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Distributive congruence lattices of congruence-permutable algebras ☆

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Abstract

We prove that every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the normal subgroup lattice of some group and to the submodule lattice of some right module. The $\aleph_1$ bound is optimal, as we find a distributive algebraic lattice D with $\aleph_2$ compact elements that is not isomorphic to the congruence lattice of any algebra with almost permutable congruences (hence neither of any group nor of any module), thus solving negatively a problem of E.T. Schmidt from 1969. Furthermore, D may be taken as the congruence lattice of the free bounded lattice on $\aleph_2$ generators in any non-distributive lattice variety.

Some of our results are obtained *via* a functorial approach of the semilattice-valued 'distances' used by B. Jónsson in his proof of Whitman's Embedding Theorem. In particular, the semilattice of compact elements of D is not the range of any distance satisfying the V-condition of type $3/2$. On the other hand, every distributive $\langle \vee, 0\rangle$-semilattice is the range of a distance satisfying the V-condition of type 2. This can be done *via* a functorial construction.

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Introduction

Representing algebraic lattices as congruence lattices of algebras often gives rise to very hard open problems. The most well-known of those problems, the *Congruence Lattice Problem*, usually abbreviated CLP, asks whether every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice, see the survey paper [22]. This problem has been solved recently by the third author in [26]. For algebraic lattices that are not necessarily distributive, there are several deep results, one of the most remarkable, due to W.A. Lampe [13], stating that *every algebraic lattice with compact unit is isomorphic to the congruence lattice of some groupoid*. This result is further extended to join-complete, unit-preserving, compactness preserving maps between two algebraic lattices [14].

Although some of our methods are formally related to Lampe's, for example, the proof of Theorem 7.1 *via* Proposition 2.6, we shall be concerned only about *distributive* algebraic lattices. This topic contains some not so well-known but also unsolved problems, as, for example, whether every distributive algebraic lattice is isomorphic to the congruence lattice of an algebra in some *congruence-distributive variety*.

If one drops congruence-distributivity, then one would expect the problems to become easier. Consider, for example, the two following problems:

CGP. *Is every distributive algebraic lattice isomorphic to the normal subgroup lattice of some group*?

CMP. *Is every distributive algebraic lattice isomorphic to the submodule lattice of some module*?

The problem CGP was originally posed for *finite* distributive (semi)lattices by E.T. Schmidt as [19, Problem 5]. A positive solution was provided by H.L. Silcock, who proved in particular that every finite distributive lattice D is isomorphic to the normal subgroup lattice of some *finite* group G (see [20]). P.P. Pálfy proved later that G may be taken *finite solvable* (see [16]). However, the general question seemed open until now. Similarly, the statement of CMP has been communicated to the authors by Jan Trlifaj, and nothing seemed to be known about the general case.

A common feature of the varieties of all groups and of all modules over a given ring is that they are *congruence-permutable*, for example, any two congruences of a group are permutable. Thus both CGP and CMP are, in some sense, particular instances of the following question:

CPP. (See [19, Problem 3].) *Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebra with permuting congruences*?

Although the exact formulation of [19, Problem 3] asked whether every *Arguesian* algebraic lattice is isomorphic to the congruence lattice of an algebra with permutable congruences, it was mentioned there that even the *distributive* case was open. Meanwhile, the Arguesian case was solved negatively by M.D. Haiman [9,10], however, the distributive case remained open.

Recall that an algebra A has *almost permutable congruences* (see [21]), if $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{ab} \cup \boldsymbol{ba}$, for all congruences $\boldsymbol{a}, \boldsymbol{b} \in \operatorname{Con} A$ (where the notation $\boldsymbol{ab}$ stands for the usual composition of relations). The three-element chain is an easy example of a lattice with almost permutable congruences but not with permutable congruences. On the other hand, it is not difficult to verify that *every almost congruence-permutable variety of algebras is congruence-permutable*. The last two authors of the present paper obtained in [21] negative congruence representation results of distributive semilattices by *lattices* with almost permutable congruences, but nothing was said there about arbitrary *algebras* with permutable congruences. Furthermore, our attempts based on the "uniform refinement properties" introduced in that paper failed, as these properties turned out to be quite lattice-specific.

In the present paper, we introduce a general framework that makes it possible to extend the methods of [21] to arbitrary algebras, and thus solving CPP—and, in fact, its generalization to algebras with almost permutable congruences—negatively. Hence, both CGP and CMP also have negative solutions. In fact, the negative solution obtained in CGP for *groups* extends to *loops*, as the variety of all loops is also congruence-permutable. Another byproduct is that we also get a negative solution for the corresponding problem for *lattice-ordered groups*, see also Problem 1.

Our counterexample is the same as in [18] and in [21], namely the congruence lattice of a free lattice with at least $\aleph_2$ generators in any non-distributive variety of lattices. We also show that the size $\aleph_2$ is optimal, by showing that *every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the submodule lattice of some module, and also to the normal subgroup lattice of some locally finite group*, see Theorems 4.1 and 5.3. We also prove that *every distributive algebraic lattice with at most countably many compact elements is isomorphic to the ℓ-ideal lattice of some lattice-ordered group*, see Theorem 6.3.

In order to reach our negative results, the main ideas are the following.

(1) Forget about the algebraic structure, just keep the partition lattice representation.
(2) State a weaker "uniform refinement property" that settles the negative result.

For Point (1), we are looking for a very special sort of lattice homomorphism of a given lattice into some partition lattice, namely, the sort that is induced, as in Proposition 1.2, by a *semilattice-valued distance*, see Definition 1.1. For a $\langle\vee, 0\rangle$-semilattice S and a set X, an S-valued distance on X is a map $\delta : X \times X \to S$ satisfying the three usual statements characterizing distances (see Definition 1.1). Every such δ induces a map φ from S to the partition lattice of X (see Proposition 1.2), and if δ satisfies the so-called *V-condition*, then φ is a join-homomorphism. Furthermore, the V-condition of type n says that the equivalences in the range of φ are pairwise $(n+1)$-permutable. Those "distances" have been introduced by B. Jónsson for providing a simple proof of Whitman's Theorem that every lattice can be embedded into some partition lattice, see [11] or Theorems IV.4.4 and IV.4.8 in [6].

While it is difficult to find a suitable notion of morphism between partition lattices, it is easy to do such a thing with our distances, see Definition 1.1. This makes it possible to define what it means for a commutative diagram of $\langle\vee, 0\rangle$-semilattices to have a lifting, modulo the forgetful functor, by distances. In particular, we prove, in Theorem 7.2, that the cube $\mathcal{D}_{\mathrm{ac}}$ considered in [21, Section 7] does not have a lifting by any diagram of V-distances "of type 3/2," that is, the equivalences in the ranges of the corresponding partition lattice representations cannot all be almost permutable. This result had been obtained only for lattices in [21].

The original proof of Theorem 7.2 was our main inspiration for getting a weaker "uniform refinement property," that we denote here by $\mathrm{WURP}^{=}$ (see Definition 2.1). First, we prove that if $\delta : X \times X \to S$ is an S-valued V-distance of type $3/2$ with range generating S, then S satisfies $\mathrm{WURP}^=$ (see Theorem 2.3). Next, we prove that for any free lattice F with at least $\aleph_2$ generators in any non-distributive variety of lattices, the compact congruence semilattice $\mathrm{Con_c}\, F$ does not satisfy $\mathrm{WURP}^=$ (see Corollary 3.8). Therefore, $\mathrm{Con}\, F$ is not isomorphic to $\mathrm{Con}\, A$, for any algebra A with almost permutable congruences (see Corollary 3.7).

On the positive side, we explain why all previous attempts at finding similar negative results for representations of type 2 (and above) failed. We prove, in particular, that *for every distributive $\langle \vee, 0\rangle$-semilattice S, there exists a surjective V-distance $\delta_S : X_S \times X_S \twoheadrightarrow S$ of type 2, which, moreover, depends functorially on S* (see Theorem 7.1). In particular, the diagram $\mathcal{D}_{\bowtie}$ considered in [23], which is not liftable, with respect to the congruence lattice functor, in any variety whose congruence lattices satisfy a non-trivial identity, is nevertheless liftable by V-distances of type 2.

Basic concepts

For elements x and y in an algebra A, we denote by $\Theta_A(x, y)$, or $\Theta(x, y)$ if A is understood, the least congruence of A that identifies x and y. Furthermore, in case A is a lattice, we put $\Theta_A^+(x, y) = \Theta_A(x \wedge y, x)$. We denote by $\mathrm{Con}\, A$ (respectively, $\mathrm{Con_c}\, A$) the lattice (respectively, semilattice) of all compact (i.e., finitely generated) congruences of A.

For join-semilattices S and T, a join-homomorphism $\mu : S \to T$ is *weakly distributive* (see [24]), if for every $c \in S$ and $a, b \in T$, if $\mu(c) \leqslant a \vee b$, then there are $x, y \in S$ such that $c \leqslant x \vee y$, $\mu(x) \leqslant a$, and $\mu(y) \leqslant b$.

A *diagram* in a category $\mathcal{C}$ is a functor $\mathbf{D} : \mathcal{I} \to \mathcal{C}$, for some category $\mathcal{I}$. For a functor $\mathbf{F} : \mathcal{A} \to \mathcal{C}$, a *lifting* of $\mathbf{D}$ with respect to $\mathbf{F}$ is a functor $\Phi : \mathcal{I} \to \mathcal{A}$ such that the composition $\mathbf{F} \circ \Phi$ is naturally equivalent to $\mathbf{D}$.

For a set X and a natural number n, we denote by $[X]^n$ the set of all n-elements subsets of X, and we put $[X]^{<\omega} = \bigcup([X]^n \mid n < \omega)$. The following statement of infinite combinatorics can be found in C. Kuratowski [12].

The Kuratowski Free Set Theorem. *Let n be a positive integer and let X be a set. Then $|X| \geqslant \aleph_n$ iff for every map $\Phi : [X]^n \to [X]^{<\omega}$, there exists $U \in [X]^{n+1}$ such that $u \notin \Phi(U \setminus \{u\})$, for any $u \in U$.*

As in [18,24], only the case $n = 2$ will be used.

We identify every natural number n with the set $\{0, 1, \ldots, n-1\}$, and we denote by ω the set of all natural numbers.

1. V-distances of type *n*

Definition 1.1. Let S be a $\langle \vee, 0\rangle$-semilattice and let X be a set. A map $\delta : X \times X \to S$ is an *S-valued distance* on X, if the following statements hold:

(i) $\delta(x, x) = 0$, for all $x \in X$.
(ii) $\delta(x, y) = \delta(y, x)$, for all $x, y \in X$.
(iii) $\delta(x, z) \leqslant \delta(x, y) \vee \delta(y, z)$, for all $x, y, z \in X$.

The *kernel* of δ is defined as $\{\langle x, y\rangle \in X \times X \mid \delta(x, y) = 0\}$. The *V-condition* on δ is the following condition:

For all $x, y \in X$ and all $\boldsymbol{a}, \boldsymbol{b} \in S$ such that $\delta(x, y) \leqslant \boldsymbol{a} \vee \boldsymbol{b}$, there are $n \in \omega \setminus \{0\}$ and $z_0 = x, z_1, \dots, z_{n+1} = y$ such that for all $i \leqslant n$, $\delta(z_i, z_{i+1}) \leqslant \boldsymbol{a}$ in case i is even, while $\delta(z_i, z_{i+1}) \leqslant \boldsymbol{b}$ in case i is odd.

In case n is the same for all x, y, $\boldsymbol{a}$, $\boldsymbol{b}$, we say that the distance δ satisfies the *V-condition of type n*, or is a *V-distance of type n*.

We say that δ satisfies the *V-condition of type* $3/2$, or is a *V-distance of type* $3/2$, if for all $x, y \in X$ and all $\boldsymbol{a}, \boldsymbol{b} \in S$ such that $\delta(x, y) \leqslant \boldsymbol{a} \vee \boldsymbol{b}$, there exists $z \in X$ such that either $(\delta(x, z) \leqslant \boldsymbol{a}$ and $\delta(z, y) \leqslant \boldsymbol{b})$ or $(\delta(x, z) \leqslant \boldsymbol{b}$ and $\delta(z, y) \leqslant \boldsymbol{a})$.

We say that a *morphism* from $\lambda : X \times X \to A$ to $\mu : Y \times Y \to B$ is a pair $\langle f, \boldsymbol{f} \rangle$, where $\boldsymbol{f} : A \to B$ is a $\langle \vee, 0 \rangle$-homomorphism and $f : X \to Y$ is a map such that $\boldsymbol{f}(\lambda(x, y)) = \mu(f(x), f(y))$, for all $x, y \in X$. The *forgetful functor* sends $\lambda : X \times X \to A$ to A and $\langle f, \boldsymbol{f} \rangle$ to $\boldsymbol{f}$.

Denote by $\operatorname{Eq} X$ the lattice of all equivalence relations on a set X. For a positive integer n, we say as usual that $\alpha, \beta \in \operatorname{Eq} X$ are *$(n+1)$-permutable*, if $\gamma_0\gamma_1 \cdots \gamma_n = \gamma_1\gamma_2 \cdots \gamma_{n+1}$, where γ_k is defined as α if k is even and as β if k is odd, for every natural number k. In particular, 2-permutable is the same as permutable. With every distance is associated a homomorphism to some $\operatorname{Eq} X$, as follows.

Proposition 1.2. *Let S be a $\langle \vee, 0 \rangle$-semilattice and let $\delta : X \times X \to S$ be an S-valued distance. Then one can define a map $\varphi : S \to \operatorname{Eq} X$ by the rule*

$$\varphi(\boldsymbol{a}) = \bigl\{ \langle x, y \rangle \in X \times X \bigm| \delta(x, y) \leqslant \boldsymbol{a} \bigr\}, \quad \textit{for all } \boldsymbol{a} \in S.$$

Furthermore,

(i) *the map φ preserves all existing meets.*
(ii) *If δ satisfies the V-condition, then φ is a join-homomorphism.*
(iii) *If the range of δ join-generates S, then φ is an order-embedding.*
(iv) *If the distance δ satisfies the V-condition of type n, then all equivalences in the range of φ are pairwise $(n+1)$-permutable.*

Any algebra gives rise to a natural distance, namely the map $\langle x, y \rangle \mapsto \Theta(x, y)$ giving the principal congruences.

Proposition 1.3. *Let n be a positive integer and let A be an algebra with $(n+1)$-permutable congruences. Then the semilattice $\operatorname{Con_c} A$ of compact congruences of A is join-generated by the range of a V-distance of type n.*

Proof. Let $\delta : A \times A \to \operatorname{Con_c} A$ be defined by $\delta(x, y) = \Theta_A(x, y)$, the principal congruence generated by $\langle x, y \rangle$, for all $x, y \in A$. The assumption that A has $(n+1)$-permutable congruences means exactly that δ is a V-distance of type n. $\square$

Of course, A has almost permutable congruences if and only if the canonical distance $\Theta_A : A \times A \to \operatorname{Con_c} A$ satisfies the V-condition of type $3/2$.

We shall focus attention on three often encountered varieties all members of which have permutable (i.e., 2-permutable) congruences:

- The variety of all right modules over a given ring R. The congruence lattice of a right module M is canonically isomorphic to the submodule lattice $\operatorname{Sub} M$ of M. We shall denote by $\operatorname{Sub_c} M$ the $\langle\vee, 0\rangle$-semilattice of all finitely generated submodules of M.
- The variety of all groups. The congruence lattice of a group G is canonically isomorphic to the normal subgroup lattice $\operatorname{NSub} G$ of G. We shall denote by $\operatorname{NSub_c} G$ the $\langle\vee, 0\rangle$-semilattice of all finitely generated normal subgroups of G.
- The variety of all ℓ-groups (i.e., lattice-ordered groups), see [1]. The congruence lattice of an ℓ-group G is canonically isomorphic to the lattice $\operatorname{Id}^{\ell} G$ of all convex normal ℓ-subgroups, or *ℓ-ideals*, of G. We shall denote by $\operatorname{Id}_c^{\ell} G$ the $\langle\vee, 0\rangle$-semilattice of all finitely generated ℓ-ideals of G.

Hence we obtain immediately the following result.

Corollary 1.4.

(i) *Let M be a right module over any ring R. Then* $\operatorname{Sub_c} M$ *is join-generated by the range of a V-distance of type* 1 *on M.*
(ii) *Let G be a group. Then* $\operatorname{NSub_c} G$ *is join-generated by the range of a V-distance of type* 1 *on G.*
(iii) *Let G be an ℓ-group. Then* $\operatorname{Id}_c^{\ell} G$ *is join-generated by the range of a V-distance of type* 1 *on G.*

The V-distances corresponding to (i), (ii), and (iii) above are, respectively, given by $\delta(x, y) = (x - y)R$, $\delta(x, y) = [xy^{-1}]$ (the normal subgroup of G generated by xy^{-1}), and $\delta(x, y) = G(xy^{-1})$ (the ℓ-ideal of G generated by xy^{-1}).

The assignments $M \mapsto \operatorname{Sub_c} M$, $G \mapsto \operatorname{NSub_c} G$, and $G \mapsto \operatorname{Id}_c^{\ell} G$ can be canonically extended to direct limits preserving functors to the category of all $\langle\vee, 0\rangle$-semilattices with $\langle\vee, 0\rangle$-homomorphisms.

2. An even weaker uniform refinement property

The following infinitary axiom $\mathrm{WURP}^{=}$ is a weakening of all the various "uniform refinement properties" considered in [18,21,24]. Furthermore, the proof that follows, aimed at obtaining Theorem 3.6, is very similar to the proofs of [18, Theorem 3.3] and [21, Theorem 2.1].

Definition 2.1. Let $\boldsymbol{e}$ be an element in a $\langle\vee, 0\rangle$-semilattice S. We say that S satisfies $\mathrm{WURP}^{=}(\boldsymbol{e})$, if there exists a positive integer m such that for all families $\langle \boldsymbol{a}_i \mid i \in I\rangle$ and $\langle \boldsymbol{b}_i \mid i \in I\rangle$ of elements of S such that $\boldsymbol{e} \leqslant \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for all $i \in I$, there are a m-sequence $\langle I_u \mid u < m\rangle$ of subsets of I such that $\bigcup(I_u \mid u < m) = I$ and a family $\langle \boldsymbol{c}_{i,j} \mid \langle i, j\rangle \in I \times I\rangle$ of elements of S such that the following statements hold:

(i) $\boldsymbol{c}_{i,j} \leqslant \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{c}_{i,j} \leqslant \boldsymbol{b}_i \vee \boldsymbol{b}_j$, for all $u < m$ and all $i, j \in I_u$.
(ii) $\boldsymbol{e} \leqslant \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$, for all $u < m$ and all $i, j \in I_u$.
(iii) $\boldsymbol{c}_{i,k} \leqslant \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$, for all $i, j, k \in I$.

Say that S satisfies $\mathrm{WURP}^{=}$, if S satisfies $\mathrm{WURP}^{=}(\boldsymbol{e})$ for all $\boldsymbol{e} \in S$.

The following easy lemma is instrumental in the proof of Corollary 3.7.

Lemma 2.2. *Let S and T be $\langle\vee,0\rangle$-semilattices, let $\mu: S \to T$ be a weakly distributive $\langle\vee,0\rangle$-homomorphism, and let $\boldsymbol{e} \in S$. If S satisfies* $\mathrm{WURP}^{=}(\boldsymbol{e})$*, then T satisfies* $\mathrm{WURP}^{=}(\mu(\boldsymbol{e}))$.

Theorem 2.3. *Let S be a $\langle\vee,0\rangle$-semilattice and let $\delta: X \times X \to S$ be a V-distance of type* $3/2$ *with range join-generating S. Then S satisfies* $\mathrm{WURP}^{=}$.

Proof. Let $\boldsymbol{e} \in S$. As S is join-generated by the range of δ, there are a positive integer n and elements $x_\ell, y_\ell \in X$, for $\ell < n$, such that $\boldsymbol{e} = \bigvee(\delta(x_\ell, y_\ell) \mid \ell < n)$. For all $i \in I$ and all $\ell < n$, from $\delta(x_\ell, y_\ell) \leqslant \boldsymbol{a}_i \vee \boldsymbol{b}_i$ and the assumption on δ it follows that there exists $z_{i,\ell} \in X$ such that

$$\begin{aligned} &\text{either} \quad \delta(x_\ell, z_{i,\ell}) \leqslant \boldsymbol{a}_i \quad \text{and} \quad \delta(z_{i,\ell}, y_\ell) \leqslant \boldsymbol{b}_i, \\ &\text{or} \quad \delta(x_\ell, z_{i,\ell}) \leqslant \boldsymbol{b}_i \quad \text{and} \quad \delta(z_{i,\ell}, y_\ell) \leqslant \boldsymbol{a}_i. \end{aligned} \tag{2.1}$$

For all $i \in I$ and all $\ell < n$, denote by $P(i,\ell)$ and $Q(i,\ell)$ the following statements:

$$\begin{aligned} &P(i,\ell)\colon \quad \delta(x_\ell, z_{i,\ell}) \leqslant \boldsymbol{a}_i \quad \text{and} \quad \delta(z_{i,\ell}, y_\ell) \leqslant \boldsymbol{b}_i; \\ &Q(i,\ell)\colon \quad \delta(x_\ell, z_{i,\ell}) \leqslant \boldsymbol{b}_i \quad \text{and} \quad \delta(z_{i,\ell}, y_\ell) \leqslant \boldsymbol{a}_i. \end{aligned}$$

We shall prove that $m = 2^n$ is a suitable choice for witnessing $\mathrm{WURP}^{=}(\boldsymbol{e})$. So let U denote the powerset of n, and put

$$I_u = \bigl\{ i \in I \bigm| (\forall \ell \in u)\ P(i,\ell) \text{ and } (\forall \ell \in n \setminus u)\ Q(i,\ell) \bigr\}, \quad \text{for all } u \in U.$$

We claim that $I = \bigcup(I_u \mid u \in U)$. Indeed, let $i \in I$, and put $u = \{\ell < n \mid P(i,\ell)\}$. It follows from (2.1) that $Q(i,\ell)$ holds for all $\ell \in n \setminus u$, whence $i \in I_u$. Now we put

$$\boldsymbol{c}_{i,j} = \bigvee \bigl(\delta(z_{i,\ell}, z_{j,\ell}) \bigm| \ell < n \bigr), \quad \text{for all } i, j \in I,$$

and we prove that the family $\langle \boldsymbol{c}_{i,j} \mid \langle i,j \rangle \in I \times I \rangle$ satisfies the required conditions, with respect to the family $\langle I_u \mid u \in U \rangle$ of 2^n subsets of I. So, let $i, j, k \in I$. The inequality $\boldsymbol{c}_{i,k} \leqslant \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$ holds trivially.

Now suppose that $i, j \in I_u$, for some $u \in U$.

Let $\ell < n$. If $\ell \in u$, then

$$\begin{aligned} \delta(z_{i,\ell}, z_{j,\ell}) &\leqslant \delta(z_{i,\ell}, x_\ell) \vee \delta(x_\ell, z_{j,\ell}) \leqslant \boldsymbol{a}_i \vee \boldsymbol{a}_j, \\ \delta(x_\ell, y_\ell) &\leqslant \delta(x_\ell, z_{j,\ell}) \vee \delta(z_{j,\ell}, z_{i,\ell}) \vee \delta(z_{i,\ell}, y_\ell) \leqslant \boldsymbol{a}_j \vee \boldsymbol{c}_{i,j} \vee \boldsymbol{b}_i, \end{aligned}$$

while if $\ell \in n \setminus u$,

$$\begin{aligned} \delta(z_{i,\ell}, z_{j,\ell}) &\leqslant \delta(z_{i,\ell}, y_\ell) \vee \delta(y_\ell, z_{j,\ell}) \leqslant \boldsymbol{a}_i \vee \boldsymbol{a}_j, \\ \delta(x_\ell, y_\ell) &\leqslant \delta(x_\ell, z_{i,\ell}) \vee \delta(z_{i,\ell}, z_{j,\ell}) \vee \delta(z_{j,\ell}, y_\ell) \leqslant \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \vee \boldsymbol{a}_j, \end{aligned}$$

whence both inequalities $\delta(z_{i,\ell}, z_{j,\ell}) \leqslant \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\delta(x_\ell, y_\ell) \leqslant \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$ hold in any case. Hence $\boldsymbol{c}_{i,j} \leqslant \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{e} \leqslant \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$. Exchanging x and y in the argument leading to the first inequality also yields that $\boldsymbol{c}_{i,j} \leqslant \boldsymbol{b}_i \vee \boldsymbol{b}_j$. $\square$

Corollary 2.4. *Let A be an algebra with almost permutable congruences. Then* $\mathrm{Con_c}\, A$ *satisfies* $\mathrm{WURP}^=$.

Remark 2.5. In case the distance δ satisfies the V-condition of type 1, the statement $\mathrm{WURP}^=$ in Theorem 2.3 can be strengthened by taking $m = 1$ in Definition 2.1. Similarly, if A is an algebra with permutable congruences, then $\mathrm{Con_c}\, A$ satisfies that strengthening of $\mathrm{WURP}^=$. In particular, as any group, respectively any module, has permutable congruences, both $\mathrm{NSub_c}\, G$, for a group G, and $\mathrm{Sub_c}\, M$, for a module M, satisfy the strengthening of $\mathrm{WURP}^=$ obtained by taking $m = 1$ in Definition 2.1.

As we shall see in Theorem 3.6, not every distributive $\langle \vee, 0\rangle$-semilattice can be join-generated by the range of a V-distance of type 3/2. The situation changes dramatically for type 2. It is proved in [8] that any modular algebraic lattice is isomorphic to the congruence lattice of an algebra with 3-permutable congruences. This easily implies the following result; nevertheless, we provide a much more direct argument, which will be useful for the proof of Theorem 7.1.

Proposition 2.6. *Any distributive $\langle \vee, 0\rangle$-semilattice is the range of some V-distance of type* 2.

Proof. Let S be a distributive $\langle \vee, 0\rangle$-semilattice. We first observe that the map $\mu_S : S \times S \to S$ defined by the rule

$$\mu_S(x, y) = \begin{cases} x \vee y, & \text{if } x \neq y, \\ 0, & \text{if } x = y \end{cases} \tag{2.2}$$

is a surjective S-valued distance on S. Now suppose that we are given a surjective S-valued distance $\delta : X \times X \to S$, and let $x, y \in X$ and $\boldsymbol{a}, \boldsymbol{b} \in S$ such that $\delta(x, y) \leqslant \boldsymbol{a} \vee \boldsymbol{b}$. Since S is distributive, there are $\boldsymbol{a}' \leqslant \boldsymbol{a}$ and $\boldsymbol{b}' \leqslant \boldsymbol{b}$ such that $\delta(x, y) = \boldsymbol{a}' \vee \boldsymbol{b}'$. We put $X' = X \cup \{u, v\}$, where u and v are two distinct outside points, and we extend δ to a distance δ' on X' by putting $\delta'(z, u) = \delta(z, x) \vee \boldsymbol{a}'$ and $\delta'(z, v) = \delta(z, y) \vee \boldsymbol{a}'$, for all $z \in X$, while $\delta'(u, v) = \boldsymbol{b}'$. It is straightforward to verify that δ' is an S-valued distance on X' extending δ. Furthermore, $\delta'(x, u) = \boldsymbol{a}' \leqslant \boldsymbol{a}$, $\delta'(u, v) = \boldsymbol{b}' \leqslant \boldsymbol{b}$, and $\delta'(v, y) = \boldsymbol{a}' \leqslant \boldsymbol{a}$. Iterating this construction transfinitely, taking direct limits at limit stages, yields an S-valued V-distance of type 2 extending δ. $\square$

3. Failure of $\mathrm{WURP}^=$ in $\mathrm{Con_c}\, F$, for F free bounded lattice

The main proof of the present section, that is, the proof of Theorem 3.6, follows the lines of the proofs of [18, Theorem 3.3] and [21, Corollary 2.1]. However, there are a few necessary changes, mainly due to the new "uniform refinement property" not being the same as the previously considered ones. As the new result extends to any algebra, and not only lattices (see Corollary 3.7), we feel that it is still worthwhile to show the main lines of the proof in some detail.

From now on until Lemma 3.5, we shall fix a non-distributive lattice variety $\mathcal{V}$. For every set X, denote by $\mathbf{B}_{\mathcal{V}}(X)$ (or $\mathbf{B}(X)$ in case $\mathcal{V}$ is understood) the *bounded* lattice in $\mathcal{V}$ freely generated by chains $s_i < t_i$, for $i \in X$. Note that if Y is a subset of X, then there is a unique retraction from $\mathbf{B}(X)$ onto $\mathbf{B}(Y)$, sending each s_i to 0 and each t_i to 1, for every $i \in X \setminus Y$. Thus, we shall often identify $\mathbf{B}(Y)$ with the bounded sublattice of $\mathbf{B}(X)$ generated by all s_i and t_i $(i \in Y)$. Moreover, the above mentioned retraction from $\mathbf{B}(X)$ onto $\mathbf{B}(Y)$ induces a retraction from $\mathrm{Con_c}\,\mathbf{B}(X)$ onto $\mathrm{Con_c}\,\mathbf{B}(Y)$. Hence, we shall also identify $\mathrm{Con_c}\,\mathbf{B}(Y)$ with the corresponding subsemilattice of $\mathrm{Con_c}\,\mathbf{B}(X)$.

Now we fix a set X such that $|X| \geqslant \aleph_2$. We denote, for all $i \in X$, by $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ the compact congruences of $\mathbf{B}(X)$ defined by

$$\boldsymbol{a}_i = \Theta(0, s_i) \vee \Theta(t_i, 1); \qquad \boldsymbol{b}_i = \Theta(s_i, t_i). \tag{3.1}$$

In particular, note that $\boldsymbol{a}_i \vee \boldsymbol{b}_i = \mathbf{1}$, the largest congruence of $\mathbf{B}(X)$.

Now, towards a contradiction, suppose that there are a positive integer n, a decomposition $X = \bigcup(X_k \mid k < n)$, and a family $\langle \boldsymbol{c}_{i,j} \mid \langle i, j\rangle \in X \times X\rangle$ of elements of $\mathrm{Con_c}\,\mathbf{B}(X)$ witnessing the statement that $\mathrm{Con_c}\,\mathbf{B}(X)$ satisfies $\mathrm{WURP}^{=}(\mathbf{1})$, where $\mathbf{1}$ denotes the largest congruence of $\mathbf{B}(X)$. We pick $k < n$ such that $|X_k| = |X|$. By "projecting everything on $\mathbf{B}(X_k)$" (as in [21, p. 224]), we might assume that $X_k = X$.

Since the $\mathrm{Con_c}$ functor preserves direct limits, for all $i, j \in X$, there exists a finite subset $F(\{i, j\})$ of X such that both $\boldsymbol{c}_{i,j}$ and $\boldsymbol{c}_{j,i}$ belong to $\mathrm{Con_c}\,\mathbf{B}(F(\{i, j\}))$. By Kuratowski's Theorem, there are distinct elements $0, 1, 2$ of X such that $0 \notin F(\{1, 2\})$, $1 \notin F(\{0, 2\})$, and $2 \notin F(\{0, 1\})$. Denote by $\pi : \mathbf{B}(X) \twoheadrightarrow \mathbf{B}(\{0, 1, 2\})$ the canonical retraction. For every $i \in \{0, 1, 2\}$, denote by i' and i'' the other two elements of $\{0, 1, 2\}$, arranged in such a way that $i' < i''$. We put $\boldsymbol{d}_i = (\mathrm{Con_c}\,\pi)(\boldsymbol{c}_{i',i''})$, for all $i \in \{0, 1, 2\}$.

Applying the semilattice homomorphism $\mathrm{Con_c}\,\pi$ to the inequalities satisfied by the elements $\boldsymbol{c}_{i,j}$ yields

$$\boldsymbol{d}_0 \subseteq \boldsymbol{a}_1 \vee \boldsymbol{a}_2, \boldsymbol{b}_1 \vee \boldsymbol{b}_2; \qquad \boldsymbol{d}_1 \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_2, \boldsymbol{b}_0 \vee \boldsymbol{b}_2; \qquad \boldsymbol{d}_2 \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_1, \boldsymbol{b}_0 \vee \boldsymbol{b}_1; \tag{3.2}$$

$$\boldsymbol{d}_0 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_1 = \boldsymbol{d}_1 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_0 = \boldsymbol{d}_2 \vee \boldsymbol{a}_1 \vee \boldsymbol{b}_0 = \mathbf{1}; \tag{3.3}$$

$$\boldsymbol{d}_1 \subseteq \boldsymbol{d}_0 \vee \boldsymbol{d}_2. \tag{3.4}$$

As in [18, Lemma 2.1], it is not hard to prove the following.

Lemma 3.1. *The congruence $\boldsymbol{d}_i$ belongs to* $\mathrm{Con_c}\,\mathbf{B}(\{i', i''\})$*, for all $i \in \{0, 1, 2\}$.*

Since $\mathcal{V}$ is a non-distributive variety of lattices, it follows from a classical result of lattice theory that $\mathcal{V}$ contains as a member some lattice $M \in \{M_3, N_5\}$. Decorate the lattice M with three 2-element chains $x_i < y_i$ (for $i \in \{0, 1, 2\}$) as in [18], which we illustrate on Fig. 1.

The relevant properties of these decorations are summarized in the two following straightforward lemmas.

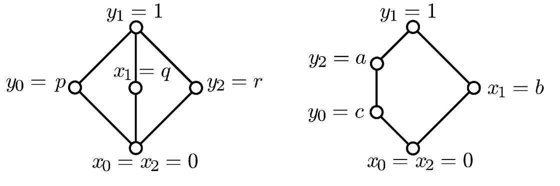

Fig. 1. The decorations of M_3 and N_5.

Lemma 3.2. *The decorations defined above satisfy the following inequalities*:

$$\begin{array}{llll} x_0 \wedge y_1 \leqslant x_1; & y_1 \leqslant x_1 \vee y_0; & x_1 \wedge y_0 \leqslant x_0; & y_0 \leqslant x_0 \vee y_1; \\ x_1 \wedge y_2 \leqslant x_2; & y_2 \leqslant x_2 \vee y_1; & x_2 \wedge y_1 \leqslant x_1; & y_1 \leqslant x_1 \vee y_2, \end{array}$$

but $y_2 \nleqslant x_2 \vee y_0$.

Lemma 3.3. *The sublattice of M generated by $\{x_{i'}, x_{i''}, y_{i'}, y_{i''}\}$ is distributive, for all $i \in \{0, 1, 2\}$.*

Now we shall denote by D be the free product (i.e., the coproduct) of two 2-element chains, say $u_0 < v_0$ and $u_1 < v_1$, in the variety of all *distributive* lattices. The lattice D is diagrammed on Fig. 2.

The join-irreducible elements of D are u_0, u_1, v_0, v_1, $u'_0 = u_0 \wedge v_1$, $u'_1 = u_1 \wedge v_0$, and $w = v_0 \wedge v_1$. Since D is finite distributive, its congruence lattice is finite Boolean, with seven atoms $\boldsymbol{p} = \Theta_D(p_*, p)$, for $p \in \mathrm{J}(D)$ (where p_* denotes the unique lower cover of p in D), that is,

$$\begin{array}{ll} \boldsymbol{u}_0 = \Theta_D^+(u_0, v_1); & \boldsymbol{u}_1 = \Theta_D^+(u_1, v_0); \\ \boldsymbol{v}_0 = \Theta_D^+(v_0, u_0 \vee v_1); & \boldsymbol{v}_1 = \Theta_D^+(v_1, u_1 \vee v_0); \\ \boldsymbol{u}'_0 = \Theta_D^+(u_0 \wedge v_1, u_1); & \boldsymbol{u}'_1 = \Theta_D^+(u_1 \wedge v_0, u_0); \\ \boldsymbol{w} = \Theta_D\big((u_0 \wedge v_1) \vee (u_1 \wedge v_0), v_0 \wedge v_1\big). & \end{array}$$

For all $i \in \{0, 1, 2\}$, let $\pi_i : \mathbf{B}(\{i', i''\}) \to D$ be the unique lattice homomorphism sending $s_{i'}$ to u_0, $t_{i'}$ to v_0, $s_{i''}$ to u_1, $t_{i''}$ to v_1. Furthermore, denote by $\rho : \mathbf{B}(\{0, 1, 2\}) \to M$ the unique lattice homomorphism sending s_i to x_i and t_i to y_i (for all $i \in \{0, 1, 2\}$); denote by ρ_i the restriction of ρ to $\mathbf{B}(\{i', i''\})$.

We shall restate [18, Lemma 3.1] here for convenience.

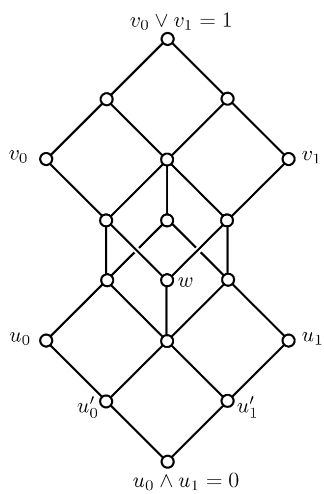

Fig. 2. The distributive lattice D.

Lemma 3.4. *Let L be any distributive lattice, let a, b, a', b' be elements of L. Then the equality $\Theta_L^+(a,b) \cap \Theta_L^+(a',b') = \Theta_L^+(a \wedge a', b \vee b')$ holds.*

Now we put $\boldsymbol{e}_i = (\mathrm{Con_c}\,\pi_i)(\boldsymbol{d}_i)$, for all $i \in \{0,1,2\}$.

Lemma 3.5. *The containments $\boldsymbol{e}^- \subseteq \boldsymbol{e}_i \subseteq \boldsymbol{e}^+$ hold for all $i \in \{0,1,2\}$, where we put*

$$\begin{aligned}
\boldsymbol{e}^- &= \Theta_D^+(u_0 \wedge v_1, u_1) \vee \Theta_D^+(v_1, u_1 \vee v_0),\\
\boldsymbol{e}^+ &= \Theta_D^+(u_0 \wedge v_1, u_1) \vee \Theta_D^+(v_1, u_1 \vee v_0) \vee \Theta_D^+(u_1 \wedge v_0, u_0) \vee \Theta_D^+(v_0, u_0 \vee v_1).
\end{aligned}$$

Proof. Applying $\mathrm{Con_c}\,\pi_i$ to the inequalities (3.2) and (3.3) yields the following inequalities:

$$\boldsymbol{e}_i \subseteq \Theta(0,u_0) \vee \Theta(0,u_1) \vee \Theta(v_0,1) \vee \Theta(v_1,1), \tag{3.5}$$

$$\boldsymbol{e}_i \subseteq \Theta(u_0,v_0) \vee \Theta(u_1,v_1), \tag{3.6}$$

$$\boldsymbol{e}_i \vee \Theta(0,u_1) \vee \Theta(v_1,1) \vee \Theta(u_0,v_0) = \mathbf{1}. \tag{3.7}$$

By using Lemma 3.4 and the distributivity of $\mathrm{Con}\,D$, we obtain, by meeting (3.5) and (3.6), the inequality $\boldsymbol{e}_i \subseteq \boldsymbol{e}^+$. On the other hand, by using (3.7) together with the equality

$$\Theta(0,u_1) \vee \Theta(v_1,1) \vee \Theta(u_0,v_0) = \boldsymbol{u}_0 \vee \boldsymbol{u}_1 \vee \boldsymbol{u}_1' \vee \boldsymbol{v}_0 \vee \boldsymbol{w}$$

(see Fig. 2), we obtain that $\boldsymbol{e}^- = \boldsymbol{u}_0' \vee \boldsymbol{v}_1 \subseteq \boldsymbol{e}_i$. $\square$

Now, for all $i \in \{0,1,2\}$, it follows from Lemma 3.3 that there exists a unique lattice homomorphism $\varphi_i : D \to M$ such that $\varphi_i \circ \pi_i = \rho_i$. Since $\mathrm{Con_c}$ is a functor, we get from this and from Lemma 3.5 that for all $i \in \{0,1,2\}$,

$$\begin{aligned}
&(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_i)\\
&\quad= (\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}_i) \subseteq (\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}^+)\\
&\quad= (\mathrm{Con_c}\,\varphi_i)\bigl(\Theta^+(u_0 \wedge v_1, u_1) \vee \Theta^+(v_1, u_1 \vee v_0) \vee \Theta^+(u_1 \wedge v_0, u_0) \vee \Theta^+(v_0, u_0 \vee v_1)\bigr)\\
&\quad= \Theta^+(x_{i'} \wedge y_{i''}, x_{i''}) \vee \Theta^+(y_{i''}, x_{i''} \vee y_{i'}) \vee \Theta^+(x_{i''} \wedge y_{i'}, x_{i'}) \vee \Theta^+(y_{i'}, x_{i'} \vee y_{i''}),
\end{aligned} \tag{3.8}$$

while

$$\begin{aligned}
(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_i) &= (\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}_i) \supseteq (\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}^-)\\
&= (\mathrm{Con_c}\,\varphi_i)\bigl(\Theta^+(u_0 \wedge v_1, u_1) \vee \Theta^+(v_1, u_1 \vee v_0)\bigr)\\
&= \Theta^+(x_{i'} \wedge y_{i''}, x_{i''}) \vee \Theta^+(y_{i''}, x_{i''} \vee y_{i'}).
\end{aligned} \tag{3.9}$$

In particular, we obtain, using Lemma 3.2,

$$\begin{aligned}
(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_0) &= \mathbf{0},\\
(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_2) &= \mathbf{0},
\end{aligned}$$

while

$$(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_1) \supseteq \Theta^+(x_0 \wedge y_2, x_2) \vee \Theta^+(y_2, x_2 \vee y_0) \neq \mathbf{0}.$$

On the other hand, by applying $\mathrm{Con_c}\,\rho$ to (3.4), we obtain that

$$(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_1) \subseteq (\mathrm{Con_c}\,\rho)(\boldsymbol{d}_0) \vee (\mathrm{Con_c}\,\rho)(\boldsymbol{d}_2),$$

a contradiction. Therefore, we have proved the following theorem.

Theorem 3.6. *Let $\mathcal{V}$ be any non-distributive variety of lattices, let X be any set such that $|X| \geqslant \aleph_2$. Denote by $\mathbf{B}_{\mathcal{V}}(X)$ the free product in $\mathcal{V}$ of X copies of a two-element chain with a least and a largest element added. Then $\mathrm{Con_c}\,\mathbf{B}_{\mathcal{V}}(X)$ does not satisfy $\mathrm{WURP}^=$ at its largest element.*

A "local" version of Theorem 3.6 is presented in Theorem 7.2.

Observe that $\mathrm{Con_c}\,\mathbf{B}_{\mathcal{V}}(X)$, being the semilattice of compact congruences of a lattice, is distributive.

As in [18, Corollary 4.1], we obtain the following.

Corollary 3.7. *Let L be any lattice that admits a lattice homomorphism onto a free bounded lattice in the variety generated by either M_3 or N_5 with $\aleph_2$ generators. Then $\mathrm{Con_c}\,L$ does not satisfy $\mathrm{WURP}^=$. In particular, there exists no V-distance of type $3/2$ with range join-generating $\mathrm{Con_c}\,L$. Hence there is no algebra A with almost permutable congruences such that $\mathrm{Con}\,L \cong \mathrm{Con}\,A$.*

Proof. The first part of the proof goes like the proof of [18, Corollary 4.1], using Lemma 2.2. The rest of the conclusion follows from Theorem 2.3. □

Corollary 3.8. *Let $\mathcal{V}$ be any non-distributive variety of lattices and let F be any free (respectively, free bounded) lattice with at least $\aleph_2$ generators in $\mathcal{V}$. Then there exists no V-distance of type $3/2$ with range join-generating $\mathrm{Con_c}\,F$. In particular, there is no algebra A with almost permutable congruences such that $\mathrm{Con}\,F \cong \mathrm{Con}\,A$.*

By using Corollary 1.4, we thus obtain the following.

Corollary 3.9. *Let $\mathcal{V}$ be a non-distributive variety of lattices, let F be any free (respectively, free bounded) lattice with at least $\aleph_2$ generators in $\mathcal{V}$, and put $D = \mathrm{Con}\,F$—a distributive, algebraic lattice with $\aleph_2$ compact elements. Then there is no module M (respectively, no group G, no ℓ-group G) such that $\mathrm{Sub_c}\,M \cong D$ (respectively, $\mathrm{NSub}\,G \cong D$, $\mathrm{Id}^{\ell}\,G \cong D$).*

Hence, not every distributive algebraic lattice is isomorphic to the submodule lattice of some module, or to the normal subgroup lattice of some group. However, our proof of this negative result requires at least $\aleph_2$ compact elements. As we shall see in Sections 4 and 5, the $\aleph_2$ bound is, in both cases of modules and groups, optimal.

4. Representing distributive algebraic lattices with at most $\aleph_1$ compact elements as submodule lattices of modules

In this section we deal with congruence lattices of right modules over rings.

Theorem 4.1. *Every distributive $\langle\vee,0\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to the submodule lattice of some right module.*

Proof. Let S be a distributive $\langle\vee,0\rangle$-semilattice of size at most $\aleph_1$. If S has a largest element, then it follows from the main result of [25] that S is isomorphic to the semilattice $\mathrm{Id_c}\, R$ of all finitely generated two-sided ideals of some (unital) von Neumann regular ring R.

In order to reduce ideals to submodules, we use a well-known trick. As R is a bimodule over itself, the tensor product $\overline{R} = R^{\mathrm{op}} \otimes R$ can be endowed with a structure of (unital) ring, with multiplication satisfying $(a\otimes b)\cdot(a'\otimes b') = (a'a)\otimes(bb')$ (both $a'a$ and bb' are evaluated in R). Then R is a right $\overline{R}$-module, with scalar multiplication given by $x\cdot(a\otimes b) = axb$, and the submodules of $R_{\overline{R}}$ are exactly the two-sided ideals of R. Hence, $\mathrm{Sub_c}\, R_{\overline{R}} = \mathrm{Id_c}\, R \cong S$.

In case S has no unit, it is an ideal of the distributive $\langle\vee,0,1\rangle$-homomorphism $S' = S\cup\{1\}$ for a new largest element 1. By the previous paragraph, $S' \cong \mathrm{Sub_c}\, M$ for some right module M, hence $S \cong \mathrm{Sub_c}\, N$ where N is the submodule of M consisting of those elements $x\in M$ such that the submodule generated by x is sent to an element of S by the isomorphism $\mathrm{Sub_c}\, M \cong S'$. $\square$

The commutative case is quite different. For example, for a commutative von Neumann regular ring R, if $\mathrm{Id}\, R$ is finite, then, as it is distributive and complemented, it must be Boolean. In particular, the three-element chain is not isomorphic to the ideal lattice of any commutative von Neumann regular ring. Even if regularity is removed, not every finite distributive lattice is allowed. For example, one can prove the following result: *A finite distributive lattice D is isomorphic to the submodule lattice of a module over some commutative ring iff D is isomorphic to the ideal lattice of some commutative ring, iff D is a product of chains*. In particular, the square $\mathbf{2}\times\mathbf{2}$ with a new bottom (respectively, top) element added is not isomorphic to the submodule lattice of any module over a commutative ring.

5. Representing distributive algebraic lattices with at most $\aleph_1$ compact elements as normal subgroup lattices of groups

Every non-abelian simple group is "neutral" in the sense of [3]. Hence, the direction $(1)\Rightarrow(5)$ in [3, Theorem 8.5] yields the following well-known result, which holds despite the failure of congruence-distributivity in the variety of all groups.

Lemma 5.1. *Let $n<\omega$ and let $\langle G_i \mid i<n\rangle$ be a finite sequence of simple non-abelian groups. Then the normal subgroups of $\prod_{i<n} G_i$ are exactly the trivial ones, namely the products of the form $\prod_{i<n} H_i$, where H_i is either G_i or $\{1_{G_i}\}$, for all $i<n$. Consequently,* $\mathrm{NSub}(\prod_{i<n} G_i) \cong \mathbf{2}^n$.

We denote by $\mathcal{F}$ the class of all finite products of alternating groups of the form $\mathfrak{A}_n$, for $n\geqslant 5$. For a group homomorphism $f : G\to H$, we denote by $\mathrm{NSub}\, f : \mathrm{NSub}\, G \to \mathrm{NSub}\, H$ the $\langle\vee,0\rangle$-homomorphism that with any normal subgroup X of G associates the normal subgroup of H generated by $f[X]$. The following square amalgamation result is crucial. It is an analogue for groups of [7, Theorem 1] (for lattices) or [25, Theorem 4.2] (for regular algebras over a division ring).

Lemma 5.2. *Let G_0, G_1, G_2 be groups in $\mathcal{F}$ and let $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be group homomorphisms. Let B be a finite Boolean semilattice, and, for $i \in \{1,2\}$, let $\boldsymbol{g}_i : \mathrm{NSub}\, G_i \to B$ be $\langle\vee,0\rangle$-homomorphisms such that*

$$\boldsymbol{g}_1 \circ \mathrm{NSub}\, f_1 = \boldsymbol{g}_2 \circ \mathrm{NSub}\, f_2. \tag{5.1}$$

Then there are a group G in $\mathcal{F}$, group homomorphisms $g_i : G_i \to G$, for $i \in \{1,2\}$, and an isomorphism $\alpha : \mathrm{NSub}\, G \to B$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $\alpha \circ \mathrm{NSub}\, g_i = \boldsymbol{g}_i$ for all $i \in \{1,2\}$.

Outline of proof. We follow the lines of the proofs of [7, Theorem 1] or [25, Theorem 4.2]. First, by decomposing B as a finite power of $\mathbf{2}$, observing that $\mathcal{F}$ is closed under finite direct products, and using Lemma 5.1, we reduce to the case where $B = \mathbf{2}$, the two-element chain. Next, denoting by $\boldsymbol{h}$ the $\langle\vee,0\rangle$-homomorphism appearing on both sides of (5.1), we put $G_0' = \{x \in G_0 \mid \boldsymbol{h}([x]) = 0\}$ (where $[x]$ denotes, again, the normal subgroup generated by x), and, similarly, $G_i' = \{x \in G_i \mid \boldsymbol{g}_i([x]) = 0\}$, for $i \in \{1,2\}$. So G_i' is a normal subgroup of G_i, for all $i \in \{0,1,2\}$, and replacing G_i by G_i/G_i' makes it possible to reduce to the case where both $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ separate zero while both f_1 and f_2 are group embeddings.

Hence the problem that we must solve is the following: given group embeddings $f_i : G_0 \hookrightarrow G_i$, for $i \in \{1,2\}$, we must find a finite, simple, non-abelian group G with group embeddings $g_i : G_i \hookrightarrow G$ such that $g_1 \circ f_1 = g_2 \circ f_2$. By the positive solution of the amalgamation problem for finite groups (see [15, Section 15]), followed by embedding the resulting group into some alternating group with index at least 5, this is possible. □

Now every distributive $\langle\vee,0\rangle$-semilattice of size at most $\aleph_1$ is the direct limit of some direct system of finite Boolean $\langle\vee,0\rangle$-semilattices and $\langle\vee,0\rangle$-homomorphisms; furthermore, we may assume that the indexing set of the direct system is a 2-*ladder*, that is, a lattice with zero where every interval is finite and every element has at most two immediate predecessors. Hence, by imitating the method of proof used in [7, Theorem 2] or [25, Theorem 5.2], it is not difficult to obtain the following result.

Theorem 5.3. *Every distributive $\langle\vee,0\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to the finitely generated normal subgroup semilattice of some group which is a direct limit of members of $\mathcal{F}$.*

Reformulating the result in terms of algebraic lattices rather than semilattices, together with the observation that all direct limits of groups in $\mathcal{F}$ are locally finite, gives the following.

Corollary 5.4. *Every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the normal subgroup lattice of some locally finite group.*

6. Representing distributive algebraic lattices with at most $\aleph_0$ compact elements as ℓ-ideal lattices of ℓ-groups

The variety of ℓ-groups is quite special, as it is both congruence-distributive and congruence-permutable. The following lemma does not extend to the commutative case (for example, $\mathbb{Z} \times \mathbb{Z}$ cannot be embedded into any simple commutative ℓ-group).

Lemma 6.1. *Every ℓ-group can be embedded into some simple ℓ-group.*

Proof. It follows from [17, Corollary 5.2] that every ℓ-group G embeds into an ℓ-group H in which any two positive elements are conjugate. In particular, H is simple. $\square$

The following result is a "one-dimensional" analogue for ℓ-groups of Lemma 5.2.

Lemma 6.2. *For any ℓ-group G, any finite Boolean semilattice B, and any $\langle\vee,0\rangle$-homomorphism $\boldsymbol{f}:\mathrm{Id}_{\mathrm{c}}^{\ell} G \to B$, there are an ℓ-group H, an ℓ-homomorphism $f:G\to H$, and an isomorphism $\alpha:\mathrm{Id}_{\mathrm{c}}^{\ell} H \to B$ such that $\boldsymbol{f}=\alpha\circ\mathrm{Id}_{\mathrm{c}}^{\ell} f$.*

Proof. Suppose first that $B=\mathbf{2}$. Observing that $I=\{x\in G \mid \boldsymbol{f}(G(x))=0\}$ is an ℓ-ideal of G, we let H be any simple ℓ-group extending G/I (see Lemma 6.1), we let $f:G\to H$ be the composition of the canonical projection $G \twoheadrightarrow G/I$ with the inclusion map $G/I \hookrightarrow H$, and we let $\alpha:\mathrm{Id}_{\mathrm{c}}^{\ell} H\to\mathbf{2}$ be the unique isomorphism.

Now suppose that $B=\mathbf{2}^n$, for a natural number n. For each $i<n$, we apply the result of the paragraph above to the ith component $\boldsymbol{f}_i:\mathrm{Id}_{\mathrm{c}}^{\ell} G\to\mathbf{2}$ of $\boldsymbol{f}$, getting a simple ℓ-group H_i, an ℓ-homomorphism $f_i:G\to H_i$, and the isomorphism $\alpha_i:\mathrm{Id}_{\mathrm{c}}^{\ell} H_i\to\mathbf{2}$. Then we put $H=\prod_{i<n} H_i$, $f:x\mapsto\langle f_i(x)\mid i<n\rangle$, and we let $\alpha:\mathrm{Id}_{\mathrm{c}}^{\ell} H\to\mathbf{2}^n$ be the canonical isomorphism. $\square$

Theorem 6.3. *Every distributive at most countable $\langle\vee,0\rangle$-semilattice is isomorphic to the semilattice of all finitely generated ℓ-ideals of some ℓ-group.*

Equivalently, every distributive algebraic lattice with (at most) countably many compact elements is isomorphic to the ℓ-ideal lattice of some ℓ-group.

Proof. It follows from [2, Theorem 3.1] (see also [4, Theorem 6.6]) that every distributive at most countable $\langle\vee,0\rangle$-semilattice S can be expressed as the direct limit of a sequence $\langle B_n\mid n<\omega\rangle$ of finite Boolean semilattices, with all transition maps $\boldsymbol{f}_n:B_n\to B_{n+1}$ and limiting maps $\boldsymbol{g}_n:B_n\to S$ being $\langle\vee,0\rangle$-homomorphisms. We fix an ℓ-group G_0 with an isomorphism $\alpha_0:\mathrm{Id}_{\mathrm{c}}^{\ell} G_0\twoheadrightarrow B_0$. Suppose having constructed an ℓ-group G_n with an isomorphism $\alpha_n:\mathrm{Id}_{\mathrm{c}}^{\ell} G_n\to B_n$. Applying Lemma 6.2 to $\boldsymbol{f}_n\circ\alpha_n$, we obtain an ℓ-group G_{n+1}, an ℓ-homomorphism $f_n:G_n\to G_{n+1}$, and an isomorphism $\alpha_{n+1}:\mathrm{Id}_{\mathrm{c}}^{\ell} G_{n+1}\to B_{n+1}$ such that $\boldsymbol{f}_n\circ\alpha_n=\alpha_{n+1}\circ\mathrm{Id}_{\mathrm{c}}^{\ell} f_n$. Defining G as the direct limit of the sequence

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots,$$

an elementary categorical argument yields an isomorphism from $\mathrm{Id}_{\mathrm{c}}^{\ell} G$ onto the direct limit S of the sequence $\langle B_n\mid n<\omega\rangle$. $\square$

7. Functorial representation by V-distances of type 2

Observe that the argument of Proposition 2.6 is only a small modification (with a more simple-minded proof) of B. Jónsson's proof that every modular lattice has a type 2 representation, see [11] or [6, Theorem IV.4.8]. It follows from Corollary 3.7 that "type 2" cannot be improved to "type 1." In view of Proposition 1.2, this is somehow surprising, as every distributive lattice has an embedding with permutable congruences into some partition lattice. This illustrates the observation that one can get much more from a distance than from an embedding into a partition lattice.

We shall now present a strengthening of Proposition 2.6 that shows that the construction can be made *functorial*. We introduce notations for the following categories:

(1) $\mathcal{S}$, the category of all distributive $\langle\vee,0\rangle$-semilattices with $\langle\vee,0\rangle$-embeddings.
(2) $\mathcal{D}$, the category of all surjective distances of the form $\delta : X \times X \twoheadrightarrow S$ with kernel the identity and S a distributive $\langle\vee,0\rangle$-semilattice, with morphisms (see Definition 1.1) of the form $\langle f, \boldsymbol{f}\rangle : \langle X, \lambda\rangle \to \langle Y, \mu\rangle$ with both f and $\boldsymbol{f}$ one-to-one.
(3) $\mathcal{D}_2$, the full subcategory of $\mathcal{D}$ consisting of all V-distances of type 2.

Furthermore, denote by $\Pi : \mathcal{D} \to \mathcal{S}$ the forgetful functor (see Definition 1.1).

Theorem 7.1. *There exists a direct limits preserving functor $\Phi : \mathcal{S} \to \mathcal{D}_2$ such that the composition $\Pi \circ \Phi$ is equivalent to the identity.*

Hence the functor Φ assigns to each distributive $\langle\vee,0\rangle$-semilattice S a set X_S together with a surjective S-valued V-distance $\delta_S : X_S \times X_S \twoheadrightarrow S$ of type 2.

Proof. The proof of Proposition 2.6 depends of the enumeration order of a certain transfinite sequence of quadruples $\langle x, y, \boldsymbol{a}, \boldsymbol{b}\rangle$, which prevents it from being functorial. We fix this by adjoining all such quadruples simultaneously, and by describing the corresponding extension. So, for a distance $\delta : X \times X \to S$, we put $S^- = S \setminus \{0\}$, and

$$\mathcal{H}(\delta) = \bigl\{\langle x, y, \boldsymbol{a}, \boldsymbol{b}\rangle \in X \times X \times S^- \times S^- \bigm| \delta(x, y) = \boldsymbol{a} \vee \boldsymbol{b}\bigr\}.$$

For $\xi = \langle x, y, \boldsymbol{a}, \boldsymbol{b}\rangle \in \mathcal{H}(\delta)$, we put $x_\xi^0 = x$, $x_\xi^1 = y$, $\boldsymbol{a}_\xi = \boldsymbol{a}$, and $\boldsymbol{b}_\xi = \boldsymbol{b}$. Now we put $X' = X \cup \{u_\xi^i \mid \xi \in \mathcal{H}(\delta) \text{ and } i \in \{0,1\}\}$, where the elements u_ξ^i are pairwise distinct symbols outside X. We define a map $\delta' : X' \times X' \to S$ by requiring δ' to extend δ, with value zero on the diagonal, and by the rule

$$\delta'\bigl(u_\xi^i, u_\eta^j\bigr) = \begin{cases} |i-j| \cdot \boldsymbol{b}_\xi, & \text{if } \xi = \eta, \\ \boldsymbol{a}_\xi \vee \boldsymbol{a}_\eta \vee \delta(x_\xi^i, x_\eta^j), & \text{if } \xi \neq \eta, \end{cases}$$

$$\delta'\bigl(u_\xi^i, z\bigr) = \delta'\bigl(z, u_\xi^i\bigr) = \delta\bigl(z, x_\xi^i\bigr) \vee \boldsymbol{a}_\xi,$$

for all $\xi, \eta \in \mathcal{H}(\delta)$, all $i, j \in \{0,1\}$, and all $z \in X$.

It is straightforward, though somewhat tedious, to verify that δ' is an S-valued distance on X', that it extends δ, and that its kernel is the identity of X' in case the kernel of δ is the identity of X (because the semilattice elements $\boldsymbol{a}_\xi$ and $\boldsymbol{b}_\xi$ are non-zero). Furthermore, if S is distributive, then every V-condition problem for δ of the form $\delta(x, y) \leqslant \boldsymbol{a} \vee \boldsymbol{b}$ can be refined to a problem of the form $\delta(x, y) = \boldsymbol{a}' \vee \boldsymbol{b}'$, for some $\boldsymbol{a}' \leqslant \boldsymbol{a}$ and $\boldsymbol{b}' \leqslant \boldsymbol{b}$ (because S is distributive), and such a problem has a solution of type 2 for δ'. Namely, in case both $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are non-zero (otherwise the problem can be solved in X), put $\xi = \langle x, y, \boldsymbol{a}', \boldsymbol{b}'\rangle$, and observe that $\delta'(x, u_\xi^0) = \boldsymbol{a}'$, $\delta'(u_\xi^0, u_\xi^1) = \boldsymbol{b}'$, and $\delta'(u_\xi^1, y) = \boldsymbol{a}'$.

Hence, if we put $\langle X_0, \delta_0\rangle = \langle X, \delta\rangle$, then $\langle X_{n+1}, \delta_{n+1}\rangle = \langle (X_n)', (\delta_n)'\rangle$ for all $n < \omega$, and finally $\overline{X} = \bigcup(X_n \mid n < \omega)$ and $\bar{\delta} = \bigcup(\delta_n \mid n < \omega)$, the pair $\Psi(\langle X, \delta\rangle) = \langle \overline{X}, \bar{\delta}\rangle$ is an S-valued V-distance of type 2 extending $\langle X, \delta\rangle$. Every morphism $\langle f, \boldsymbol{f}\rangle : \langle X, \lambda\rangle \to \langle Y, \mu\rangle$ in $\mathcal{S}$ extends canonically to a morphism $\langle f', \boldsymbol{f}\rangle : \langle X', \lambda'\rangle \to \langle Y', \mu'\rangle$ (the underlying semilattice map $\boldsymbol{f}$ is the same), by defining

$$f'\big(u^i_\xi\big) = u^i_{f\xi}, \quad \text{for all } \xi \in \mathcal{H}(\lambda) \text{ and all } i < 2,$$

where we put, of course,

$$f\langle x, y, \boldsymbol{a}, \boldsymbol{b}\rangle = \big\langle f(x), f(y), \boldsymbol{f}(\boldsymbol{a}), \boldsymbol{f}(\boldsymbol{b})\big\rangle, \quad \text{for all } \langle x, y, \boldsymbol{a}, \boldsymbol{b}\rangle \in \mathcal{H}(\lambda).$$

Hence, by an easy induction argument, $\langle f, \boldsymbol{f}\rangle$ extends canonically to a morphism $\Psi(\langle f, \boldsymbol{f}\rangle) = \langle \overline{f}, \boldsymbol{f}\rangle : \langle \overline{X}, \bar{\lambda}\rangle \to \langle \overline{Y}, \bar{\mu}\rangle$, and the correspondence $\langle f, \boldsymbol{f}\rangle \mapsto \langle \overline{f}, \boldsymbol{f}\rangle$ is itself a functor. As the construction defining the correspondence $\langle X, \delta\rangle \mapsto \langle X', \delta'\rangle$ is local, the functor Ψ preserves direct limits.

It remains to find something to start with, to which we can apply Ψ. A possibility is to use the distance μ_S, given by (2.2), introduced in the proof of Proposition 2.6. The correspondence $S \mapsto \mu_S$ defines a functor, in particular, if $f : S \hookrightarrow T$ is an embedding of distributive $\langle \vee, 0\rangle$-semilattices, then the equality $\mu_T(f(x), f(y)) = f(\mu_S(x, y))$ holds, for all $x, y \in S$. The desired functor Φ is given by $\Phi(S) = \Psi(\langle S, \mu_S\rangle)$, for any distributive $\langle \vee, 0\rangle$-semilattice S. $\square$

In contrast with the result of Theorem 7.1, we shall isolate a finite, "combinatorial" reason for the forgetful functor from V-distances of type 3/2 to distributive $\langle \vee, 0\rangle$-semilattices not to admit any left inverse. By contrast, we recall that for V-distances of type 2, the corresponding result is positive, see Theorem 7.1. In order to establish the negative result, we shall use the example $\mathcal{D}_{\mathrm{ac}}$ of [21, Section 7], and extend the corresponding result from lattices with almost permutable congruences to arbitrary V-distances of type 3/2.

We recall that $\mathcal{D}_{\mathrm{ac}}$ is the (commutative) cube of finite Boolean semilattices represented on Fig. 3, where $\mathfrak{P}(X)$ denotes the powerset algebra of a set X and $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_0$, $\boldsymbol{h}_1$, and $\boldsymbol{h}_2$ are the $\langle \vee, 0\rangle$-homomorphisms (and, in fact, $\langle \vee, 0, 1\rangle$-embeddings) defined by their values on atoms as follows:

$$\boldsymbol{e}(1) = \{0, 1\},$$

$$\boldsymbol{f}\colon \begin{cases} \{0\} \mapsto \{0, 1\}, \\ \{1\} \mapsto \{2, 3\}, \end{cases} \qquad \boldsymbol{g}\colon \begin{cases} \{0\} \mapsto \{0, 2\}, \\ \{1\} \mapsto \{1, 3\}, \end{cases}$$

$$\boldsymbol{h}_0\colon \begin{cases} \{0\} \mapsto \{0, 4, 7\}, \\ \{1\} \mapsto \{3, 5, 6\}, \\ \{2\} \mapsto \{2, 5, 6\}, \\ \{3\} \mapsto \{1, 4, 7\}, \end{cases} \qquad \boldsymbol{h}_1\colon \begin{cases} \{0\} \mapsto \{0, 4, 5, 7\}, \\ \{1\} \mapsto \{1, 4, 6, 7\}, \\ \{2\} \mapsto \{2, 5, 6, 7\}, \\ \{3\} \mapsto \{3, 4, 5, 6\}, \end{cases} \qquad \boldsymbol{h}_2\colon \begin{cases} \{0\} \mapsto \{0, 4, 6\}, \\ \{1\} \mapsto \{1, 5, 7\}, \\ \{2\} \mapsto \{3, 5, 7\}, \\ \{3\} \mapsto \{2, 4, 6\}, \end{cases}$$

Theorem 7.2. *The diagram $\mathcal{D}_{\mathrm{ac}}$ has no lifting, with respect to the forgetful functor, by distances, surjective at level* 0 *and satisfying the V-condition of type* 3/2 *at level* 1.

$$\begin{tikzcd}
 & \mathfrak{P}(8) & \\
\mathfrak{P}(4) \arrow[ur, "\boldsymbol{h}_2"] & \mathfrak{P}(4) \arrow[u, "\boldsymbol{h}_1"] & \mathfrak{P}(4) \arrow[ul, "\boldsymbol{h}_0"'] \\
\mathfrak{P}(2) \arrow[u, "\boldsymbol{f}"] \arrow[ur, "\boldsymbol{f}"] & \mathfrak{P}(2) \arrow[ul, "\boldsymbol{g}"] \arrow[ur, "\boldsymbol{f}"] & \mathfrak{P}(2) \arrow[ul, "\boldsymbol{g}"] \arrow[u, "\boldsymbol{g}"'] \\
 & \mathbf{2} \arrow[ul, "\boldsymbol{e}"] \arrow[u, "\boldsymbol{e}"] \arrow[ur, "\boldsymbol{e}"'] &
\end{tikzcd}$$

Fig. 3. The cube $\mathcal{D}_{\mathrm{ac}}$, unliftable by V-distances of type 3/2.

$$\begin{tikzcd}
 & \mu & \\
\mu_2 \arrow[ur] & \mu_1 \arrow[u] & \mu_0 \arrow[ul] \\
\lambda_0 \arrow[u] \arrow[ur] & \lambda_1 \arrow[ul] \arrow[ur] & \lambda_2 \arrow[ul] \arrow[u] \\
 & \lambda \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Fig. 4. A commutative diagram of semilattice-valued distances.

Proof. Suppose that the diagram of Fig. 3 is lifted by a diagram of distances, with distances $\lambda : X \times X \to \mathbf{2}$, $\lambda_i : X_i \times X_i \to \mathfrak{P}(2)$, $\mu_i : Y_i \times Y_i \to \mathfrak{P}(4)$, and $\mu : Y \times Y \to \mathfrak{P}(8)$, for all $i \in \{0, 1, 2\}$, see Fig. 4.

We assume that λ is surjective and that λ_i is a V-distance of type 3/2, for all $i \in \{0, 1, 2\}$. Denote by $f_{U,V}$ the canonical map from U to V given by this lifting, for U below V among X, X_0, X_1, X_2, Y_0, Y_1, Y_2, Y. After having replaced each of those sets U by its quotient by the kernel of the corresponding distance, and then by its image in Y under $f_{U,Y}$, we may assume that $f_{U,V}$ is the inclusion map from U into V, for all U below V among X, X_0, X_1, X_2, Y_0, Y_1, Y_2, Y.

Since λ is surjective, there are $x, y \in X$ such that $\lambda(x, y) = 1$. For all $i \in \{0, 1, 2\}$,

$$\lambda_i(x, y) = \boldsymbol{e}\big(\lambda(x, y)\big) = \boldsymbol{e}(1) = \{0, 1\} = \{0\} \cup \{1\},$$

thus, since λ_i satisfies the V-condition of type 3/2, there exists $z_i \in X_i$ such that

$$\begin{aligned} \text{either} \quad & \lambda_i(x, z_i) = \{0\} \quad \text{and} \quad \lambda_i(z_i, y) = \{1\} \quad \big(\text{say, } P(i)\big), \\ \text{or} \quad & \lambda_i(x, z_i) = \{1\} \quad \text{and} \quad \lambda_i(z_i, y) = \{0\} \quad \big(\text{say, } Q(i)\big). \end{aligned} \tag{7.1}$$

So we have eight cases to consider, according to which combination of P and Q occurs in (7.1) for $i \in \{0, 1, 2\}$. In each case, we shall obtain the inequality

$$\mu(z_0, z_2) \nsubseteq \mu(z_0, z_1) \cup \mu(z_1, z_2), \tag{7.2}$$

which will contradict the triangular inequality for μ.

Case 1. $P(0)$, $P(1)$, and $P(2)$ hold. Then $\mu_2(z_0, x) = \boldsymbol{f}\lambda_0(x, z_0) = \{0, 1\}$ and $\mu_2(x, z_1) = \boldsymbol{g}(\lambda_1(x, z_1)) = \{0, 2\}$, whence $\mu_2(z_0, z_1) \subseteq \{0, 1, 2\}$. Similarly, replacing x by y in the argument above, $\mu_2(z_0, y) = \boldsymbol{f}\lambda_0(z_0, y) = \{2, 3\}$ and $\mu_2(y, z_1) = \boldsymbol{g}(\lambda_1(z_1, y)) = \{1, 3\}$, whence $\mu_2(z_0, z_1) \subseteq \{1, 2, 3\}$. Therefore, $\mu_2(z_0, z_1) \subseteq \{1, 2\}$. On the other hand, from $\mu_2(x, z_0) \cup \mu_2(z_0, z_1) = \mu_2(x, z_1) \cup \mu_2(z_0, z_1)$ the converse inclusion follows, whence $\mu_2(z_0, z_1) = \{1, 2\}$. Similar computations yield that $\mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{1, 2\}$.

Hence, we obtain the equalities

$$\mu(z_0, z_1) = \boldsymbol{h}_2\mu_2(z_0, z_1) = \{1, 3, 5, 7\},$$
$$\mu(z_0, z_2) = \boldsymbol{h}_1\mu_1(z_0, z_2) = \{1, 2, 4, 5, 6, 7\},$$
$$\mu(z_1, z_2) = \boldsymbol{h}_0\mu_0(z_1, z_2) = \{2, 3, 5, 6\}.$$

Observe that 4 belongs to $\mu(z_0, z_2)$ but not to $\mu(z_0, z_1) \cup \mu(z_1, z_2)$.

Case 2. $P(0)$, $P(1)$, and $Q(2)$ hold. As in Case 1, we obtain

$$\mu_2(z_0, z_1) = \{1, 2\} \quad \text{and} \quad \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{0, 3\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$, which confirms (7.2) and thus causes a contradiction.

Case 3. $P(0)$, $Q(1)$, and $P(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_0(z_1, z_2) = \{0, 3\} \quad \text{and} \quad \mu_1(z_0, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 4. $P(0)$, $Q(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \{0, 3\} \quad \text{and} \quad \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$.

Case 5. $Q(0)$, $P(1)$, and $P(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \{0, 3\} \quad \text{and} \quad \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$.

Case 6. $Q(0)$, $P(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_0(z_1, z_2) = \{0, 3\} \quad \text{and} \quad \mu_1(z_0, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 7. $Q(0)$, $Q(1)$, and $P(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \{1, 2\} \quad \text{and} \quad \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{0, 3\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 8. $Q(0)$, $Q(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$.

In all cases, we obtain a contradiction. $\square$

A "global" version of Theorem 7.2 is presented in Theorem 3.6.

The following corollary extends [21, Theorem 7.1] from lattices to arbitrary algebras.

Corollary 7.3. *The diagram $\mathcal{D}_{\mathrm{ac}}$ has no lifting, with respect to the congruence lattice functor, by algebras with almost permutable congruences.*

About other commonly encountered structures, we obtain the following.

Corollary 7.4. *The diagram $\mathcal{D}_{\mathrm{ac}}$ has no lifting by groups with respect to the* NSub *functor, and no lifting by modules with respect to the* $\mathrm{Sub}_{\mathrm{c}}$ *functor.*

The following example offers a significant difference between the situations for groups and modules.

Example 7.5. The diagonal map $\mathbf{2} \hookrightarrow \mathbf{2}^2$ has no lifting, with respect to the Sub functor, by modules over any ring. Indeed, suppose that $A \hookrightarrow B \times C$ is such a lifting, with A, B, and C simple modules. Projecting on B and on C yields that A is isomorphic to a submodule of both B and C, whence, by simplicity, A, B, and C are pairwise isomorphic. But then, $B \times C \cong B \times B$ has the diagonal as a submodule, so its submodule lattice cannot be isomorphic to $\mathbf{2}^2$.

By contrast, every square of finite Boolean $\langle\vee, 0\rangle$-semilattices can be lifted, with respect to the NSub functor, by *groups*, see Lemma 5.2.

8. Open problems

Although we do know that the negative result of Corollary 3.8 applies to ℓ-groups (for every ℓ-group has permutable congruences), we do not know whether the positive results proved here for modules (Theorem 4.1) or for groups (Theorem 5.3) extend to ℓ-groups. The problem is that the class of all ℓ-groups does not satisfy the amalgamation property, see [17, Theorem 3.1], so the proof of Lemma 5.2 cannot be used in this context, and so we do not know how to extend Theorem 6.3 to the first uncountable level.

Problem 1. Is every distributive algebraic lattice with $\aleph_1$ compact elements isomorphic to the ℓ-ideal lattice of some ℓ-group?

Our next question is related to the functor Φ obtained in the statement of Theorem 7.1.

Problem 2. Does there exist a functor Φ as in Theorem 7.1 that sends finite semilattices to distances with finite underlying sets?

That is, can we assign *functorially* (with respect to $\langle\vee,0\rangle$-embeddings), to each finite distributive $\langle\vee,0\rangle$-semilattice S, a surjective V-distance $\langle X_S,\delta_S\rangle$ of type 2 with $\delta_S\colon X_S\times X_S\twoheadrightarrow S$ and X_S *finite*?

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