in $\dot{L}^2_{\mathrm{per}}(D)$ [12, p. 52].

Define the solution operator S_{t, t_0}: $L^2 \to L^2$ by $S_{t, t_0}\omega_0 := \omega(t)$ for $t \geq t_0$, where $\omega(t)$ is the solution of the QG equations in L^2 starting at $\omega_0 \in L^2$ at time t_0. Since the dissipative system (1)–(3) is strictly parabolic, the solution operators S_{t, t_0} exist and are compact for all $t > t_0$; see, for example, [12]. In fact, the S_{t, t_0} are compact in H^k_0 for all $k \geq 0$ and so, in particular, $S_{t, t_0}B$ is a compact subset of L^2 for each $t > t_0$ and every closed and bounded subset B of L^2.

We now show that this system is a dissipative system in the sense [11] that all solutions $C(x, y, t)$ approach a bounded set in $\dot{L}^2_{\mathrm{per}}(D)$ as time goes to infinity. Multiplying (1) by $C(x, y, t)$ and integrating over D, we get

$$\frac{1}{2}\frac{d}{dt}\|C\|^2 + \iint u(y) C_x C\, dx\, dy = -\kappa \iint |\nabla C|^2\, dx\, dy + \iint f(x, y, t) C\, dx\, dy. \tag{9}$$

Note that, using the double-periodic boundary conditions (2),

$$\iint u(y) C_x C\, dx\, dy = 0. \tag{10}$$

We further assume that the square-integral of $f(x, y, t)$ with respect to x, y is bounded in time, i.e., $\|f\| \leq M$ ($M > 0$ is a constant independent of

t). Then, by the Young inequality,

$$\iint f(x,y,t)C\,dx\,dy \le \frac{1}{2\epsilon}\iint |f(x,y,t)|^2\,dx\,dy + \frac{\epsilon}{2}\iint |C|^2\,dx\,dy$$
$$\le \frac{M}{2\epsilon} + \frac{\epsilon}{2}\iint |C|^2\,dx\,dy, \tag{11}$$

where $\epsilon > 0$ is an arbitrary positive number.

Since C has a zero mean, we can use the Poincaré inequality [10, p. 164] to obtain

$$\|C\|^2 \le \frac{2}{\pi}\|\nabla C\|^2. \tag{12}$$

Putting (10), (11), and (12) into (9), we obtain

$$\frac{1}{2}\frac{d}{dt}\|C\|^2 \le \left(\frac{\epsilon}{2} - \frac{\kappa\pi}{2}\right)\|C\|^2 + \frac{M}{2\epsilon} \tag{13}$$

or

$$\frac{d}{dt}\|C\|^2 \le (\epsilon - \pi\kappa)\|C\|^2 + \frac{M}{\epsilon}. \tag{14}$$

We now fix $\epsilon > 0$ so small that $\epsilon - \kappa/\pi < 0$. By the Gronwall inequality [12] we finally get

$$\|C\|^2 \le \left(\|C_0\|^2 + \frac{M}{\epsilon(\epsilon - \kappa/\pi)}\right)e^{(\epsilon-\kappa/\pi)t} + \frac{M}{\epsilon(\kappa/\pi - \epsilon)}. \tag{15}$$

Hence all of the solutions $C(x,y,t)$ enter a bounded set in $\dot{L}^2_{\mathrm{per}}$,

$$\mathscr{B} = \left\{C\colon \|C\| \le \sqrt{\frac{M}{\epsilon(\kappa/\pi - \epsilon)}}\right\},$$

as time goes to infinity. The system (1) is therefore a dissipative system.

We now consider the strong contraction property. Assume that $C^{(i)}$ are two trajectories corresponding to initial values $C_0^{(i)} \in \mathscr{B}$, $i = 1$ and 2. Note that these trajectories remain inside $\mathscr{B}$. Their difference $\delta C = C^{(1)} - C^{(2)}$ satisfies the equation

$$\delta C_t + u(y)\,\delta C_x = \kappa(\delta C_{xx} + \delta C_{yy}).$$

Similarly to the proof above it can be shown from this equation that

$$\frac{1}{2}\frac{d}{dt}\|\delta C\|^2 + \int_D u(y)\,\delta C_x\,\delta C\,dx\,dy = -\kappa\|\nabla\delta C\|^2. \tag{16}$$

By (4) and $\int_D u(y)\delta C_x\,\delta C\,dx\,dy = 0$, Eq. (16) can be written as

$$\frac{1}{2}\frac{d}{dt}\|\delta C\|^2 + \kappa\pi\|\delta C\|^2 \le 0.$$

This gives

$$\|\delta C\|^2 \le \|\delta C_0\|^2 e^{-2\kappa\pi t} \to 0, \qquad \text{as } t \to \infty.$$

This is the strong contraction condition.

3. ALMOST PERIODIC DYNAMICS

A function $\varphi: \mathbb{R} \to X$, where (X, d_X) is a metric space, is called *almost periodic* [1] if for every $\varepsilon > 0$ there exists a relatively dense subset M_ε of $\mathbb{R}$ such that

$$d_X(\varphi(t+\tau), \varphi(t)) < \varepsilon$$

for all $t \in \mathbb{R}$ and $\tau \in M_\varepsilon$. A subset $M \subseteq \mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists a positive number $l \in \mathbb{R}$ such that for every $a \in \mathbb{R}$ the interval $[a, a+l] \cap \mathbb{R}$ of length l contains an element of M, i.e., $M \cap [a, a+l] \neq \emptyset$ for every $a \in \mathbb{R}$.

In order to study the temporally almost periodic solutions for the passive tracer convection–diffusion equation (1), we need some results from the theory of nonautonomous dynamical systems. Consider first an autonomous dynamical system on a metric space P described by a group $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ of mappings of P into itself.

Let X be a complete metric space and consider a continuous mapping

$$\Phi: \mathbb{R}^+ \times P \times X \to X$$

satisfying the properties

$$\Phi(0, p, \cdot) = \mathrm{id}_X, \qquad \Phi(\tau + t, p, x) = \Phi(\tau, \theta_t p, \Phi(t, p, x))$$

for all $t, \tau \in \mathbb{R}^+$, $p \in P$, and $x \in X$. The mapping Φ is called a cocycle on X with respect to θ on P.

The appropriate concept of an attractor for a nonautonomous cocycle system is the *pullback attractor*. In contrast to autonomous attractors it consists of a family of subsets of the original state space X that are indexed by the cocycle parameter set.

A family $\hat{A} = \{A_p\}_{p \in P}$ of nonempty compact sets of X is called a pullback attractor of the cocycle Φ on X with respect to θ_t on P if it is Φ-invariant, i.e.,

$$\Phi(t, p, A_p) = A_{\theta_t p} \qquad \text{for all } t \in \mathbb{R}^+, p \in P,$$

and pullback attracting, i.e.,

$$\lim_{t \to \infty} H_X^*\big(\Phi(t, \theta_{-t} p, D), A_p\big) = 0 \qquad \text{for all } D \in K(X), p \in P,$$

where $K(X)$ is the space of all nonempty compact subsets of the metric space (X, d_X) that are semi-metric between nonempty compact subsets of X, i.e., $H_X^*(A, B) := \max_{a \in A} \operatorname{dist}(a, B) = \max_{a \in A} \min_{b \in B} d_X(A, b)$ for $A, B \in K(X)$.

The following theorem combines several known results. See Crauel and Flandoli [6], Flandoli and Schmalfuß [9], and Cheban [2], as well as [3, 14], for this and various related proofs.

THEOREM 2. *Let Φ be a continuous cocycle on a metric space X with respect to a group θ of continuous mappings on a metric space P. In addition, suppose that there is a nonempty compact subset B of X and that for every D there exists a $T(D) \in \mathbb{R}^+$ which is independent of $p \in P$ such that*

$$\Phi(t, p, D) \subset B \qquad \textit{for all } t > T(D). \tag{17}$$

Then there exists a unique pullback attractor $\hat{A} = \{A_p\}_{p \in P}$ of the cocycle Φ on X, where

$$A_p = \bigcap_{\tau \in \mathbb{R}^+} \overline{\bigcup_{\substack{t > \tau \\ t \in \mathbb{R}^+}} \Phi(t, \theta_{-t} p, B)}. \tag{18}$$

Moreover, if the cocycle Φ is strongly contracting inside the absorbing set B then the pullback attractor consists of a singleton valued component, i.e., $A_p = \{a^*(p)\}$, and the mapping $p \mapsto a^*(p)$ is continuous.

The solution operators S_{t, t_0} for (1) form a cocycle mapping on $X =$ parameter set $P = \mathbb{R}$, where $p = t_0$, the initial time, and $\theta_t t_0 = t_0 + t$, the left shift by time t. However, the space $P = \mathbb{R}$ is not compact here. Although more complicated, it is more useful to consider P to be the closure of the subset $\{\theta_t f, t \in \mathbb{R}\}$, i.e., the hull of f, in the metric space $L^2_{\text{loc}}(\mathbb{R}, \dot{L}^2_{\text{per}}(D))$ of the locally $L^2(\mathbb{R})$-functions $f\colon \mathbb{R} \to \dot{L}^2_{\text{per}}(D)$ with the metric

$$d_P(f,g) := \sum_{N=1}^{\infty} 2^{-N}\min\left\{1, \sqrt{\int_{-N}^{N} \|f(t) - g(t)\|^2\, dt}\right\},$$

with θ_t defined to be the left shift operator, i.e., $\theta_t f(\cdot) := f(\cdot + t)$. By a classical result [1, 18], a function f in the above metric space is almost periodic if and only if the hull of f is compact and minimal. Here minimal means nonempty, closed, and invariant with respect to the autonomous dynamical system generated by the shift operators θ_t such that no proper subset has these properties. The cocycle mapping is defined to be the solution $C(t)$ of (1) starting at C_0 at time $t_0 = 0$ for a given forcing

$$\Phi(t, f, \omega_0) := S^f_{t,0}\,\omega_0,$$

where we have included a superscript f on S to denote its dependence on the forcing term f. (This dependence is in fact continuous.) The cocycle property here follows $S^f_{t,t_0}\omega_0 = S^{\theta_{t_0} f}_{t-t_0,0}\,\omega_0$ for all $t \geq t_0$, $t_0 \in \mathbb{R}$, $C_0 \in \dot{L}^2_{\text{per}}$, and $f \in P$.

Following Theorem 2 and the dissipativity and contractivity results which we have obtained in the last section, we conclude that the passive tracer convection–diffusion model (1)–(3) has the unique pullback attractor that consists of the singleton-valued component $\{a^*(p)\}$, and that the mapping $p \mapsto a^*(p)$ is continuous on P. As in Duan and Kloeden [7], we now show that this singleton attractor $a^*(p)$ defines an almost periodic solution.

In fact, the mapping $p \mapsto a^*(p)$ is uniformly continuous on P because P is a compact subset of $L^2_{\text{loc}}(\mathbb{R}, L^2(D))$ due to the assumed almost periodicity. That is, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|a^*(p) - a^*(q)\| < \varepsilon$ whenever $d_P(p,q) < \delta$. Now let the point $\bar{p}$ ($= f$, the given temporal forcing function) be almost periodic and for $\delta = \delta(\varepsilon) > 0$ denote by M_δ the relatively dense subset of $\mathbb{R}$ such that $d_P(\theta_{t+\tau}\bar{p}, \theta_t\bar{p}) < \delta$ for all $\tau \in M_\delta$ and $t \in \mathbb{R}$. From this and the uniform continuity we have

$$\|a^*(\theta_{t+\tau}\bar{p}) - a^*(\theta_t\bar{p})\| < \varepsilon$$

for all $t \in \mathbb{R}$ and $\tau \in M_{\delta(\varepsilon)}$. Hence $t \mapsto C^*(t) := a^*(\theta_t\bar{p})$ is almost periodic, and it is a solution of the passive tracer convection–diffusion model. It is unique as the single-trajectory pullback attractor is the only trajectory that exists and is bounded for the entire time line. Therefore, the conclusion in Theorem 1 follows.

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