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The max-flow min-cut property of two-dimensional affine convex geometries

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Abstract

In a matroid, (X, e) is a rooted circuit if X is a set not containing element e and $X \cup \{e\}$ is a circuit. We call X a broken circuit of e. A broken circuit clutter is the collection of broken circuits of a fixed element. Seymour [The matroids with the max-flow min-cut property, J. Combinatorial Theory B 23 (1977) 189–222] proved that a broken circuit clutter of a binary matroid has the max-flow min-cut property if and only if it does not contain a minor isomorphic to Q_6. We shall present an analogue of this result in affine convex geometries. Precisely, we shall show that a broken circuit clutter of an element e in a convex geometry arising from two-dimensional point configuration has the max-flow min-cut property if and only if the configuration has no subset forming a 'Pentagon' configuration with center e.

Firstly we introduce the notion of closed set systems. This leads to a common generalization of rooted circuits both of matroids and convex geometries (antimatroids). We further study some properties of affine convex geometries and their broken circuit clutters.

Keywords: Max-flow min-cut property; Packing; Broken-circuit clutter

1. Introduction

A convex geometry is a combinatorial abstraction of convex sets of ordinary affine space $\mathbb{R}^n$. A closed set system is a collection of sets of a finite set E such that it contains the entire set E and is closed under intersection. A closed set system naturally gives rise to a closure operator on 2^E. And if the closure operator satisfies the anti-exchange property, the closed set system is called a convex geometry. A set belonging to a convex geometry is called a convex set. The complement of a convex set is a feasible set, and the collection of the feasible sets is called an antimatroid. An antimatroid is usually defined through 'shelling' on combinatorial objects. Since antimatroids and convex geometries are simply the complement of each other and completely equivalent mathematical objects, we refer to this object as a 'convex geometry' throughout this paper, although the words of antimatroid and convex geometry have a rather different flavor and we need both approaches when considering a problem on convex geometries.

In a lattice theoretic terminology, a convex geometry is equivalent to a meet-distributive lattice, and an antimatroid is equivalent to a join-distributive lattice (which is also called a locally free lattice). Monjardet [19] described the history of the study of meet-distributive lattices, the origin of which can be traced back to Dilworth's result in 1940 [6].

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Edelman [7] established the equivalence between the class of meet-distributive lattices and the anti-exchange property of the corresponding closure operators. An antimatroid, in the meanwhile, was studied under the name of a shelling structure or an alternative precedence structure by Korte and Lovász [15]. Antimatroids as well as matroids are special cases of greedoids. For the theory of greedoids, we refer to [2,4,16].

A lot of examples of convex geometries arise from various combinatorial structures such as posets, graphs, point configurations, and so on. For a finite set in an affine space, an ordinary convex hull determines a convex geometry on it. Another typical example of convex geometry is the collection of monophonically convex sets of a chordal graph [10]. The shelling corresponding to this convex geometry is known as 'simplicial shelling'. The family of ideals of a poset is a simple example of convex geometry. It is also known that the collection of all the sub-semilattices of a semilattice forms a meet-distributive lattice [18], and that the lattice consisting of all the partial orders on a finite set is meet-distributive [9,12]. In addition, an acyclic oriented matroid gives rise to a convex geometry [3,8].

In matroid theory, quite a few 'nice' properties can be characterized by a rather small set of 'forbidden minors', while in the theory of convex geometry these types of nice characterizations by 'forbidden substructures' seem to be unknown until the works by the second author [20,22], in which the class of node-search convex geometries on rooted graphs and digraphs are shown to be characterized by certain forbidden minors.

For a matroid on E, if $X \cup \{e\}$ is a circuit and $e \notin X \subseteq E$, we shall call (X, e) a *rooted circuit* with *root* e, and X a *broken circuit* of e. The collection of all the broken circuits of e forms a clutter, which is called a *broken circuit clutter* and denoted by $\mathbb{C}(e)$. Seymour [24] proved that in a binary matroid, a broken circuit clutter $\mathbb{C}(e)$ has the max-flow min-cut property if and only if $\mathbb{C}(e)$ contains no minor isomorphic to Q_6.

In this paper, introducing the notion of 'closed set systems', we present a common generalization of rooted circuits of matroids and convex geometries (antimatroids). In light of this point of view, it seems worth trying to investigate and seek for some analogue of Seymour's result in convex geometry. And we have established that a broken circuit clutter $\mathbb{C}(e)$ of an affine convex geometry in two-dimensional space $\mathbb{R}^2$ has the max-flow min-cut property if and only if there is no set of five points that constitutes a 'Pentagon' configuration with center e.

In Section 2 the terms of set systems and packing theory needed in this paper are described. In Section 3 we shall introduce the notions of closed set systems, and the associated rooted circuits and cocircuits. In Section 4 the axiom sets for convex geometries and antimatroids are given. And in the succeeding sections we shall study affine convex geometries. In particular, we shall fully characterize affine convex geometries with kernel settled in two-dimensional space $\mathbb{R}^2$, and their broken circuit clutters. Also we shall establish a 'forbidden-minor' condition for a broken circuit of two-dimensional affine convex geometries to possess the max-flow min-cut property.

2. Set systems and packing of clutters

We shall describe the notions and the definitions that will be used in this paper.

Let E be a finite nonempty set. A *set system* is a pair $(\mathbb{S}, E)$ such that $\mathbb{S}$ is a family of subsets of E. There are three ways of defining a 'subfamily' of $\mathbb{S}$: For $A \subseteq E$, we can define subfamilies $\mathbb{S}/A$, $\mathbb{S}\backslash A$ and $\mathbb{S} - A$ by

$$\mathbb{S}/A = \{X\backslash A : X \in \mathbb{S}, A \subseteq X\}, \tag{2.1}$$

$$\mathbb{S}\backslash A = \{X : X \in \mathbb{S}, X \cap A = \emptyset\}, \tag{2.2}$$

$$\mathbb{S} - A = \{X\backslash A : X \in \mathbb{S}\}, \tag{2.3}$$

respectively. We say that $(\mathbb{S}/A, E\backslash A)$ is a *contraction-minor*, $(\mathbb{S}\backslash A, E\backslash A)$ is a *deletion-minor*, and $(\mathbb{S} - A, E\backslash A)$ is a *reduction-minor*. We also use a notation $\mathbb{S}|A = \mathbb{S}\backslash(E\backslash A)$, called a *restriction*, and $\mathbb{S} : A$ is defined as $\mathbb{S} - (E\backslash A)$, and called a *trace*. The collection of the complements of elements of $\mathbb{S}$ is denoted by $\mathbb{S}^C$.

A *clutter* is a family $\mathbb{L} \subseteq 2^E$ such that any element of $\mathbb{L}$ does not contain other elements as a proper subset.

The collection of minimal sets which intersect every member of $\mathbb{L}$ is called a *blocker* of $\mathbb{L}$, and denoted by $b(\mathbb{L})$. It is easy to check that $b(b(\mathbb{L})) = \mathbb{L}$ holds.

Besides the deletion and the contraction of set systems, those of clutters are defined as follows. For a subset $A \subseteq E$, $\mathbb{L}/A$ is the collection consisting of the minimal elements of $\{X\backslash A : X \in \mathbb{L}\}$, which is a *contraction-minor* of $\mathbb{L}$. And $\mathbb{L}\backslash A = \{X \in \mathbb{L} : X \cap A = \emptyset\}$ is a *deletion-minor* of $\mathbb{L}$. A family obtained by repeating contraction and deletion is called a *minor* of a clutter. The contraction and the deletion of clutters are dual in the blocking relation. That is, $b(\mathbb{L}/A) = \mathbb{L}\backslash A$ and $b(\mathbb{L}\backslash A) = \mathbb{L}/A$.

A subfamily $\{X_i : i = 1, \ldots, k\}$ of a clutter $\mathbb{L}$ is a *packing family* if X_i's are disjoint with each other. The *packing number* of $\mathbb{L}$ is the maximum size of the packing families. The minimum size of the elements of $b(\mathbb{L})$ is the *blocking number* of $\mathbb{L}$. Clearly, a packing number does not exceed the blocking number. A clutter $\mathbb{L}$ is said to *pack* if the packing number equals to the blocking number. An odd cycle $\mathbb{C}_{2k+1}=\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{2k}, a_{2k+1}\}, \{a_{2k+1}, a_1\}\}$ $(k \geqslant 1)$ is a typical example of clutters that do not pack.

Let M be a 0, 1-matrix whose columns are indexed by E and whose rows are the characteristic vectors of the elements of $\mathbb{L}$. We consider the following pair of dual linear programs:

$$\min\{wx : x \geqslant \mathbf{0}, Mx \geqslant \mathbf{1}\} \tag{2.4}$$

$$= \max\{y\mathbf{1} : y \geqslant \mathbf{0}, yM \leqslant w\}. \tag{2.5}$$

Here $\mathbf{1}$ denotes a column vector whose components are all equal to one. In terms of this linear program, $\mathbb{L}$ packs if and only if when $w = \mathbf{1}$, both (2.4) and (2.5) have optimal integral solutions x, y. Every deletion-minor of $\mathbb{L}$ packs if and only if (2.4) and (2.5) have optimal integral solutions for an arbitrary 0, 1-vector w. Every minor of $\mathbb{L}$ packs if and only if (2.4) and (2.5) have optimal integral solutions for an arbitrary $w \in \{0, 1, +\infty\}^n$. $\mathbb{L}$ is said to have the *max-flow min-cut property* if both (2.4) and (2.5) have optimal integral solutions x and y for an arbitrary nonnegative integral vector w.

We shall introduce the notion of amplification of set systems, and the notion of replication and duplication of clutters. Let E and E' be nonempty finite sets, and $\psi : E' \to E$ be a surjection.

For a set system $(\mathbb{S}, E)$, the set system $(\psi^*(\mathbb{S}), E')$ defined as below

$$\psi^*(\mathbb{S}) = \{X \subseteq E' : \psi(X) \in \mathbb{S}\}$$

is the *amplification* of $(\mathbb{S}, E)$ by ψ. Then

Lemma 2.1. $(\psi^*(\mathbb{S}))^C = \psi^*(\mathbb{S}^C)$.

For a clutter $(\mathbb{L}, E)$, the *replication* of $(\mathbb{L}, E)$ by ψ, denoted by $\psi^r(\mathbb{L})$, is the collection of all the minimal sets of $\psi^*(\mathbb{L})$. The *duplication* of $(\mathbb{L}, E)$ by ψ, denoted by $\psi^d(\mathbb{L})$, is the collection of all the maximal sets of $\psi^*(\mathbb{L})$.

Then:

Proposition 2.1 (*Cornúejols [5]*). *For a clutter* $(\mathbb{L}, E)$, *the following conditions are equivalent.*

(1) $(\mathbb{L}, E)$ *has the max-flow min-cut property.*
(2) *For every deletion-minor* $(\mathbb{L} \backslash A)$ *where* $A \subseteq E$, *the replication* $\psi^r(\mathbb{L} \backslash A)$ *arising from an arbitrary surjection* $\psi : E' \to E$ *packs.*

3. Closed set systems and rooted circuits

In this section, we introduce the notion of closed set systems and rooted circuits.

A set system $(\mathbb{K}, E)$ is a *closed set system* if

(1) $E \in \mathbb{K}$,
(2) if $X, Y \in \mathbb{K}$, then $X \cap Y \in \mathbb{K}$.

A set in $\mathbb{K}$ is a *closed set*, and the complement of a closed set is an *open set*.

A closed set system defines a closure operator σ on 2^E by

$$\sigma(A) = \cap\{X : X \in \mathbb{K}, A \subseteq X\}. \tag{3.1}$$

In other words $\sigma(A)$ is the smallest closed set containing A.

Let $\mathbb{O}$ be the collection of all the open sets. Then $(\mathbb{O}, E)$ is called an *open set system*. For each $Z \subseteq E$, $B = \bigcup_{Y \in \mathbb{O}, Y \subseteq Z} Y$ is the maximum open set in Z, called a *basis* of Z. For an arbitrary set $A \subseteq E$, let B be the basis of $E \backslash A$.

Then it is obvious that

$$\sigma(A) = \bigcap_{A\subseteq X, X\in\mathbb{K}} X = \bigcup_{Y\subseteq E\backslash A, Y\in\mathbb{O}} Y = E\backslash B. \tag{3.2}$$

An element x in A is an *extremal element* of A if $x \notin \sigma(A - x)$. Let $\mathrm{Ex}(A)$ denote the set of extremal elements of A. We say that A is *independent* if $A = \mathrm{Ex}(A)$.

Lemma 3.1. *If $Y = \mathrm{Ex}(Y)$ and $X \subseteq Y$, then $X = \mathrm{Ex}(X)$. In other words, any subset of an independent set is independent.*

Proof. By definition, for each $x \in X \subseteq Y$, we have $x \notin \sigma(Y - x)$. Since $\sigma(X - x) \subseteq \sigma(Y - x)$, $x \notin \sigma(X - x)$ follows and $x \in \mathrm{Ex}(X)$. Hence $X = \mathrm{Ex}(X)$. □

Hence the collection of all the independent sets forms a simplicial complex. A set is *dependent* if it is not independent, and we shall call a minimal dependent set a *circuit*.

Typical examples of closed set systems are the family of flats (closed sets) of a matroid and the family of convex sets of a convex geometry, the definition of which will be presented in the next section. The open set system of flats of a matroid coincides with the collection of unions of cocircuits, and the open set system of a convex geometry is an antimatroid.

Following the terminology of Klain [14] and Edelman and Reiner [11], we say that a set is *free* if it is both closed and independent. In Korte et al. [16], a set A is called *'free'* if $A = \mathrm{Ex}(E\backslash B)$ where B is a basis (maximum open set) in $E\backslash A$. Let us call a set *α-free* if it is 'free' in the sense of [16]. Then as will be shown later in Proposition 4.2, in a convex geometry (and an antimatroid), α-freeness is equivalent to the independency defined in this paper.

Let $(\mathbb{K}, E)$ be a closed set system, and σ be the associated closure operator. Let $(\mathbb{O}, E)$ be the open-set system that is the complement of $(\mathbb{K}, E)$. For a fixed element $e \in E$, which we call a *root*, let us denote by $\mathbb{C}(e)$ the collection of all the minimal sets $X \subseteq E\backslash e$ which satisfies $e \in \sigma(X)$, and call a set in $\mathbb{C}(e)$ a *broken circuit* of e. And a pair (X, e) of a broken circuit and a root is called a *rooted circuit*. Let us denote by $\mathbb{D}(e)$ the collection of all the minimal sets $Y \subseteq E\backslash\{e\}$ such that $Y \cup e$ is an open set in $\mathbb{O}$. For $Y \in \mathbb{D}(e)$, (Y, e) is a *rooted cocircuit*, and Y is a *cut-set* of e. $\mathbb{C}(e)$ and $\mathbb{D}(e)$ are called a *broken circuit clutter* and a *cut-set clutter*, respectively. (For more details, see [21].)

In case that $(\mathbb{K}, E)$ is the collection of flats of a matroid, it is easily observed that (X, e) is a rooted circuit if and only if $e \notin X$ and $X \cup e$ is a circuit in the ordinary sense of matroid theory. Similarly, $(Y, e) \in \mathbb{D}(e)$ if and only if $e \notin Y$ and $Y \cup e$ is a cocircuit of a matroid. (See, for instance, Oxley [23].)

Proposition 3.1. *Suppose that a closed set system is given. Then for $e \in E$ and $X \subseteq E\backslash e$,*

$$e \in \sigma(X) \quad \textit{if and only if} \quad A \in \mathbb{O}, e \in A \Rightarrow X \cap A \neq \emptyset. \tag{3.3}$$

In particular, $\mathbb{D}(e)$ and $\mathbb{C}(e)$ are the blockers of each other.

Proof. ($\Rightarrow$) Suppose that $A \in \mathbb{O}$, $e \in A$ and $X \cap A = \emptyset$. Since $X \subseteq E - A \in \mathbb{K}$, we have $\sigma(X) \subseteq E - A$, which leads to $e \notin \sigma(X)$, a contradiction.

($\Leftarrow$) Suppose contrarily e is not in $\sigma(X)$. Set $A = E - \sigma(X)$. Then we have $e \in A$ and $A \in \mathbb{O}$. Since $X \subseteq \sigma(X)$, this implies $X \cap A = \emptyset$, a contradiction. □

Let E and E' be nonempty finite sets, and $\psi : E' \to E$ be a surjection. Fix an element $e \in E$, and suppose $|\psi^{-1}(e)| = 1$ and $\psi(e') = \{e\}(e' \in E')$. Let $\mathbb{C}(e)$ and $\mathbb{D}(e)$ be the broken circuit clutter and the cut-set clutter of $(\mathbb{K}, E)$ with respect to root e, and let $\mathbb{C}^*(e')$ and $\mathbb{D}^*(e')$ be the broken circuit clutter and the cut-set clutter of $(\psi^*(\mathbb{K}), E')$ with respect to root e', respectively. Then:

Proposition 3.2. *$\mathbb{C}^*(e')$ is the replication of $\mathbb{C}(e)$ by ψ, and $\mathbb{D}^*(e')$ is the duplication of $\mathbb{D}(e)$ by ψ.*

Proof. Let σ and σ^* be the closure operator of $(\mathbb{K}, E)$ and $(\mathbb{K}^*, E') = (\psi^*(\mathbb{K}), E')$, respectively. Define

$$\mathbb{A} = \{X' \subseteq E'\backslash e' : e \in \sigma(\psi(X'))\}, \quad \mathbb{B} = \{X' \subseteq E'\backslash e' : e' \in \sigma^*(X')\},$$

and we shall show $\mathbb{A}=\mathbb{B}$. Since $\psi^r(\mathbb{C}(e))$ is the collection of minimal elements of $\mathbb{A}$ and $\mathbb{C}^*(e')$ is the collection of minimal elements of $\mathbb{B}$, the assertion of the proposition follows.

Let us first show $\mathbb{A}\subseteq\mathbb{B}$. Take a set X' in $\mathbb{A}$. By definition, $e\in\sigma(\psi(X'))$. We would like to show $e'\in\sigma^*(X')$ and hence $X'\in\mathbb{B}$. Suppose contrarily that $e'\notin\sigma^*(X')$. Then there exists $W'\in\mathbb{K}$ such that $X'\subseteq W'$ and $e'\notin W'$. Hence, we have $\psi(X')\subseteq\psi(W')$ and $e\notin\psi(W')\in\mathbb{K}$, which contradicts the assumption that $e\in\sigma(\psi(X'))$.

Next we shall show $\mathbb{B}\subseteq\mathbb{A}$. Suppose $X'\in\mathbb{B}$ and $e'\in\sigma^*(X')$. Take an arbitrary closed set $W\in\mathbb{K}$ with $\psi(X')\subseteq W$. When we set $W'=\psi^{-1}(W)$, W' is in $\mathbb{K}^*$. By assumption, $X'\subseteq W'$. The assumption that $e'\in\sigma^*(X')$ implies $e'\in W'$. Hence $e=\psi(e')\in\psi(W')=W$, and since W is arbitrarily chosen, this implies $e\in\sigma(\psi(X'))$. Hence $\mathbb{B}\subseteq\mathbb{A}$ is shown, and the proof is completed. □

4. Convex geometries and antimatroids

A *convex geometry* is a closed set system $(\mathbb{K}, E)$ satisfying (3) in addition to (1) and (2):

(1) $E\in\mathbb{K}$.
(2) If $X, Y\in\mathbb{K}$, then $X\cap Y\in\mathbb{K}$.
(3) For each closed set $X\in\mathbb{K}$ with $X\neq E$, there exists an element $x\in E\backslash X$ such that $X\cup x\in\mathbb{K}$ [accessibility to E].

A closed set of a convex geometry is called a *convex set*, and the complement of a convex set is called a *feasible set*. Let $\mathbb{F}$ be the collection of all the feasible sets. Then an open set system $(\mathbb{F}, E)$ is called an *antimatroid*.

There are several ways of formulating convex geometries.

Proposition 4.1 (*Ando [1], Edelman [7,9], Edelman and Jamison [10]*). *For a closed set system* $(\mathbb{K}, E)$, *the following conditions are equivalent*.

(1) $(\mathbb{K}, E)$ *is a convex geometry*.
(2) σ *satisfies the anti-exchange property*, *i.e. it holds that*

$$x, y\notin\sigma(A),\quad x\neq y,\quad x\in\sigma(A\cup y)\Rightarrow y\notin\sigma(A\cup x).$$

(3) *For each closed set* $X\in\mathbb{K}$, $X=\sigma(\mathrm{Ex}(X))$ [Minkowski–Klein–Milman property].
(4) *For each* $X\subseteq E$, $\mathrm{Ex}(\sigma(X))=\mathrm{Ex}(X)$ (Ando [1]).

For a convex geometry, each contraction-minor is again a convex geometry as well as each trace (i.e. reduction-minor) is a convex geometry. However, a deletion-minor is not necessarily a convex geometry. Although $\mathbb{K}|X=\mathbb{K}-(E-X)$ is a convex geometry provided that X is a convex set.

The axioms for convex geometries can be restated as a set of axioms for antimatroids as below.

(A1) $\emptyset\in\mathbb{F}$,
(A2) $X\in\mathbb{F}, X\neq\emptyset\Rightarrow$ there exists an element $e\in X$ such that $X\backslash e\in\mathbb{F}$,
(A3) $X, Y\in\mathbb{F}\Rightarrow X\cup Y\in\mathbb{F}$.

An apparently weaker but equivalent set of axioms for antimatroids is

(A1) $\emptyset\in\mathbb{F}$,
(A2) $X\in\mathbb{F}, X\neq\emptyset\Rightarrow$ there exists an element $e\in X$ such that $X\backslash e\in\mathbb{F}$,
(LF) For each $X\in\mathbb{F}$ and $x, y\in E$, if $X\cup x\in\mathbb{F}$ and $X\cup y\in\mathbb{F}$, then $X\cup\{x, y\}\in\mathbb{F}$ [locally free].

When we assume (A3), (LF) readily follows, while (A3) follows from (A1), (A2) and (LF). Hence these two sets of axioms for antimatroids are equivalent. The notion of alternative precedence structures (or shelling structures) introduced by Korte and Lovász [15] is equivalent to antimatroid.

Since a convex geometry is a closed set system, its rooted circuits and rooted cocircuits are correspondingly determined following the definition in Section 3. However, in the former literature [15,16], etc., the rooted circuits of convex geometries are introduced as 'minimal nonfree' sets. We describe the equivalence of their definitions and ours.

Proposition 4.2. *Let* $(\mathbb{K}, E)$ *be a convex geometry, and* $(\mathbb{F}, E)$ *be the associated complementary antimatroid. For a subset X of E, the following statements are equivalent.*

(1) *X is an independent set, i.e.* $X = \mathrm{Ex}(X)$.
(2) $X = \mathrm{Ex}(E\backslash B) = \{e \in X : B \cup \{e\} \in \mathbb{F}\}$ *where B is the maximum feasible set (basis) in* $E\backslash X$.
(3) $\mathbb{F} : X \equiv \{X \cap A : A \in \mathbb{F}\} = 2^X$.

Proof. We first prove (1) ⇔ (2). By (3.2), we have $E\backslash B = \sigma(X)$. Then it follows from (4) of Proposition 4.1, that

$$X = \mathrm{Ex}(X) \iff X = \mathrm{Ex}(\sigma(X)) \iff X = \mathrm{Ex}(E\backslash B)$$

Hence (1) and (2) are equivalent.

(2) ⇔ (3) is obvious since the lattice of open sets of a convex geometry is join-distributive, and satisfies (LF). □

Lemma 4.1. *Let C be a circuit of a convex geometry. Then there exists uniquely an element* $e \in C$ *such that* $C\backslash e = \mathrm{Ex}(C)$.

Proof. By definition, there exists a nonextremal element e in C, i.e. $e \in \sigma(C\backslash e)$. We show that this e is unique. Otherwise suppose there is another element x of C with $x \neq e$ and $x \in \sigma(C\backslash x)$. Let $A = C\backslash\{e, x\}$. By the definition of circuit, $A \cup \{e\}$ and $A \cup \{x\}$ are both independent. Hence $e, x \notin \sigma(A)$, while $e \in \sigma(C\backslash e) = \sigma(A \cup \{x\})$. By the anti-exchange property, we have $x \notin \sigma(A \cup \{e\}) = \sigma(C\backslash x)$, which is a contradiction. This completes the proof. □

Now we shall call a distinguished element e of Lemma 4.1 the *root* of a circuit C, and $C\backslash e$ a *broken circuit*. And $(C\backslash e, e)$ is a rooted circuit. In Section 3, we have already defined broken circuit clutters $\mathbb{C}(e)$ and rooted circuits for closed set systems generally. We shall show that the definitions here for convex geometries are consistent with the previous ones.

Proposition 4.3. *Let* $(\mathbb{K}, E)$ *be a convex geometry, and e be an element of E. Then for a subset* $X \subseteq E\backslash e$, *the following statements are equivalent.*

(1) $C = X \cup \{e\}$ *is a circuit, and e is its root, i.e.* $\mathrm{Ex}(C) = X$.
(2) $\mathbb{F} : C = 2^C - \{e\}$, *and* $X = C\backslash e$ *is minimal with respect to this property.*
(3) $X \in \mathbb{C}(e)$ *where the broken circuit clutter* $\mathbb{C}(e)$ *is defined with respect to the closed set system* $(\mathbb{K}, E)$, *that is, X is a minimal set in* $E\backslash e$ *satisfying* $e \in \sigma(X)$.

(Note: Proposition 4.1 in Chapter 3 of [15] is implicitly included in our Proposition 4.3.)

Proof. (1) ⇒ (2). Take a maximal open set B in $E\backslash C$. Take any $x \in C\backslash e$. Then since $\Gamma(B) = \mathrm{Ex}(C) = C\backslash e$, we have $\{x\} \in \mathbb{F} : C$. By assumption, $C\backslash x$ is an independent set. Hence $B \cup \{x, e\}$ is an open set in $\mathbb{F}$. These imply that $\mathbb{F} : C = 2^C - \{e\}$. And since any proper subset C' of C is independent, $\mathbb{F} : C' = 2^{C'}$ holds, which implies the minimality of X in (2).

(2) ⇒ (3). Suppose contrarily $e \notin \sigma(X)$. Then we have $Y = E\backslash\sigma(X) \in \mathbb{F}$ and $e \in Y$, which gives $Y \cap (X \cup e) = \{e\}$, a contradiction. The minimality of X in (3) readily follows from (2).

(3) ⇒ (1). We shall firstly show that for each $a \in X$ there exists $A \in \mathbb{F}$ such that $(X \cup e) \cap A = \{a\}$.

From the minimality of X, $e \notin \sigma(X - a)$ readily follows. Put $X' = X - a$.

Now suppose, contrarily, $a \in \sigma(X')$. Then we have $X = X' \cup a \subseteq \sigma(X')$. Hence $\sigma(X) = \sigma(X')$ and $e \in \sigma(X')$ follows, which contradicts the minimality of X. Hence we have $a \notin \sigma(X')$.

From assumption, we have $e \in \sigma(X) = \sigma(X' \cup a)$. By the above argument, we have $e, a \notin \sigma(X')$. Hence from the anti-exchange property, it follows that $a \notin \sigma(X' \cup e)$. Now set $A = E\backslash\sigma(X' \cup e)$. Then A is an open set in $\mathbb{F}$ and satisfies that $(X \cup e) \cap A = \{a\}$.

Next we shall show that there does not exist such $A' \in \mathbb{F}$ that $(X \cup e) \cap A' = \{e\}$. Suppose that such an open set $A' \in \mathbb{F}$ exists. We have $X \subseteq E \backslash A'$ and since $E \backslash A'$ is a convex set, $\sigma(X) \subseteq E \backslash A'$ follows. On the other hand, $e \notin E \backslash A'$ holds, which contradicts the assumption that $e \in \sigma(X)$.

Now it is shown that $\mathrm{Ex}(C) = \mathrm{Ex}(X) = X$ and that $C = X \cup e$ is a dependent set.

We shall further show that for each $x \in C = X \cup e$, $\mathrm{Ex}(C \backslash x) = C \backslash x$ and $C \backslash x$ is an independent set. This will show that C minimally dependent.

In case of $x = e$, we have already established $\mathrm{Ex}(X) = X$, which implies that $C \backslash e = X$ is an independent set.

In case of $x \in X$, set $X' = X \backslash x$, and we shall show X' is an independent set. By assumption, $e \in \sigma(X)$ and henceforth $\sigma(X \cup e) = \sigma(X)$, from which it follows that

$$\sigma(X') \subseteq \sigma(X' \cup e) \subseteq \sigma(X \cup e) = \sigma(X).$$

Setting $A = E - \sigma(X)$ and $A' = E - \sigma(X')$, we have $A, A' \in \mathbb{F}$. Now $A \subseteq A'$ is obvious. Here the locally free property of antimatroids and Proposition 4.1 (4) gives

$$\Gamma(A) \backslash A' \subseteq \Gamma(A') = \mathrm{Ex}(\sigma(X')) = \mathrm{Ex}(X').$$

At the same time, we have $\Gamma(A) = \mathrm{Ex}(\sigma(X)) = \mathrm{Ex}(X) = X$. Hence, $X' \subseteq X = \Gamma(A)$, while $X' \cap A' = \emptyset$. These induce $X' \subseteq \mathrm{Ex}(X')$. Since $\mathrm{Ex}(X') \subseteq X'$ holds by definition, we have $X' = \mathrm{Ex}(X')$. Hence X' is now shown to be an independent set.

The minimality of X in (1) readily follows that in (3). This completes the proof of (3) $\Rightarrow$ (1). $\square$

5. Affine convex geometries

An ordinary convex hull in an affine space gives rise to an associated convex geometry. Let E be a finite set in an n-dimensional affine space $\mathbb{R}^n$. Then

$$\sigma(X) = E \cap \mathrm{conv.hull}(X) \quad (\text{conv.hull is an ordinary convex hull in } \mathbb{R}^n). \tag{5.1}$$

is a closure operator on 2^E satisfying the anti-exchange property and hence gives rise to a convex geometry $(\mathbb{K}, E)$. We shall call it an *affine convex geometry* defined from a point configuration E.

Take a nonempty finite set $T \subseteq \mathbb{R}^n$. Then we can also define a closure operator σ_T by

$$\sigma_T(X) = E \cap \mathrm{conv.hull}(X \cup T), \tag{5.2}$$

which determines a convex geometry $(\mathbb{K}_T, E)$, called a *kernelled affine convex geometry*. The nonempty set T is called its *kernel*. That is, a kernelled affine convex geometry arises from a pair (E, T) of finite sets in $\mathbb{R}^n$ such that T is nonempty. And it is easy to verify that every affine convex geometry in $\mathbb{R}^n$ is isomorphic to some kernelled affine convex geometry in $\mathbb{R}^{n+1}$.

A reduction-minor $(\mathbb{K} - A, E \backslash A)$ is obviously equal to the affine convex geometry arising from the point configuration $E \backslash A$. And the class of affine convex geometries and that of kernelled affine convex geometries are both closed under taking reduction-minor. In contrast, for a nonempty subset A of E, a contraction-minor $(\mathbb{K}/A, E \backslash A)$ of an affine convex geometry of a point configuration E is equal to the kernelled affine convex geometry of the point configuration $E \backslash A$ with kernel A and is not necessarily an affine convex geometry. Hence, the class of affine convex geometries is not closed under the operation of taking minors. On the contrary, the class of kernelled affine convex geometries is obviously closed under taking minors since a contraction-minor $(\mathbb{K}_T/A, E \backslash A)$ of a kernelled affine convex geometry is a kernelled affine convex geometry on $E \backslash A$ with the kernel $T \cup A$.

The class of kernelled affine convex geometries is 'universal' in the sense that every convex geometry can be represented by a certain kernelled affine convex geometry. In fact, let $(\mathbb{K}, E)$ be an arbitrary convex geometry. Set $n = |E|$, and let Δ_{n-1} be an $(n-1)$-dimensional simplex with vertices $v_1, \ldots, v_n$ in $\mathbb{R}^{n-1}$. Take arbitrary bijection $\psi : E \to \{v_1, \ldots, v_n\}$. With each rooted circuit (X, e) of $\mathbb{K}$, we associate a point $r_{(X,e)}$ defined by

$$r_{(X,e)} = (|X| + 1)\psi(e) - \sum_{x \in X} \psi(x). \tag{5.3}$$

We use the set $T = \{r_{(X,e)} : (X, e) \text{ is a rooted circuit of } \mathbb{K}\}$ as a kernel.

Then:

Theorem 5.1 (*Kashiwabara et al. [13]*). *A convex geometry $\mathbb{K}$ on E is isomorphic to the kernelled affine convex geometry given by the pair $(\psi(E), T)$ in $\mathbb{R}^{n-1}$ where $n = |E|$.*

The maximum size of broken circuits of a convex geometry is known as Caratheodory number [16]. The Caratheodory number of a kernelled affine convex geometry is bounded by the dimension of the space of point configurations. That is:

Proposition 5.1. *Let (E, T) be a disjoint pair of finite sets in $\mathbb{R}^n$, and suppose that T is not empty. Then the size of broken circuits of the kernelled affine convex geometry derived from (E, T) is at most n.*

Proof. Take an arbitrary rooted circuit (X, e). If $e \in \text{conv.hull}(T)$, then $\mathbb{C}(e) = \{\emptyset\}$ and the assertion trivially follows. Hence we can assume $e \notin \text{conv.hull}(T)$. Now we suppose $e \in \text{conv.hull}(X \cup T)$, $X \subseteq E \backslash e$, and X is minimal with respect to this property. Let T' be a minimal set of T such that $e \in \text{conv.hull}(X \cup T')$. It is obvious that all the elements of T' are the vertices of $P = \text{conv.hull}(X \cup T')$. By the minimality of definition, each element of X is also a vertex of P, and the set of vertices of P coincides with the set $X \cup T'$.

First we consider the case $T' = \emptyset$. Since T is not empty, we can take an element $w \in T$. Let d be the dimension of the polytope of $\text{conv.hull}(X, w)$. Here we can construct a triangulation of $\text{conv.hull}(X, w)$ to d-simplexes each of which contains w as a vertex. (Actually, this can be done by "pulling [17, p. 272]" all the vertices in a sequence starting from w.) Hence, there is a subset X' of X such that $X' \cup w$ is a d-simplex and $e \in \text{conv.hull}(X' \cup w)$. We have $|X'| = d \leqslant n$ by the definition, while the minimality of circuits implies $X' = X$, Hence $|X| = |X'| \leqslant n$ follows.

In case that $T' \neq \emptyset$, take an element w' in T', which is automatically a vertex of P. As is similar to the first case, P can be triangulated to d-simplexes so that each d-simplex contains w'. Hence the same argument gives the assertion that $|X| \leqslant n$. $\square$

6. Broken circuit clutters of two-dimensional kernelled affine convex geometries

Let S be a finite nonempty set in $\mathbb{R}^n$, and v be a point in $\mathbb{R}^n$ such that $v \notin \text{conv.hull}(S)$. Suppose $C^+(v, S)$ and $C^-(v, S)$ to be a convex cone with apex v defined by

$$C^+(v, S) = \{(1-\lambda)v + \lambda a \mid \lambda \geqslant 0, a \in \text{conv.hull}(S)\}, \tag{6.1}$$

$$C^-(v, S) = \{(1-\lambda)v + \lambda a \mid \lambda \leqslant 0, a \in \text{conv.hull}(S)\}, \tag{6.2}$$

respectively.

Lemma 6.1. *Let (E, T) be a disjoint pair of finite sets in $\mathbb{R}^n$, and suppose that T is not empty. For a point $v \in E$ and a subset $X \subseteq E \backslash v$, the following statements are equivalent.*

(1) *(X, v) is a rooted circuit of a kernelled affine convex geometry defined from (E, T).*
(2) *The intersection of* $\text{conv.hull}(X)$ *and $C^-(v, T)$ is not empty, and X is minimal with respect to this property.*

Proof. It is obvious that $v \in \text{conv.hull}(X \cup T)$ if and only if $\text{conv.hull}(X) \cap C^-(v, T) \neq \emptyset$. The assertion directly follows from this and the definition of rooted circuits. $\square$

Lemma 6.2. *A clutter $E_{2+2} = \{\{v_1, v_2\}, \{v_3, v_4\}\}$ on four-element set $\{v_1, v_2, v_3, v_4\}$ cannot be realized as a broken circuit clutter of a kernelled affine convex geometry in two-dimensional space.*

Proof. Suppose, contrarily, that (E, T) is a pair of point configurations in $\mathbb{R}^2$ with $E = \{v, v_1, v_2, v_3, v_4\}$, and that the broken circuit clutter with root v is $\mathbb{C} = \{\{v_1, v_2\}, \{v_3, v_4\}\}$.

Deleting $C^+(v, T) \cup C^-(v, T)$ from $\mathbb{R}^2$, we have two disjoint open regions S_1 and S_2. Since $\{v_1, v_2\} \in \mathbb{C}$, either $C_1 \in S_1$, $v_2 \in S_2$ or $v_1 \in S_2$, $v_2 \in S_1$ holds. Here we suppose $v_1 \in S_1$ and $v_2 \in S_2$. We can apply the same argument to $\{v_3, v_4\} \in \mathbb{C}$, and so we may assume $v_1, v_3 \in S_1$ and $v_2, v_4 \in S_2$ without loss of generality.

Let L_1 be a line going through v and v_1. Then $\mathbb{R}^2 - L_1$ is a union of a pair of open half spaces. Let H be the one of these two which contains T. Then since $\{v_1, v_4\}$ is not a broken circuit of v, v_4 must belong to H. Similarly, let L_2 be the line passing through points v and v_2, and H' be the one of the two half spaces in $\mathbb{R}^2\backslash L_2$ which contains T. Since $\{v_2, v_3\}$ is not in $\mathbb{C}$, v_3 must belong to H'. Here note that $v \notin H, H'$.

Now $H \cap H'$ is a convex set and contains v_3, v_4, and T. Hence we have $v \notin \text{conv.hull}(\{v_3, v_4\} \cup T)$, but this contradicts our assumption $\{v_3, v_4\} \in \mathbb{C}$ and $v \in \sigma(\{v_3, v_4\}) = E \cap \text{conv.hull}(\{v_3, v_4\} \cup T)$. This completes the proof. $\square$

In a two-dimensional kernelled affine convex geometry, from Proposition 5.1, the size of broken circuits is at most two. Hence by considering each element of size 2 in $\mathbb{C}(e)$ as an edge, and a singleton as an isolated vertex, we can consider a graph $G(\mathbb{C}(e))$ on vertex set E. Then

Lemma 6.3. *$G(\mathbb{C}(e))$ is a bipartite graph.*

Proof. This is obvious from the proof of Lemma 6.2. $\square$

Corollary 6.1. *A broken circuit clutter of a two-dimensional kernelled affine convex geometry necessarily has the max-flow min-cut property.*

Proof. This is obvious from Lemma 6.3 due to the fact that 0, 1-matrix representing a broken circuit clutter here is a node-arc incidence matrix of a bipartite graph and hence is totally unimodular. $\square$

The following properties determine the same class of bipartite graphs.

Lemma 6.4. *Let G be a bipartite graph without isolated vertices, and $\mathbb{L}$ be the clutter consisting of edges of G. Then the following statements are equivalent.*

(1) *$\mathbb{L}$ does not contain a minor isomorphic to E_{2+2}.*
(2) *$\mathbb{L}$ does not contain a deletion-minor isomorphic to E_{2+2}.*
(3) *G does not contain an induced subgraph isomorphic to $K_2 \oplus K_2$ (a pair of nonadjacent edges).*
(4) *Let W_1, W_2 denote the partition of the vertex set of the bipartite graph G. Suppose $W_1 = \{u_1, \ldots, u_k\}$. Then after a suitable permutation of indices, $\{\text{Adj}(u_i) : i = 1, \ldots, k\}$ becomes a monotone nondecreasing sequence, i.e.*

$$\text{Adj}(u_1) \subseteq \text{Adj}(u_2) \subseteq \cdots \subseteq \text{Adj}(u_k), \tag{6.3}$$

where $\text{Adj}(u_i)$ denotes the set of vertices in W_2 incident to u_i.

Proof. (1) $\Rightarrow$ (2) is obvious.

Proof of (2) $\Rightarrow$ (1). Let $E = A \cup X \cup Y$ be a partition such that $|A| = 4$ and $\mathbb{L}^0 \equiv (\mathbb{L}/X)\backslash Y = E_{2+2}$.

Now suppose $\mathbb{L}\backslash(X \cup Y)$ contains an element e other than those in $\mathbb{L}^0 = \{e_1, e_2\}$. From the definition of E_{2+2}, e is never a singleton. Then $e \cap (X \cup Y) = \emptyset$ holds, and it implies $\mathbb{L}^0 = (\mathbb{L}/X)\backslash Y$ contains e, which contradicts the assumption that $\mathbb{L}^0$ is isomorphic to E_{2+2}. Hence, we have $\mathbb{L}^0 = (\mathbb{L}/X)\backslash Y = \mathbb{L}\backslash(X \cup Y)$.

It follows from this that every minor can be obtained as a deletion-minor.

(3) is a mere restatement of (2), and they are equivalent.

(3) $\Rightarrow$ (4). In order to show (4), it is sufficient to prove that for each $1 \leqslant i, j \leqslant k$, either $\text{Adj}(a_i) \subseteq \text{Adj}(a_j)$ or $\text{Adj}(a_j) \subseteq \text{Adj}(a_i)$ holds. Suppose contrarily that there is a pair i, j for which this assertion fails. Then there exist $b_1 \in \text{Adj}(a_i)\backslash\text{Adj}(a_j)$ and $b_2 \in \text{Adj}(a_j)\backslash\text{Adj}(a_i)$. The subgraph induced by $\{a_1, a_2, b_1, b_2\}$ is isomorphic to $K_2 \oplus K_2$, a contradiction.

(4) $\Rightarrow$ (3). Suppose contrarily that G contains $K_2 \oplus K_2$ as an induced subgraph. Let a_1b_1, a_2b_2 be the edges of the induced subgraph $K_2 \oplus K_2$ with $\{a_1, a_2\} \subseteq W_1$, $\{b_1, b_2\} \subseteq W_2$. Without loss of generality, we can assume that a_1 precedes a_2 in the monotone sequence of (4). Then

$$b_1 \in \text{Adj}(a_1) \subseteq \text{Adj}(a_2)$$

holds. But this implies G has an edge a_2b_1, a contradiction. $\square$

By Lemma 6.2, the condition of Lemma 6.4 is a necessary condition for a clutter to be realized as a broken circuit clutter of a kernelled two-dimensional affine convex geometry. From Lemma 6.3, a broken circuit clutter of a two-dimensional kernelled affine convex geometry never contains an odd cycles $C_{2k+1}(k \geqslant 1)$. We prove conversely that this set of conditions is sufficient.

Theorem 6.1. *Let $\mathbb{L}$ be a clutter on a finite set E. A necessary and sufficient condition for $\mathbb{L}$ to be realized as a broken circuit clutter $\mathbb{C}(e)$ of a kernelled affine convex geometry in two-dimensional space is that the size of elements of $\mathbb{L}$ is at most two, and $\mathbb{L}$ neither contains odd cycles $C_{2k+1}(k \geqslant 1)$ nor E_{2+2} as a minor of clutter.*

Proof. Necessity is already shown. We shall show the sufficiency. Suppose a clutter $\mathbb{L}$ is given which satisfies the conditions. We construct a point configuration in two-dimensional affine space such that the resultant broken circuit clutter is isomorphic to $\mathbb{L}$.

Let E_0 be the set of elements of E not contained in any member of $\mathbb{L}$, and E_1 be the set of elements which form singletons in $\mathbb{L}$. Set $E' = E - (E_0 \cup E_1)$. Then a graph G on the vertex set E' with the edge set $\mathbb{L}$ is a bipartite graph without isolated vertices such that the condition of Lemma 6.4 holds. Let (W_1, W_2) be the partition of the bipartite graph G. By assumption, we can assume that $W_1 = \{u_1, \ldots, u_k\}$ satisfies (6.3).

Let us define a one-to-one map $\varphi : E \to \mathbb{R}^2$. For the root e, we set $\varphi(e) = (0, (k+1)/2)$. We let φ map the elements of W_1 into the line $l_1 = \{(-1, \theta) : \theta \geqslant 0\}$, and those of W_2 into the line $l_2 = \{(1, \theta) : \theta \geqslant 0\}$.

For each element $u_i \in W_1$, we assign $\varphi(u_i) \in \mathbb{R}^2$ by

$$u_i \mapsto \varphi(u_i) = (-1, i) \in \mathbb{R}^2.$$

Let us divide l_2 to the intervals

$$L_1 = \{(1, \theta) \in \mathbb{R}^2 | k \leqslant \theta\},$$
$$L_i = \{(1, \theta) \in \mathbb{R}^2 | k - (i-1) \leqslant \theta < k - (i-2)\} \quad (2 \leqslant i \leqslant k).$$

Then for each element v of W_2, from the monotonicity of (6.3), there uniquely exists an index j such that $\mathrm{Adj}(v) = \{u_j, u_{j+1}, \ldots, u_k\}$. Take a point p arbitrarily on the segment L_j, and set $\varphi(v) = p \in \mathbb{R}^2$ so that no two elements of W_2 correspond to the same point in $\mathbb{R}^2$. (That is clearly always possible. See Fig. 1.)

For each singleton $\{x\} \in \mathbb{L}$, $\varphi(x)$ is set to be a point arbitrarily in $\{(0, t) : (k+1)/2 < t\}$. For elements in E_0, their images are taken arbitrarily in

$\{(0, t) : t < (k+1)/2\}$.

Take $T = \{(-1, 0), (1, 0)\}$ as a kernel.

It is easy to see that the broken circuit clutter with respect to the root $\varphi(e) = (0, (k+1)/2)$ of the kernelled affine convex geometry derived from $(\varphi(E), T)$ in $\mathbb{R}^2$ is isomorphic to $\mathbb{L}$. □

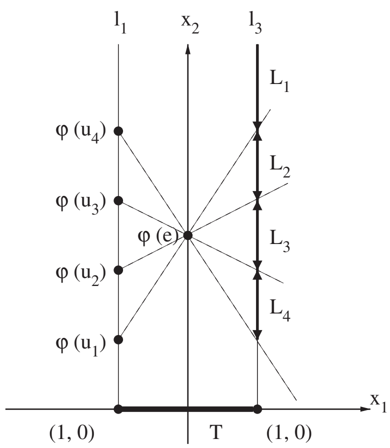

Fig. 1. Proof of Theorem 6.1.

7. Max-flow min-cut property of broken circuit clutters of two-dimensional affine convex geometries

Let E be a finite set in two-dimensional space $\mathbb{R}^2$. Fix an element $e \in E$, and set $E' = E \backslash e$, and let $(\mathbb{K}, E)$ be an affine convex geometry derived from $(E, \mathbb{R}^2)$. $\mathbb{C}(e)$ and $\mathbb{D}(e)$ are the broken circuit clutter and the cut-set clutter with root e, respectively.

Let S_e^1 be the unit circle in $\mathbb{R}^2$ with center e. We shall define a map $\varphi_e : E \to S_e^1$ as follows: For each $x \in E'$, the line starting from e and going through x intersects with S_e^1 uniquely at a point, which we set to be $\varphi_e(x)$. We define $\varphi_e(e) = e$.

Let $T = \varphi_e(E') \subseteq S_e^1$. Clearly, $\varphi_e : E \to T \cup \{e\}$ is a surjection, and $|\varphi_e^{-1}(e)| = 1$.

Let us denote by $(\mathbb{K}_T, T \cup \{e\})$ the convex geometry of point configuration $T \cup \{e\}$ in $\mathbb{R}^2$. Let $\mathbb{C}_T$ and $\mathbb{D}_T$ denote the broken circuit clutter and the cut-set clutter of e in $\mathbb{K}_T$, respectively.

Since $\varphi_e : E \to S_e^1$ is a surjection, it gives rise to an amplification of set system $(\mathbb{K}_T, T \cup \{e\})$, which is denoted by $(\varphi_e^*(\mathbb{K}_T), E)$. Although $(\varphi_e^*(\mathbb{K}_T), E)$ is in general not equal to $(\mathbb{K}, E)$, $\mathbb{C}(e)$ coincides with the replication of $\mathbb{C}_T$ by φ_e, and $\mathbb{D}(e)$ with the duplication of $\mathbb{D}_T$ by φ_e.

For a point t on the circle S_e^1, $\tilde{t}$ denotes its *antipodal point*, that is, $\tilde{t}$ is a point such that $\frac{1}{2}t + \frac{1}{2}\tilde{t} = e$.

For two points t_1, t_2 on S_e^1, $[t_1, t_2)$ is the arc of S_e^1 traced from t_1 to t_2 clockwise, including t_1, and not including t_2. And $(t_1, t_2]$ denotes an arc similarly defined not including t_1 but including t_2. $[t_1, t_2]$ and (t_1, t_2) are defined in the same manner.

For each point t on S_e^1, its multiplicity degree of φ_e is

$$\deg_e(t) = |\varphi_e^{-1}(t)|.$$

For a subset A of S_e^1, the total degree of multiplicity is

$$\mathrm{Deg}(A) = \sum_{t \in A} \deg_e(t).$$

An arc of the form of $[t, \tilde{t})$ or $(t, \tilde{t}]$ for a point $t \in S_e^1$ is called a *half circle*. For a half circle $C \subseteq S_e^1$, the region

$$H = \{x \in \mathbb{R}^2 \backslash \{e\} : \varphi_e(x) \in C\}$$

is called a *half space*. We set $\tilde{C} = \{\tilde{t} : t \in C\}$ and $\tilde{H} = \{x \in \mathbb{R}^2 \backslash \{e\} : \varphi_e(x) \in \tilde{C}\}$. Then H and $\tilde{H}$ constitute a partition of $\mathbb{R}^2 \backslash \{e\}$.

For a half space H, the set $H \cap E'$ is called a *residual set*, and $\mathrm{res}(H) = |H \cap E'|$ is called its *residual number*. In case that H is an extension of a half circle C, it is obvious that $\mathrm{res}(H) = \mathrm{Deg}(C) = \sum \deg_e(\alpha) : \alpha \in C$.

Then it is easily seen that:

Lemma 7.1. *The collection of minimal residual sets equals to $\mathbb{D}(e)$. In particular, the minimum of the residual numbers is equal to the blocking number of $\mathbb{C}(e)$.*

Proof. Let Z be a residual set. That is, there exists a half space H such that $Z = H \cap E'$. Take any $X \in \mathbb{C}(e)$. By definition $e \in \mathrm{conv.hull}(X)$ holds, and this implies $X \cap Z = X \cap (E' \cap H) = X \cap H \neq \emptyset$. Hence Z must contain a cut-set $Y \in \mathbb{D}(e)$.

On the opposite side, take an arbitrary $Y \in \mathbb{D}(e)$, and we shall show there exists a residual set Z with $Z \subseteq Y$. Suppose contrarily that there does not exist such a residual set. Then for an arbitrary half space H, $H \cap Y \neq \emptyset$ and $X' \cap (\mathbb{R}^2 \backslash (H \cup e)) \neq \emptyset$. This implies that there does not exist a separating hyperplane between $\mathrm{conv.hull}(X')$ and e. Equivalently, the point e is in $\mathrm{conv.hull}(X')$. By definition, there is a set X in $\mathbb{C}(e)$ such that $X \subseteq X'$. Hence $Y \cap X = \emptyset$, which contradicts the assumption that $Y \in \mathbb{C}(e)$. This completes the proof. $\square$

Five points $u_1, \ldots, u_5$ in $\mathbb{R}^2 \backslash \{e\}$ are said to form a *Pentagon configuration* with center e if

(1) $\varphi_e(u_1), \ldots, \varphi_e(u_5)$ are lying on S_e^1 (clockwise) in this order.
(2) For each $i = 1, \ldots, 5$,

$$\varphi_e(u_i) \in (\varphi_e(u_{i+2}), \varphi_e(u_{i+3})) \quad \text{(the indices are taken in mod 5).}$$

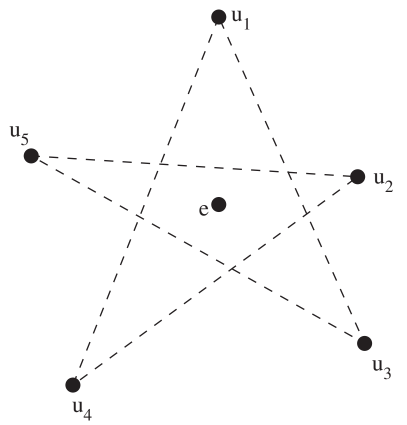

Fig. 2. Pentagon configuration with root e.

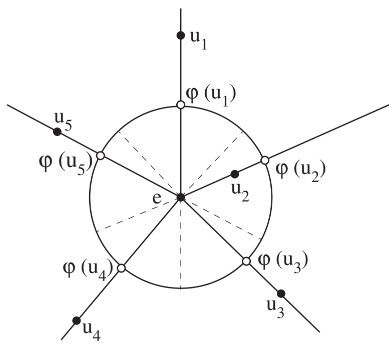

Fig. 3. A unit circle S_e^1 and $\varphi(u_i)$ $(i = 1, \ldots, 5)$.

A Pentagon configuration looks like as in Fig. 2. And Fig. 3 shows the projections $\varphi(u_i)$ of u_i on the unit circle S_e^1.

The broken circuit clutter of this point configuration $\{v_1, v_2, v_3, v_4, v_5, e\}$ with root e is

$$T_3^5 = \{\{v_1, v_3, v_4\}, \{v_2, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_5\}\}. \tag{7.1}$$

T_3^5 is just a 5×5 circulant 0, 1-matrix with three ones in each row. Clearly, T_3^5 is the blocker of an odd hole of length 5 and is minimally nonpacking.

The Pentagon configuration T_3^5 plays a crucial role in our argument.

Theorem 7.1. *Let E be a finite set in two-dimensional space $\mathbb{R}^2$, and $(\mathbb{K}, E)$ the associated affine convex geometry. Let e be an element of E, and $E' = E \backslash e$. $\mathbb{C}(e)$ is the broken circuit clutter of e. Then the following statements are equivalent.*

(1) *Every minor of $\mathbb{C}(e)$ packs.*
(2) *$\mathbb{C}(e)$ does not contain a minor isomorphic to T_3^5.*
(3) *$\mathbb{C}(e)$ does not contain a deletion-minor isomorphic to T_3^5.*
(4) *There does not exist a subset of E' composing a Pentagon configuration with center e.*
(5) *Every deletion-minor of $\mathbb{C}(e)$ packs.*

In advance to proceed to the proof of Theorem 7.1, we shall present some lemmata.

Let $(\mathbb{K}, E)$ be an affine convex geometry in $\mathbb{R}^d$. Set $\hat{E} = \{(v, 0) : v \in E\} \subseteq \mathbb{R}^{d+1}$, and $\hat{e} = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$. Taking $\{\hat{e}\}$ as a kernel, the kernelled affine convex geometry associated with configuration $(\hat{E}, \{\hat{e}\})$ is equal to $\mathbb{K}$. Hence if E is a set in $\mathbb{R}^2$, by Proposition 5.1, the sizes of broken circuits of $(\mathbb{K}, E)$ are at most three. We shall show that the broken circuits of size two can be neglected when considering packing of broken circuit clutters in this case.

Lemma 7.2. *Let $\mathbb{L}$ be a subfamily of $\mathbb{C}(e)$ of a two-dimensional affine convex geometry $(\mathbb{K}, E)$, and $X \in \mathbb{L}$ with $|X|=2$. Then $\mathbb{L}$ packs if and only if $\mathbb{L}' = \mathbb{L} - \{X\}$ packs.*

Proof. Let $X = \{x_1, x_2\}$, and $E' = E \backslash X$. Let b be the blocking number of $\mathbb{L}$, and b' that of $\mathbb{L}'$. For an arbitrary half space H, $|H \cap X| = 1$ holds, and hence

$$|H \cap E| = |H \cap E'| + 1. \tag{7.2}$$

So it follows from Lemma 7.1 that

$$b = b' + 1 \tag{7.3}$$

Here we shall show that $\mathbb{L}$ always has a maximum packing family including X. Suppose this is already shown. If there exists a maximum packing family of $\mathbb{L}$ including X, say $\{X, X_2, \ldots, X_p\}$, obviously $\{X_2, \ldots, X_p\}$ is a packing family of $\mathbb{L}'$. And if $\mathbb{L}$ packs, p is equal to b. Hence by (7.3), $\mathbb{L}'$ also packs. If $\mathbb{L}'$ packs, adding X to a packing family of $\mathbb{L}'$ obviously results in a packing family again, and (7.3) implies $\mathbb{L}$ also packs.

Now suppose $\{X_1, X_2, \ldots, X_p\}$ is a maximum packing family of $\mathbb{L}$ which does not contain X. Let $U = \cup X_i : i = 1, \ldots, p$. By assumption, $U \cap X \neq \emptyset$. Hence we have either $|U \cap X| = 1$ or $|U \cap X| = 2$.

In case of $|U \cap X| = 1$, we can suppose $X \cap X_1 = \{x_1\}$ without loss of generality. Then clearly $\{X, X_2, \ldots, X_p\}$ is a packing family of size p, it is proven that there exists a maximum packing family of $\mathbb{L}$ including X.

In case of $|U \cap X| = 2$, we can suppose $X = \{x_1, x_2\}$, $X \cap X_1 = \{x_1\}$, and $X \cap X_2 = \{x_2\}$ without loss of generality. Let $W = X_1 \cup X_2$. Since the sizes of broken circuits are at most two, we only have to consider the three cases below.

(1) If $|W| = 4$, then $|X_1| = |X_2| = 2$. Now suppose $X_1 = \{x_1, y_1\}$ and $X_2 = \{x_2, y_2\}$. Then from the definition, it follows that $Y = \{y_1, y_2\} \in \mathbb{C}(e)$. Hence, $\{X, Y, X_3, \ldots, X_p\}$ is a packing family of $\mathbb{L}$.
(2) If $|W| = 5$, then we may suppose $|X_1| = 2$ and $|X_2| = 3$. Suppose $X_1 = \{x_1, y_1\}$ and $X_2 = \{x_2, y_2, w_2\}$. Then $X = \{x_1, x_2\} \in \mathbb{L}$ as well as $X_1 = \{x_1, y_1\} \in \mathbb{L}$ implies that $\varphi_e(x_2)$ and $\varphi_e(y_1)$ are the antipodal point of $\varphi_e(x_1)$. Hence, $\varphi_e(x_2) = \varphi_e(y_1)$, while we have $Y = \{y_1, y_2, w_2\} \in \mathbb{C}(e)$. Thus $\{X, Y, X_3, \ldots, X_p\}$ is a packing family of $\mathbb{L}$.
(3) Suppose $|W| = 6$. Then since $\mathbb{L} \backslash (E' - W)$ contains X_1, X_2, the packing number and the blocking number of $\mathbb{L} \backslash (E' - W)$ are at least two. Hence the intersection of $W' = W \backslash \{x_1, x_2\}$ and each member of $\mathbb{C}(e)$ is nonempty. And so e is included in conv.hull(W'). Let $Y \subseteq W'$ be a minimal set satisfying $e \in$ conv.hull(Y). Then Y belongs to $\mathbb{L}$, and $\{X, Y, X_3, \ldots, X_p\}$ is a packing family of $\mathbb{L}$. This completes the proof. □

Corollary 7.1. *Let $\mathbb{C}'$ be a subfamily of $\mathbb{C}(e)$ obtained by deleting all the elements of size at most two. Then $\mathbb{C}(e)$ packs if and only if $\mathbb{C}'$ packs.*

The next lemma is the key of the proof of Theorem 7.1.

Lemma 7.3. *Suppose that $\mathbb{C}(e)$ do not pack, and that for every nonempty subset X of $E' = E \backslash e$, the deletion-minor $\mathbb{C}(e) \backslash X$ packs. Then there exists a Pentagon configuration in E' whose center is e.*

Proof. From Corollary 7.1, we may assume $\mathbb{C}(e)$ does not include a broken circuit of size one and two, and every broken circuit in $\mathbb{C}(e)$ is of size three. Let b be the blocking number of $\mathbb{C}(e)$, and b_T be that of $\mathbb{C}_T$.

The assumption that $\mathbb{C}(e)$ does not pack implies $b_T \geqslant 1$.

(a) Suppose $b_T = 1$. From Lemma 7.1, there exists a half circle A such that $A \cap T = \{t\}$ $(t \in S_e^1)$. By the assumption that $b_T = 1$, we have $\tilde{t} \notin T$.

Now without loss of generality we may suppose there is a point $c \in S^1_e$ such that $A = [c, \tilde{c})$ and $c, \tilde{c} \notin T$. Lemma 7.1 gives

$$\mathrm{Deg}(A) = \deg_e(t) \geqslant b.$$

Hence, there are distinct elements $x_1, \ldots, x_b$ in E' such that $\varphi_e(x_i) = t$ $(i = 1, \ldots, b)$. Similarly, it follows from Lemma 7.1 that

$$\mathrm{Deg}((t, \tilde{t}]) \geqslant b, \quad \mathrm{Deg}([\tilde{t}, t)) \geqslant b.$$

Here since $[c, \tilde{c}] \cap T = \{t\}$ holds, we have

$$\mathrm{Deg}((\tilde{c}, \tilde{t})) = \mathrm{Deg}((t, \tilde{t}]) \geqslant b, \quad \mathrm{Deg}((\tilde{t}, c)) = \mathrm{Deg}([\tilde{t}, t)) \geqslant b.$$

Hence, there are distinct b points in E', say $y_1, \ldots, y_b$, such that $\varphi_e(y_i) \in (\tilde{c}, \tilde{t})$ $(i = 1, \ldots, b)$. Similarly, there are distinct b points in E', say $w_1, \ldots, w_b$, such that $\varphi_e(w_i) \in (\tilde{t}, c)$ $(i = 1, \ldots, b)$.

For each $i = 1, \ldots, b$, it is obvious that $\{x_i, y_i, w_i\}$ is a broken circuit in $\mathbb{C}(e)$. Since they are pairwise disjoint, they form a packing family of size b. In particular, $\mathbb{C}(e)$ is now shown to pack, which is a contradiction.

(b) Hence we can suppose $b_T \geqslant 2$. From Lemma 7.1, there exists $c \in S^1_e$ with $c, \tilde{c} \notin T$ such that

$$\mathrm{Deg}([c, \tilde{c}]) = \mathrm{Deg}((c, \tilde{c})) = b.$$

Now set $(c, \tilde{c}) \cap T = \{t_1, \ldots, t_{b_T}\}$. And suppose that $t_1, \ldots, t_{b_T}$ are placed on S^1_e in this order (clockwise). Let $t = t_1$ and $s = t_{b_T}$. Since $\mathrm{Deg}([t, s]) = b$, there are distinct b points, say $x_1, \ldots, x_b \in E'$, such that $\varphi_e(x_i) \in [t, s]$ $(i = 1, \ldots, b)$.

Since $\mathbb{C}(e)$ is supposed to contain no broken circuit of size two, $\tilde{t}, \tilde{s} \notin T$ must hold. From Lemma 7.1, we have

$$\mathrm{Deg}([\tilde{t}, t)) \geqslant b, \quad \mathrm{Deg}((s, \tilde{s}]) \geqslant b.$$

We shall show that

$$[\tilde{t}, \tilde{s}] \cap T \neq \emptyset.$$

On the contrary, suppose $[\tilde{t}, \tilde{s}] \cap T = \emptyset$, i.e. suppose $\mathrm{Deg}([\tilde{t}, \tilde{s}]) = 0$. Now we have $\mathrm{Deg}((s, \tilde{s}]) \geqslant b$ and $\mathrm{Deg}((s, \tilde{c}]) = 0$, which shows $\mathrm{Deg}((\tilde{c}, \tilde{s}]) \geqslant b$. Furthermore, since $\mathrm{Deg}([\tilde{t}, \tilde{s}]) = 0$, we have $\mathrm{Deg}((\tilde{c}, \tilde{t})) \geqslant b$. Similarly, $\mathrm{Deg}((\tilde{s}, c)) \geqslant b$ follows from $\mathrm{Deg}([\tilde{t}, t)) \geqslant b$ and $\mathrm{Deg}([\tilde{t}, \tilde{s}]) = \mathrm{Deg}([c, t)) = 0$. That is, there exist distinct b points $y_1, \ldots, y_b \in E'$such that $\varphi_e(y_i) \in (\tilde{c}, \tilde{t})$, and there exist distinct b points $w_1, \ldots, w_b \in E'$ such that $\varphi_e(w_i) \in (\tilde{s}, c)$. By definition, for each $i = 1, \ldots, b$, $\{x_i, y_i, w_i\}$ is a broken circuit in $\mathbb{C}(e)$, and they form a packing family of size b. Hence, $\mathbb{C}(e)$ packs, which is a contradiction.

Next we shall show

$$\mathrm{Deg}((\tilde{c}, \tilde{t}]) \geqslant 1, \quad \mathrm{Deg}([\tilde{s}, c)) \geqslant 1.$$

Lemma 7.1 gives $\mathrm{Deg}((t, \tilde{t}]) \geqslant b$. First we shall show $\mathrm{Deg}((\tilde{c}, \tilde{t}))$=$\mathrm{Deg}((\tilde{c}, \tilde{t}]) \geqslant 1$. Suppose, contrarily, that $\mathrm{Deg}((\tilde{c}, \tilde{t}])=$ 0. Then,

$$\mathrm{Deg}((t, \tilde{t}]) = \mathrm{Deg}((t, \tilde{c}]) = \mathrm{Deg}((c, \tilde{c}]) - \deg(t).$$

Since $\mathrm{Deg}((c, \tilde{c}]) = b$ and $\deg(t) \geqslant 1$, we have $\mathrm{Deg}((t, \tilde{t}]) < b$, a contradiction. Hence, $\mathrm{Deg}((\tilde{c}, \tilde{t})) = \mathrm{Deg}((\tilde{c}, \tilde{t}]) \geqslant 1$ is shown. Similarly, it can be shown that $\mathrm{Deg}([\tilde{s}, c)) = \mathrm{Deg}((\tilde{s}, c)) \geqslant 1$.

Finally, we have

$$(\tilde{c}, \tilde{t}) \cap T \neq \emptyset, \quad (\tilde{t}, \tilde{s}) \cap T \neq \emptyset, \quad (\tilde{s}, c) \cap T \neq \emptyset.$$

This implies that there exist three elements $u_3, u_4, u_5 \in E'$ such that

$$\varphi_e(u_3) \in (\tilde{c}, \tilde{t}) \cap T, \quad \varphi_e(u_4) \in (\tilde{t}, \tilde{s}) \cap T, \quad \varphi_e(u_5) \in (\tilde{s}, c) \cap T.$$

By definition, there exist elements u_1, u_2 in E' such that $\varphi_e(u_1) = t$, $\varphi_e(u_2) = s$. And it is clear that $u_1, \ldots, u_5$ form a Pentagon configuration with center e. $\square$

Obviously, a Pentagon configuration with center e exists in E' if and only if T includes a Pentagon configuration with center e. Hence, Lemma 7.3 gives:

Corollary 7.2. *$\mathbb{C}(e)$ packs if and only if every replication of $\mathbb{C}(e)$ packs.*

Each proper contraction-minor of $\mathbb{C}(e)$ is a broken circuit clutter of a proper contraction of an affine convex geometry, while as is seen in Section 5, a proper contraction of an affine convex geometry is necessarily a kernelled affine convex geometry. Hence a proper contraction-minor of $\mathbb{C}(e)$ is a broken circuit clutter of a kernelled affine convex geometry in two-dimensional space, and by Corollary 6.1, it necessarily packs. Hence we have:

Lemma 7.4. *The following statements are equivalent.*

(1) *Every deletion-minor of $\mathbb{C}(e)$ packs.*
(2) *Every minor of $\mathbb{C}(e)$ packs.*

Now we shall present the proof of Theorem 7.1.

Proof of Theorem 7.1. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. (4) $\Rightarrow$ (5) follows from Lemma 7.3. (5) $\Rightarrow$ (1) is deduced from Lemma 7.4. □

Combining Proposition 2.1, Corollary 7.2, and Theorem 7.1, we have

Theorem 7.2. *For a two-dimensional affine convex geometry $(\mathbb{K}, E)$ and an element e in E, the broken circuit clutter $\mathbb{C}(e)$ has the max-flow min-cut property if and only if it does not contain a minor isomorphic to T_3^5.*

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