The split decomposition of a tridiagonal pair

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Received 14 December 2006; accepted 17 January 2007
Available online 20 February 2007
Submitted by R.A. Brualdi

Abstract

Let \( \mathbb{K} \) denote a field and let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. We consider a pair of linear transformations \( A : V \to V \) and \( A^* : V \to V \) that satisfy (i)–(iv) below:

(i) Each of \( A, A^* \) is diagonalizable.
(ii) There exists an ordering \( V_0, V_1, \ldots, V_d \) of the eigenspaces of \( A \) such that \( A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \) for \( 0 \leq i \leq d \), where \( V_{-1} = 0, V_{d+1} = 0 \).
(iii) There exists an ordering \( V^*_0, V^*_1, \ldots, V^*_\delta \) of the eigenspaces of \( A^* \) such that \( AV^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \) for \( 0 \leq i \leq \delta \), where \( V^*_{-1} = 0, V^*_\delta+1 = 0 \).
(iv) There is no subspace \( W \) of \( V \) such that both \( AW \subseteq W, A^*W \subseteq W \), other than \( W = 0 \) and \( W = V \).

We call such a pair a tridiagonal pair on \( V \). In this note we obtain two results. First, we show that each of \( A, A^* \) is determined up to affine transformation by the \( V_i \) and \( V^*_i \). Secondly, we characterize the case in which the \( V_i \) and \( V^*_i \) all have dimension one. We prove both results using a certain decomposition of \( V \) called the split decomposition.

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AMS classification: 05E35; 05E30; 33C45; 33D45

Keywords: Leonard pair; Tridiagonal pair; \( q \)-Racah polynomial; Orthogonal polynomial

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doi:10.1016/j.laa.2007.01.028
1. Introduction

Throughout this note \( \mathbb{K} \) will denote a field and \( V \) will denote a vector space over \( \mathbb{K} \) with finite positive dimension. Let \( \text{End}(V) \) denote the \( \mathbb{K} \)-algebra of all \( \mathbb{K} \)-linear transformations from \( V \) to \( V \).

For \( A \in \text{End}(V) \) and for a subspace \( W \subseteq V \), we call \( W \) an eigenspace of \( A \) whenever \( W \neq 0 \) and there exists \( \theta \in \mathbb{K} \) such that \( W = \{ v \in V | Av = \theta v \} \). We say \( A \) is diagonalizable whenever \( V \) is spanned by the eigenspaces of \( A \). We now recall the notion of a tridiagonal pair.

Definition 1.1 ([1]). By a tridiagonal pair on \( V \) we mean an ordered pair of elements \( A, A^* \) taken from \( \text{End}(V) \) that satisfy (i)–(iv) below:

(i) Each of \( A, A^* \) is diagonalizable.
(ii) There exists an ordering \( V_0, V_1, \ldots, V_d \) of the eigenspaces of \( A \) such that
\[
A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),
\]
where \( V_{-1} = 0, V_{d+1} = 0 \).
(iii) There exists an ordering \( V_0^*, V_1^*, \ldots, V_\delta^* \) of the eigenspaces of \( A^* \) such that
\[
AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),
\]
where \( V_{-1}^* = 0, V_{\delta+1}^* = 0 \).
(iv) There is no subspace \( W \) of \( V \) such that both \( AW \subseteq W, A^*W \subseteq W \), other than \( W = 0 \) and \( W = V \).

Note 1.2. It is a common notational convention to use \( A^* \) to represent the conjugate-transpose of \( A \). We are not using this convention. In a tridiagonal pair \( A, A^* \) the linear transformations \( A \) and \( A^* \) are arbitrary subject to (i)–(iv) above.

We refer the reader to [1–3] for background on tridiagonal pairs.

Referring to Definition 1.1 we have \( d = \delta \) [1, Lemma 4.5]; we call this common value the diameter of \( A, A^* \). For \( 0 \leq i \leq d \) the dimensions of \( V_i \) and \( V_i^* \) coincide; we denote this common value by \( \rho_i \), and observe that \( \rho_i \neq 0 \). The sequence \( \rho_0, \rho_1, \ldots, \rho_d \) is symmetric and unimodal; i.e. \( \rho_i = \rho_{d-i} \) for \( 0 \leq i \leq d \) and \( \rho_i = \rho_{i-1} \) for \( 1 \leq i \leq d/2 \) [1, Corollaries 5.7, 6.6]. We call the vector \( (\rho_0, \rho_1, \ldots, \rho_d) \) the shape of \( A, A^* \). By a Leonard pair we mean a tridiagonal pair with shape \( (1, 1, \ldots, 1) \) [5, Definition 1.1]. See [5–7] for background information on Leonard pairs.

In this note we obtain the following two results. Let \( A, A^* \) denote a tridiagonal pair from Definition 1.1. First, we show that each of \( A, A^* \) is determined up to affine transformation by the \( V_i \) and \( V_i^* \). Secondly, we characterize the Leonard pairs among the tridiagonal pairs. We prove both results using a certain decomposition of the underlying vector space called the split decomposition [1, Section 4].

2. The split decomposition

In this section we recall the split decomposition [1, Section 4]. We start with a comment. Referring to Definition 1.1, since \( V_0, V_1, \ldots, V_d \) are the eigenspaces of \( A \) and since \( A \) is diagonalizable we have
\[
V = V_0 + V_1 + \cdots + V_d \quad \text{(direct sum)}.
\]
Similarly
\[ V = V_0^* + V_1^* + \cdots + V_d^* \] (direct sum).  
(2)

For \( 0 \leq i \leq d \) define
\[ U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d). \]  
(3)

By [1, Theorem 4.6],
\[ V = U_0 + U_1 + \cdots + U_d \] (direct sum),  
(4)

and for \( 0 \leq i \leq d \) both
\[ U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^*, \]  
(5)
\[ U_i + U_{i+1} + \cdots + U_d = V_i + V_{i+1} + \cdots + V_d. \]  
(6)

For \( 0 \leq i \leq d \) let \( \theta_i \) (resp. \( \theta_i^* \)) denote the eigenvalue of \( A \) (resp. \( A^* \)) associated with the eigenspace \( V_i \) (resp. \( V_i^* \)). Then by [1, Theorem 4.6] both
\[ (A - \theta_i I) U_i \subseteq U_{i+1}, \]  
(7)
\[ (A^* - \theta_i^* I) U_i \subseteq U_{i-1}, \]  
(8)
where \( U_{-1} = 0 \) and \( U_{d+1} = 0 \). The sequence \( U_0, U_1, \ldots, U_d \) is called the split decomposition of \( V \) [1, Section 4].

3. A subalgebra of \( \text{End}(V) \)

The following subalgebra of \( \text{End}(V) \) will be useful to us. Referring to Definition 1.1, let \( D \) denote the subalgebra of \( \text{End}(V) \) generated by \( A \). In what follows we often view \( D \) as a vector space over \( \mathbb{K} \). The dimension of this vector space is \( d + 1 \) since \( A \) is diagonalizable with \( d + 1 \) eigenspaces. Therefore \( \{ A^i | 0 \leq i \leq d \} \) is a basis for \( D \). There is another basis for \( D \) that is better suited to our purpose. To define it we use the following notation. Let \( \mathbb{K}[\lambda] \) denote the \( \mathbb{K} \)-algebra of all polynomials in an indeterminate \( \lambda \) that have coefficients in \( \mathbb{K} \). For \( 0 \leq i \leq d \) we define \( \tau_i \in \mathbb{K}[\lambda] \) by
\[ \tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}). \]  
(9)

We note that \( \tau_i \) is monic with degree \( i \). Therefore \( \{ \tau_i(A) | 0 \leq i \leq d \} \) is a basis for \( D \). Combining (7) and (9) we find
\[ \tau_i(A) U_0 \subseteq U_i \quad (0 \leq i \leq d). \]  
(10)

The following lemma is a variation on [1, Lemma 6.5]; we give a short proof for the convenience of the reader.

Lemma 3.1. Referring to Definition 1.1, for all nonzero \( u \in V_0^* \) and nonzero \( X \in D \), we have \( X u \neq 0 \).

Proof. It suffices to show that the vector spaces \( D \) and \( Du \) have the same dimension. We saw earlier that \( \{ \tau_i(A) | 0 \leq i \leq d \} \) is a basis for \( D \). We show that \( \{ \tau_i(A)u | 0 \leq i \leq d \} \) is a basis for \( Du \). By (4), (10), and since \( U_0 = V_0^* \), this will hold if we can show \( \tau_i(A)u \neq 0 \) for \( 0 \leq i \leq d \). Let \( i \) be given and suppose \( \tau_i(A)u = 0 \). We will obtain a contradiction by displaying a subspace \( W \)
of $V$ that violates Definition 1.1(iv). Observe that $i \neq 0$ since $\tau_0 = 1$ and $u \neq 0$; therefore $i \geq 1$.

By (9) and since $\tau_i(A)u = 0$ we find $u \in V_0 + V_1 + \cdots + V_{i-1}$, so

$$
u \in V_0^* \cap (V_0 + V_1 + \cdots + V_{i-1}).$$

(11)

Define

$$W_r = (V_0^* + V_1^* + \cdots + V_r^*) \cap (V_0 + V_1 + \cdots + V_{i-r-1})$$

(12)

for $0 \leq r \leq i - 1$ and put

$$W = W_0 + W_1 + \cdots + W_{i-1}. \quad (13)$$

We show $W$ violates Definition 1.1(iv). Observe that $W \neq 0$ since the nonzero vector $u \in W_0$ by (11) and since $W_0 \subseteq W$. Next we show $W \neq V$. By (12), for $0 \leq r \leq i - 1$ we have

$$W_r \subseteq V_0^* + V_1^* + \cdots + V_r^* \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^*.$$

By this and (13) we find

$$W \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^* \subseteq V_0^* + V_1^* + \cdots + V_{d-1}^*.$$

Combining this with (2) and using $V_d^* \neq 0$ we find $W \neq V$. We now show $AW \subseteq W$. To this end, we show that $(A - \theta_{i-r-1}I)W_r \subseteq W_{r+1}$ for $0 \leq r \leq i - 1$, where $W_i = 0$. Let $r$ be given. From the construction we have

$$(A - \theta_{i-r-1}I) \sum_{h=0}^{i-r-1} V_h = \sum_{h=0}^{i-r-2} V_h. \quad (14)$$

By Definition 1.1(iii) we have

$$(A - \theta_{i-r-1}I) \sum_{h=0}^{r} V_h^* \subseteq \sum_{h=0}^{r+1} V_h^*. \quad (15)$$

Combining (14) and (15) we find $(A - \theta_{i-r-1}I)W_r \subseteq W_{r+1}$ as desired. We have shown $AW \subseteq W$. We now show $A^*W \subseteq W$. To this end, we show that $(A^* - \theta_r^*I)W_r \subseteq W_{r-1}$ for $0 \leq r \leq i - 1$, where $W_{-1} = 0$. Let $r$ be given. From the construction we have

$$(A^* - \theta_r^*I) \sum_{h=0}^{r-1} V_h^* = \sum_{h=0}^{r-2} V_h^*. \quad (16)$$

By Definition 1.1(ii) we have

$$(A^* - \theta_r^*I) \sum_{h=0}^{i-r-1} V_h \subseteq \sum_{h=0}^{i-r} V_h. \quad (17)$$

Combining (16) and (17) we find $(A^* - \theta_r^*I)W_r \subseteq W_{r-1}$ as desired. We have shown $A^*W \subseteq W$. We have now shown that $W \neq 0$, $W \neq V$, $AW \subseteq W$, $A^*W \subseteq W$, contradicting Definition 1.1(iv). We conclude $\tau_i(A)u \neq 0$ and the result follows. \Box

4. Each of $A$, $A^*$ is determined by the eigenspaces

Let the tridiagonal pair $A$, $A^*$ be as in Definition 1.1. In this section we show that each of $A$, $A^*$ is determined up to affine transformation by the eigenspaces $V_i$, $V_i^*$. Our main result is based on the following proposition.


Proposition 4.1. Referring to Definition 1.1, assume \( d \geq 1 \). Then the following (i), (ii) are equivalent for all \( X \in \mathrm{End}(V) \).

(i) \( X \in \mathcal{D} \) and \( XV_0^* \subseteq V_0^* + V_1^* \).
(ii) There exist scalars \( r, s \in \mathbb{K} \) such that \( X = rA + sI \).

Proof. (i) \( \Rightarrow \) (ii): Assume \( X \neq 0 \); otherwise the result is trivial. Pick a nonzero \( u \in V_0^* \) and note that \( u \in U_0 \) by (5). We have \( Xu \in V_0^* + V_1^* \) by assumption so

\[
Xu \in U_0 + U_1 \tag{18}
\]

in view of (5). Recall \( \{ \tau_i(A) | 0 \leq i \leq d \} \) is a basis for \( \mathcal{D} \). We assume \( X \in \mathcal{D} \) so there exist \( \alpha_i \in \mathbb{K} \) \((0 \leq i \leq d)\) such that

\[
X = \sum_{i=0}^{d} \alpha_i \tau_i(A). \tag{19}
\]

We show \( \alpha_i = 0 \) for \( 2 \leq i \leq d \). Suppose not and define \( \eta = \max\{i | 2 \leq i \leq d, \alpha_i \neq 0\} \). We will obtain a contradiction by showing

\[
0 \neq U_\eta \cap (U_0 + U_1 + \cdots + U_{\eta-1}). \tag{20}
\]

Note that \( \tau_\eta(A)u \neq 0 \) by Lemma 3.1 and \( \tau_\eta(A)u \in U_\eta \) by (10). Also by (19) we find \( \tau_\eta(A)u \) is in the span of \( Xu \) and \( \tau_0(A)u, \tau_1(A)u, \ldots, \tau_{\eta-1}(A)u \); combining this with (10) and (18) we find \( \tau_\eta(A)u \) is contained in \( U_0 + U_1 + \cdots + U_{\eta-1} \). By these comments \( \tau_\eta(A)u \) is a nonzero element in \( U_\eta \cap (U_0 + U_1 + \cdots + U_{\eta-1}) \) and (20) follows. Line (20) contradicts (4) and we conclude \( \alpha_i = 0 \) for \( 2 \leq i \leq d \). Now \( X = \alpha_1 \tau_1(A) + \alpha_0 I \). Therefore \( X = rA + sI \) with \( r = \alpha_1 \) and \( s = \alpha_0 - \alpha_1 \theta_0 \).

(ii) \( \Rightarrow \) (i): Immediate from Definition 1.1(iii). \( \square \)

The following is our first main theorem.

Theorem 4.2. Let \( A, A^* \) denote a tridiagonal pair on \( V \), with eigenspaces \( V_i, V_i^* \) \((0 \leq i \leq d)\) as in Definition 1.1. Let \( A', A'^* \) denote a second tridiagonal pair on \( V \), with eigenspaces \( V_i', V_i'^* \) \((0 \leq i \leq d)\) as in Definition 1.1. Assume \( V_i = V_i' \) and \( V_i^* = V_i'^* \) for \( 0 \leq i \leq d \). Then \( \text{Span}\{A, I\} = \text{Span}\{A', I\} \) and \( \text{Span}\{A^*, I\} = \text{Span}\{A'^*, I\} \).

Proof. Assume \( d \geq 1 \); otherwise the result is clear. Let \( \mathcal{D} \) (resp. \( \mathcal{D}' \)) denote the subalgebra of \( \text{End}(V) \) generated by \( A \) (resp. \( A' \)). Since \( V_i = V_i' \) for \( 0 \leq i \leq d \) we find \( \mathcal{D} = \mathcal{D}' \) so \( A' \in \mathcal{D} \). Applying Proposition 4.1 to the tridiagonal pair \( A, A^* \) (with \( X = A' \)), there exist \( r, s \in \mathbb{K} \) such that \( A' = rA + sI \). Note that \( r \neq 0 \); otherwise \( A' = sI \) has a single eigenspace which contradicts \( d \geq 1 \). It follows that \( \text{Span}\{A, I\} = \text{Span}\{A', I\} \). Similarly we find \( \text{Span}\{A^*, I\} = \text{Span}\{A'^*, I\} \). \( \square \)

5. A characterization of the Leonard pairs

In this section we obtain a characterization of the Leonard pairs among the tridiagonal pairs. This characterization is based on the notion of the switching element of a Leonard pair [4]. We briefly recall this notion. Let the tridiagonal pair \( A, A^* \) be as in Definition 1.1, and assume the corresponding shape is \((1, 1, \ldots, 1)\) so that \( A, A^* \) is a Leonard pair. For \( 0 \leq i \leq d \) let \( E_i \) denote
the element of $\text{End}(V)$ such that $(E_i - \delta_{ij}I) V_j = 0$ for $0 \leq j \leq d$. We observe (i) $E_i E_j = \delta_{ij}E_i$ ($0 \leq i, j \leq d$); (ii) $\sum_{i=0}^{d} E_i = I$; (iii) $A = \sum_{i=0}^{d} \theta_i E_i$. We further observe that $E_0, E_1, \ldots, E_d$ is a basis for $D$. We define

$$S = \sum_{r=0}^{d} \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r,$$

where $\varphi_1, \varphi_2, \ldots, \varphi_d$ (resp. $\phi_1, \phi_2, \ldots, \phi_d$) is the first split sequence (resp. second split sequence) for $A, A^*$ [5, Definitions 3.10, 3.12]. The element $S$ is called the switching element for $A, A^*$ [4, Definition 5.1].

The switching element has the following property.

**Theorem 5.1** ([4, Theorem 6.7]). Let $A, A^*$ denote a Leonard pair on $V$ and let $S$ denote the switching element for $A, A^*$. Let $V_i, V_i^*$ be as in Definition 1.1, and let $\mathcal{D}$ denote the subalgebra of $\text{End}(V)$ generated by $A$. Then for $X \in \text{End}(V)$ the following (i), (ii) are equivalent.

(i) $X$ is a scalar multiple of $S$.
(ii) $X \in \mathcal{D}$ and $X V_0^* \subseteq V_d^*$.

The following is our second main result.

**Theorem 5.2.** Referring to Definition 1.1, let $\mathcal{D}$ denote the subalgebra of $\text{End}(V)$ generated by $A$. Then the following (i), (ii) are equivalent.

(i) There exists a nonzero $X \in \mathcal{D}$ such that $X V_0^* \subseteq V_d^*$.
(ii) The pair $A, A^*$ is a Leonard pair.

**Proof.** (i) $\Rightarrow$ (ii): Recall $\{\tau_i(A)|0 \leq i \leq d\}$ is a basis for $\mathcal{D}$. Therefore there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_d$ in $\mathbb{K}$, not all zero, such that

$$X = \sum_{i=0}^{d} \alpha_i \tau_i(A). \tag{21}$$

Fix a nonzero $u \in V_0^*$. Then $Xu \in V_d^*$ so $(A^* - \theta_d^* I) Xu = 0$. In this equation we evaluate $X$ using (21), rearrange terms, and use $A^* u = \theta_0^* u$ to find

$$0 = (A^* - \theta_d^* I) Xu = \sum_{i=0}^{d} \alpha_i (A^* - \theta_d^* I) \tau_i(A) u + \sum_{i=0}^{d} \alpha_i (A^* - \theta_i^* I) \tau_i(A) u = \sum_{i=0}^{d} \alpha_i \theta_i^* \tau_i(A) u + \sum_{i=0}^{d} \alpha_i (A^* - \theta_i^* I) \tau_i(A) u = \sum_{i=0}^{d} \alpha_i \theta_i^* \tau_i(A) u + \alpha_i+1(A^* - \theta_i^* I) \tau_{i+1}(A) u.$$
In the above line, for $0 \leq i \leq d - 1$ the summand at $i$ is contained in $U_i$ in view of (8) and (10), so this summand is 0 in view of (4). Therefore

$$\alpha_i \left( \theta_i - \theta_d \right) \tau_i(A) u + \alpha_{i+1} \left( A^* - \theta_{i+1} \right) \tau_{i+1}(A) u = 0 \quad (0 \leq i \leq d - 1).$$

(22)

Suppose for the moment that there exists an integer $i$ ($0 \leq i \leq d - 1$) such that $\alpha_{i+1} = 0$. Then $\alpha_i \left( \theta_i - \theta_d \right) \tau_i(A) u = 0$ by (22). But $\tau_i(A) u \neq 0$ by Lemma 3.1 and $\theta_i \neq \theta_d$ since $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, so $\alpha_i = 0$. Therefore $\alpha_{i+1} = 0$ implies $\alpha_i = 0$ for $0 \leq i \leq d - 1$. By this and since $\alpha_0, \alpha_1, \ldots, \alpha_d$ are not all zero, there exists an integer $j$ ($0 \leq j \leq d$) such that $\alpha_i = 0$ for $0 \leq i \leq j - 1$ and $\alpha_i \neq 0$ for $j \leq i \leq d$. Define

$$W = \text{Span}\{ \tau_j(A) u, \tau_{j+1}(A) u, \ldots, \tau_d(A) u \}.$$

We show that $W$ is nonzero and invariant under each of $A, A^*$. Observe that $W \neq 0$ since $W$ contains $\tau_d(A) u$, and this vector is nonzero by Lemma 3.1. Observe that $AW \subseteq W$, since using (9) we find $(A - \theta_I) \tau_i(A) u = \tau_{i+1}(A) u$ for $j \leq i \leq d - 1$ and $(A - \theta_d) \tau_d(A) u = 0$. Observe that $A^* W \subseteq W$, since by (22) the product $(A^* - \theta_i I) \tau_i(A) u = 0$ for $i = j$ and a scalar multiple of $\tau_{i-1}(A) u$ for $j + 1 \leq i \leq d$. We have now shown that $W$ is nonzero and invariant under each of $A, A^*$. Therefore $W = V$ in view of Definition 1.1(iv). We can now easily show that $A, A^*$ is a Leonard pair. By construction the dimension of $W$ is at most $d + 1$. Also, using (1) the dimension of $V$ is $\sum_{i=0}^d \rho_i$, so this dimension is at least $d + 1$ with equality if and only if $\rho_i = 1$ for $0 \leq i \leq d$. By these comments and since $W = V$ we find $V$ has dimension $d + 1$ and $\rho_i = 1$ for $0 \leq i \leq d$. Therefore the pair $A, A^*$ is a Leonard pair.

(ii) $\Rightarrow$ (i): Apply Theorem 5.1 with $X = S$, where $S$ is the switching element of $A, A^*$. $\square$

References