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A Steiner Tree Associated with Three Quarks

W. A. BEYER AND L. HELLER

Theoretical Division,
Los Alamos National Laboratory,
Los Alamos, NM 87545, U.S.A.

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The MIT bag model with three static quarks of the same charge has been discussed by Aerts & Heller. Two different geometries are given for the bag. It is shown that both geometries give the same energy. Use is made of the symbol manipulation system MACSYMA.

1. Introduction

Aerts & Heller (1981) have discussed the MIT bag model with three static quarks, the three quarks having the same charges. A tubular approximation is used for large pairwise separations of the quarks. Two different geometries are proposed (when the angles of the triangle are less than $2\pi/3$) for the tubes: the triangular bag shown in Fig. 1 and the Y-shaped bag with the vertex at the Torricelli point shown in Fig. 2. On the basis of numerical studies, Aerts & Heller conjecture that both geometries give the same energy. They also state their belief that more involved geometries will not yield lower energies.

In this paper we put the Aerts–Heller question in an algebraic form involving a system of polynomial equations, and this system is shown to yield the Aerts–Heller conjecture.

The symbol manipulation system MACSYMA played an important role in the course of verifying the conjecture. The conjecture was first reduced to proving an equality having the form

$$\sqrt{r_1(l_1, l_2)+\sqrt{r_2(l_1, l_2)}} = \omega(l_1, l_2), \tag{0.1}$$

where r_i are rational functions and ω is an algebraic function of l_1 and l_2 involving at most algebraic and square root operations. The equality in (0.1) was verified by MACSYMA in about 10 hours of computation. This involved three successive squarings. Only after this was done were we able to verify the Aerts–Heller conjecture in a fairly simple way.

The study of (0.1) has lead Stein & Zemach (1987) to the study of methods of solution of the "square root" equation

$$\sum_{i=1}^{n} (\pm)_i\sqrt{p_i(x)} = 0, \tag{0.2}$$

where $p_i(x)$ are polynomials in x. For $n \geqslant 5$, the elementary methods of solving (0.2) by successive transposition and squaring do not work. This has led Stein & Zemach to new methods of solving (0.2) by symmetric functions and symbolic computation.

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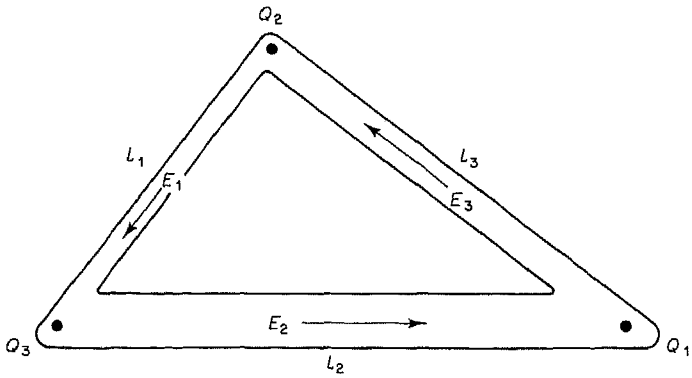

Fig. 1.

2. Model

The physics of three static quarks in the MIT bag model has been discussed by Aerts & Heller (1981). Here we summarise (for completeness) some of the equations in their paper. These quarks are thought of as contained in an open, connected set B of three space. This set B is called a bag. Its surface is called S. With each quark i there is a vector Q_i called the colour charge, having eight components: Q_i^a, $a = 1, 2, \ldots, 8$. The three vectors have the same magnitude, conventionally taken to be

$$Q_i \cdot Q_i = 4/3.$$

In the problems of physical interest the total charge is zero

$$\sum_{i=1}^{3} Q_i = 0$$

and, consequently,

$$Q_i \cdot Q_j = -2/3, \quad i \neq j.$$

Inside the bag is a scalar field ϕ^a of eight real components (the time components of the gluon field). To lowest order in the quark–gluon coupling constant g, ϕ^a is the solution to the Poisson equations

$$-\nabla^2 \phi^a(x) = g \sum_{j=1}^{3} Q_j^a \delta^{(3)}(x - x_j), \tag{1}$$

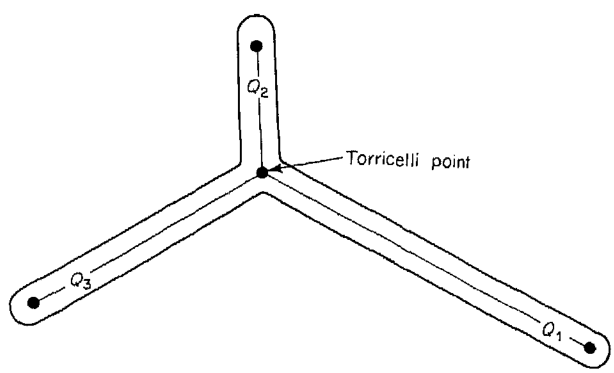

Fig. 2.

(x_j are the locations of the three quarks), satisfying the boundary conditions

$$\hat{n}\,.\,\mathrm{grad}\ \phi^a(x) = 0, \quad \text{on } S. \tag{2a}$$

and

$$\tfrac{1}{2}\sum_{a=1}^{8} |\mathrm{grad}\ \phi^a(x)|^2 = B, \quad \text{on } S. \tag{2b}$$

B is the given bag pressure, which is taken to be constant, and g is the coupling constant.

When the separations of every pair of quarks is large, in terms of $B^{-1/4}$, eqns (1) and (2) can be satisfied (neglecting end-effects) by choosing the bag to consist of a set of connected tubes, with the colour-electric field inside of any tube being parallel to the tube and having a constant magnitude.

In this note we carry out the details of the calculations in the case where the tubes lie along the sides of the triangle formed by the three quarks (see Fig. 1) and show that the energy of the system is the same as that obtained by the "Y" shape with the vertex at the Torricelli point (Fig. 2) which is the point T minimising the sum of the lengths from T to the vertices of the triangle. This point is also called the Steiner point.

3. Reduction of Model to Algebraic Equations

Consider the triangle of tubes shown in Fig. 1 (taken from Fig. 3(a) of Aerts & Heller). Suppose all angles are less than $2\pi/3$. Label the lengths of the tubes of the triangle by l_i as shown in Fig. 1 and let A_i be the corresponding cross-sectional areas. Let E_i^a be the colour-electric field vector in the ith tube with potential ϕ_i^a, so that

$$E_i^a = -\mathrm{grad}\ \phi_i^a.$$

Let u_i be a unit vector pointing in one of the directions along the axis of the ith tube, using the same orientation for all three u_i. The direction selected is that shown in Fig. 1. Define $\bar{E}_i^a$ by

$$E_i^a = \bar{E}_i^a u_i, \quad i = 1, 2, 3,$$

so that $\bar{E}_i^a$ is a real number, positive, negative, or zero. Application of the integrated version of eqn (1) (Gauss's law) at each of the three vertices gives

$$\bar{E}_1^a A_1 - \bar{E}_3^a A_3 = gQ_2^a \tag{3a}$$

$$-\bar{E}_1^a A_1 + \bar{E}_2^a A_2 = gQ_3^a \tag{3b}$$

and

$$\bar{E}_3^a A_3 - \bar{E}_2^a A_2 = gQ_1^a. \tag{3c}$$

Equations (3(a)–(c)) are consistent since the total charge is zero. On applying Faraday's law:

$$\oint E^a\,.\,\mathrm{d}s = -\oint \nabla\phi^a\,.\,\mathrm{d}s = 0 \tag{4}$$

to a path around the triangle, we obtain

$$\bar{E}_1^a l_1 + \bar{E}_2^a l_2 + \bar{E}_3^a l_3 = 0. \tag{5}$$

The algebra for solving these equations is quite trivial. First solve eqns (3) and (5) for the $\bar{E}_i^a$ as functions of the A's and Q^a's. Then square to obtain $\sum_a \bar{E}_i^{a^2}$ as functions of the A's and the inner products of the Q's. Expressing all lengths in terms of the unit $c^{-1/4}$, where $c = 2B/(4g^2/3)$, and all energies in units of $2B/c^{3/4}$, and defining

$$p_i = \frac{l_i}{A_i}, \tag{6}$$

the statement from eqn (2b) and the discussion below it that $\sum_a E_i^{a^2}$ has the same value in all tubes becomes

$$p_1^2(p_2^2+p_2p_3+p_3^2) = l_1^2(p_1+p_2+p_3)^2, \tag{7}$$

$$p_2^2(p_3^2+p_3p_1+p_1^2) = l_2^2(p_1+p_2+p_3)^2, \tag{8}$$

and

$$p_3^2(p_1^2+p_1p_2+p_2^2) = l_3^2(p_1+p_2+p_3)^2. \tag{9}$$

Equations (7)–(9) are the equations to be solved for p_1, p_2, and p_3.

(In the limiting case that an angle of the triangle becomes $2\pi/3$, say, the angle opposite l_2, p_2 becomes infinite and

$$p_1 = l_1, \quad p_3 = l_3.$$

This is also the physically correct solution for angles greater than $2\pi/3$, and the geometry is shown in Fig. 3.)

4. Energy and the Aerts & Heller Formula

The quantity of interest is the energy

$$E = \int \mathrm{d}v\,[\tfrac{1}{2}\sum_a (E^a)^2 + B] = 2BV,$$

since $\frac{1}{2}\sum_a (E^a)^2$ has the same value (B) in all tubes. In the dimensionless units discussed above, this becomes

$$E_\Delta = \sum_{i=1}^{3} l_i A_i = \sum_{i=1}^{3} \frac{l_i^2}{p_i}. \tag{10}$$

The subscript Δ on the energy denotes the fact that this energy applies to the geometry in which the tubes run along the sides of the triangle. E_Δ is now understood to be a function only of the l_i's.

For the alternative geometry in which the tubes assume a Y shape, by again solving eqns (3)–(5) and minimizing the energy Aerts & Heller (1981) show that the common junction is at the Torricelli point, and the energy for this case, denoted E_Y, is given by the sum of the lengths x_i of the Steiner tree associated with the given triangle. For a survey of the theory of Steiner trees, see Gilbert & Pollack (1968). For a triangle with angles less than $2\pi/3$, as shown in Fig. 4, with the interior vertex at the Steiner point, this energy becomes

$$E_Y = x_1 + x_1 + x_3. \tag{11}$$

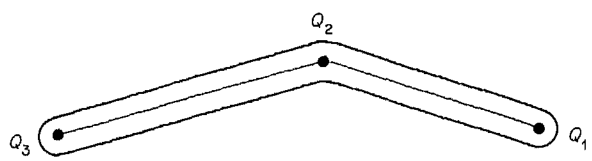

Fig. 3.

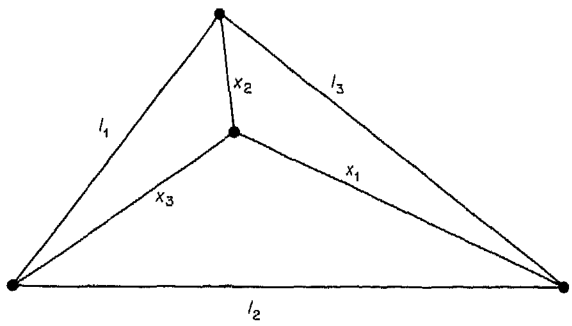

Fig. 4.

5. Verification that $E_\Delta = E_Y$

Referring to Fig. 4, the three angles at the Steiner point are $2\pi/3$. So by the law of cosines,

$$l_1^2 = x_2^2 + x_3^2 + x_2 x_3, \tag{12}$$

$$l_2^2 = x_3^2 + x_1^2 + x_3 x_1, \tag{13}$$

$$l_3^2 = x_1^2 + x_2^2 + x_1 x_2. \tag{14}$$

These equations define the Steiner lengths x_1, x_2, and x_3. They are known to be unique; see Melzak (1961). We now rewrite (7), (8), and (9) in the form

$$l_1^2 = \left(\frac{p_1 p_2}{\sum p_i}\right)^2 + \left(\frac{p_1 p_3}{\sum p_i}\right)^2 + \left(\frac{p_1 p_2}{\sum p_i}\right)\left(\frac{p_1 p_3}{\sum p_i}\right), \tag{15}$$

$$l_2^2 = \left(\frac{p_1 p_2}{\sum p_i}\right)^2 + \left(\frac{p_2 p_3}{\sum p_i}\right)^2 + \left(\frac{p_1 p_2}{\sum p_i}\right)\left(\frac{p_2 p_3}{\sum p_i}\right), \tag{16}$$

$$l_3^2 = \left(\frac{p_1 p_3}{\sum p_i}\right)^2 + \left(\frac{p_2 p_3}{\sum p_i}\right)^2 + \left(\frac{p_1 p_3}{\sum p_i}\right)\left(\frac{p_2 p_3}{\sum p_i}\right). \tag{17}$$

We can make the following association

$$x_3 = \frac{p_1 p_2}{\sum p_i}, \tag{18}$$

$$x_2 = \frac{p_1 p_3}{\sum p_i}, \tag{19}$$

and

$$x_1 = \frac{p_2 p_3}{\sum p_i} \tag{20}$$

between eqns (12)–(14) and (15)–(17). The solution to (18)–(20) for p_i, yields

$$p_i = (x_1 x_2 + x_1 x_3 + x_2 x_3)/x_i, \quad i = 1, 2, 3. \tag{21}$$

Thus, since the eqns (12)–(14) have a unique positive solution for x_i, the eqns (15)–(17)

have a unique positive solution for the p_i. Further, by use of (10)–(14),

$$\begin{aligned} E_\Delta &= \sum \frac{l_i^2}{p_i} = \frac{1}{x_1x_2+x_1x_3+x_2x_3} \sum l_i^2 x_i \\ &= [x_1(x_2^2+x_3^2+x_2x_3)+x_2(x_3^2+x_1^2+x_3x_1) \\ &\quad +x_3(x_1^2+x_2^2+x_1x_2)]/(x_1x_2+x_1x_3+x_2x_3) \\ &= x_1+x_2+x_3 \\ &= E_Y, \end{aligned} \tag{22}$$

which proves the conjecture of Aerts & Heller.

The proof that $E_\Delta = E_Y$ did not require knowledge of the actual length of the Steiner tree. A formula for this length is given in Bottema *et al.* (1969)

$$L = x_1+x_2+x_3 = \left[\tfrac{1}{2}\sum l_i^2 + 2\sqrt{3A}\right]^{\frac{1}{2}}, \tag{23}$$

where A is the area of the triangle

$$A = [s(s-l_1)(s-l_2)(s-l_3)]^{\frac{1}{2}}$$

with

$$s = \tfrac{1}{2}(l_1+l_2+l_3).$$

That (23) is the same as the formula in Aerts & Heller was also verified by MACSYMA.

6. Verification of Aerts–Heller Conjecture by Symbolic Computation

Prior to finding the simple proof given in section 5, we verified the equality $E_\Delta = E_Y$ directly by use of MACSYMA. The direct verification proceeded as follows. Equations (7)–(9) were treated as simultaneous equations for the variables p_i in terms of l_i for $i = 1, 2, 3$. The solution of this system gave a quartic equation which by factorisation yielded two quadratic equations. Finally, the solution of the system (7)–(9) yields algebraic functions for the $p_i(l_1, l_2, l_3)$ involving compositions of square roots and rational functions. These values of p_i are substituted into the expression

$$\begin{aligned} D &\equiv E_\Delta - E_Y \\ &\equiv \sum_{i=1}^{3} \frac{l_i^2}{p_i(l_1, l_2, l_3)} - L, \end{aligned} \tag{24}$$

where L is given by (23). We can select $l_3 \equiv 1$ in (24) because of form of (24). By manipulation of (24) and three squaring operations on MACSYMA requiring about 10 hours of computing time, it is shown that there exists a choice of the branches of the square root functions in (24) so that the value of (24) is zero. The physics of the problem determine which of the branches are relevant. We determined which branches were physically relevant and that it was the physically relevant and only the physically relevant branches which made (24) zero.

7. Relation between the Three-point Steiner Tree and Sides of the Triangle

A relation between x_i and l_i ($1 \leqslant i \leqslant 3$) does not seem to appear in the literature. So it seems worthwhile to give this relation.

THEOREM. *Define three parameters:*

$$\lambda_i = \left(\frac{l_i}{L}\right)^2 - \left(\frac{l_j^2 - l_k^2}{L^2}\right)^2,$$

where i, j, k are a cyclic permutation of the integers 1, 2, *and* 3. *Then*

$$\frac{x_i}{L} = \frac{\lambda_j \lambda_k}{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1},$$

where L is given in eqn (23).

PROOF. It is required to solve eqn (12) and its cyclic permutations, (13) and (14), simultaneously for the x's. Subtracting (13) from (12) gives

$$(x_2 - x_1)L = l_1^2 - l_2^2. \tag{25}$$

Dividing by L^2, squaring, and making use of (14) gives

$$3\frac{x_1}{L}\frac{x_2}{L} = \left(\frac{l_3}{L}\right)^2 - \left(\frac{l_1^2 - l_2^2}{L^2}\right)^2 = \lambda_3, \tag{26}$$

and a cyclic permutation of the indices yields

$$3\frac{x_2}{L}\frac{x_3}{L} = \lambda_1. \tag{27}$$

Dividing (26) by (27) shows that

$$\frac{x_i}{L}\lambda_i = c, \tag{28}$$

where c is the same for all i. Making use of

$$\sum \frac{x_i}{L} = 1 \tag{29}$$

determines

$$c = \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1}, \tag{30}$$

and inserting (30) in (28) proves the theorem.

The work was done with the aid of MACSYMA, a large symbolic manipulation program developed at the MIT Laboratory of Computer Science and supported from 1975 to 1983 by the National Aeronautics and Space Administration under grant NSG 1323, by the Office of Naval Research under grant N00014-77-C-0641, by the U.S. Department of Energy under grant ET-78-C-02-4697, and by the U.S. Air Force under grant F49620-79-C-020, and since 1982 by Symbolics, Inc. of Cambridge, MA.

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