PERGAMON

Applied Mathematics Letters 15 (2002) 585–590

Applied Mathematics Letters

www.elsevier.com/locate/aml

A Remark on Nonexistence of Global Solutions to Quasi-Linear Hyperbolic and Parabolic Equations

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(Received July 2000; revised and accepted June 2001)

Communicated by C. Bardos

Abstract—Sufficient conditions for global nonexistence of solutions of initial value problems for a class of second-order quasi-linear hyperbolic and parabolic equations are given.

Keywords—Quasi-linear equations, Global nonexistence.

INTRODUCTION

We consider the initial boundary value problems for the following quasi-linear hyperbolic equation:

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(d_0 + |\nabla u|^{m-2}\right)\frac{\partial u}{\partial x_i}\right) + h(u, \nabla u) = f(u), \qquad x \in \Omega, \quad t > 0, \tag{H}$$

and the following quasi-linear parabolic equation:

$$u_{t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(d_0 + |\nabla u|^{m-2}\right)\frac{\partial u}{\partial x_i}\right) + h(u, \nabla u) = f(u), \qquad x \in \Omega, \quad t > 0, \tag{P}$$

where $m \geq 2$ is a given number, h and f are continuous functions, and Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$. We assume that

$$(f(u), u) \geq 2(\alpha + 1)G(u), \quad \forall\, u \in L_\infty(\Omega), \qquad \alpha > 0, \quad G(u) = \int_\Omega \left(\int_0^u f(s)\,ds\right) dx, \tag{F}$$

and

$$|h(u,p)| \leq C\left(|u|^{m/2} + |p|^{m/2}\right), \qquad C > 0, \quad \forall\, u \in R^1, \quad \forall\, p \in R^n. \tag{G}$$

Here and below $(.,.)$ is the $L_2(\Omega)$ inner product and $||.||$ is the $L_2(\Omega)$ norm. Our aim is to find sufficient conditions for global nonexistence of solutions to initial boundary value problems for equations (P) and (H). Global nonexistence theorems for equations (P) and (H), when $h \equiv 0$

0893-9659/02/\$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX
PII: S0893-9659(01)00176-8

and $f(u) = u^{q+1}$, $q > 1$ is proven in [1] (see also [2]). Sufficient conditions of global nonexistence of solutions to equations (P) and (H) when $m = 2$ are obtained in [3]. A global nonexistence theorem for (P) when h satisfies the condition

$$|h(u,p)| \le C(|u| + |p|), \quad \forall u \in R^1, \quad \forall p \in R^n, \quad C > 0 \tag{G$_1$}$$

is established in [4].

HYPERBOLIC EQUATION

Consider the problem

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(d_0 + |\nabla u|^{m-2}\right)\frac{\partial u}{\partial x_i}\right) + h(u, \nabla u) = f(u), \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{3}$$

We assume that in Theorems 1 and 2, u_0, u_1, h, and f are sufficiently smooth functions, so that problem (1)–(3) has a local in time strong solution (about the local existence theorems; see, for example, [5,6]). Let us note that our results are also true for the corresponding weak solutions of (1)–(3) with $u_0 \in W_0^{m,1}(\Omega)$ and $u_1 \in L_2(\Omega)$ and the weak solutions of (15)–(17) with $u_0 \in W_0^{m,1}$.

THEOREM 1. *Let u be the solution of problem (1)–(3). Assume that h satisfies condition (G), f satisfies condition (F), and the following conditions are valid:*

$$\alpha > \frac{m-2}{2}, \quad m \ge 2, \qquad \|u_0\|^2 > 0, \quad (u_1, u_0) > \left(\sqrt{\alpha+2} + 1 + \alpha\right)\lambda\frac{\|u_0\|^2}{\alpha},$$

where

$$\lambda = \sqrt{\frac{C^2(1+\lambda_m)m}{2(2+\alpha)(2\alpha - m + 2)}}, \qquad \lambda_m = \inf_{w \in W_0^{1,m}(\Omega)} \frac{\int_\Omega |\nabla w(x)|^m \, dx}{\int_\Omega |w(x)|^m \, dx},$$

$$I(0) = -\frac{1}{2}\|-\lambda u_0 + u_1\|^2 - \frac{1}{m}\int_\Omega |\nabla u_0|^m \, dx - \frac{\lambda^2}{2}\|u_0\| - \frac{\lambda}{2}\|\nabla u_0\|^2 + G(u_0) \ge 0.$$

Then $\|u(.,t)\| \to \infty$ as $t \to t_1$, where

$$t_1 \le t_2 = \frac{1}{2\sqrt{\alpha+2}\lambda} \ln \frac{\left(\sqrt{\alpha+2} - 1 - \alpha\right)\lambda\|u_0\|^2 + \alpha(u_1,u_0)}{\left(-\sqrt{\alpha+2} - 1 - \alpha\right)\lambda\|u_0\|^2 + \alpha(u_1,u_0)}.$$

PROOF. In order to prove this theorem, we will use the following lemma.

LEMMA 1. *(See [3].) Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies on $t \ge 0$ the inequality*

$$\Psi''\Psi - (1+\alpha_1)(\Psi')^2 \ge -2M_1\Psi\Psi' - M_2\Psi^2,$$

where $\alpha_1 > 0$, $M_1, M_2 \ge 0$. If $\Psi(0) > 0$, $\Psi'(0) > -\gamma_2\alpha_1^{-1}\Psi(0)$, and $M_1 + M_2 > 0$, then $\Psi(t)$ tends to infinity as

$$t \to t_1 \le t_2 = \frac{1}{2\sqrt{M_1^2 + \alpha_1 M_2}} \ln \frac{\gamma_1\Psi(0) + \alpha_1\Psi'(0)}{\gamma_2\Psi(0) + \alpha_1\Psi'(0)},$$

where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \alpha_1 M_2}$. If $\Psi(0) > 0$, $\Psi'(0) > 0$, and $M_1 = M_2 = 0$, then $\Psi(t) \to \infty$ as $t \to t_1 \le t_2 = \Psi(0)/\alpha_1\Psi'(0)$.

Let us consider the function $v(x,t) = e^{-\lambda t}u(x,t)$, where u is the solution of problem (1)–(3). Then we can rewrite problem (1)–(3) as follows:

$$\begin{gathered} v_{tt} + 2\lambda v_t + \lambda^2 v - d_0 \Delta v - e^{(m-2)\lambda t} \operatorname{div}\left(|\nabla v|^{m-2} \nabla v\right) + \tilde{h}(t, v, \nabla v) = \tilde{f}(t, v), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in \Omega, \\ v(x,t) = 0, \qquad x \in \partial\Omega. \end{gathered} \tag{4}$$

Here $\tilde{f}(t,v) = e^{-\lambda t} f(e^{\lambda t} v)$, $\tilde{h}(t,v,\nabla v) = e^{-\lambda t} h(e^{\lambda t} v, e^{\lambda t} \nabla v)$. By using (F),(G), we can easily see that $\tilde{f}$ and $\tilde{G}(t,v) = e^{-2\lambda t} G(e^{\lambda t} v)$ satisfy

$$\left|\tilde{h}(t,v,\nabla v)\right| \le C e^{((m-2)/2)\lambda t} \left(|v|^{m/2} + |\nabla v|^{m/2}\right), \tag{5}$$

$$\left(\tilde{f}(t,v), v\right) \ge 2(\alpha+1)\tilde{G}(t,v), \tag{6}$$

$$\frac{d}{d\tau}\tilde{G}\left(t, v(.,\tau)\right) = \left(\tilde{f}(t,v), v_\tau\right), \tag{7}$$

$$\frac{d}{dt}\tilde{G}(t, v(.,t)) = \tilde{G}_t\left(t, v(t)\right) + \left(\tilde{f}(t,v), v_t\right), \tag{8}$$

$$\tilde{G}_t(t,v) \ge 2\lambda\alpha\tilde{G}(t,v). \tag{9}$$

Taking $L_2(\Omega)$ inner product of both sides of (4) with v_t and using the relations

$$(\Delta v, v_t) = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2\,dx \qquad \text{and} \qquad \int_\Omega \operatorname{div}\left(|\nabla v|^{(m-2)}\nabla v\right) v_t\,dx = -\frac{1}{m}\frac{d}{dt}\int_\Omega |\nabla v|^m\,dx,$$

we get

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\|v_t\|^2 + 2\lambda\|v_t\|^2 + \frac{\lambda^2}{2}\frac{d}{dt}\|v\|^2 + \frac{d_0}{2}\frac{d}{dt}\|\nabla v\|^2 \\ + \frac{e^{(m-2)\lambda t}}{m}\frac{d}{dt}\int_\Omega |\nabla v|^m\,dx + \left(\tilde{h}(t,v,\nabla v), v_t\right) = \left(\tilde{f}(t,v), v_t\right). \end{gathered} \tag{10}$$

By using (5),(8), we can obtain from (10), the following inequality:

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\|v_t\|^2 + \frac{\lambda^2}{2}\frac{d}{dt}\|v\|^2 + \frac{d_0}{2}\frac{d}{dt}\|\nabla v\|^2 \\ + \frac{e^{(m-2)\lambda t}}{m}\frac{d}{dt}\int_\Omega |\nabla v|^m\,dx - \frac{d}{dt}\tilde{G}(t,v) \le -2\lambda\|v_t\|^2 + 2\epsilon_0\|v_t\|^2 \\ + \frac{C^2}{4\epsilon_0}e^{(m-2)\lambda t}\int_\Omega |v|^m\,dx + \frac{C^2 e^{(m-2)\lambda t}}{4\epsilon_0}\int_\Omega |\nabla v|^m\,dx - \tilde{G}_t(t,v). \end{gathered}$$

By using the Poincaré inequality

$$\int_\Omega |v|^m\,dx \le \lambda_m \int_\Omega |\nabla v|^m\,dx$$

and (9) in the last inequality, we get

$$\frac{d}{dt}I(t) \ge 2(\lambda-\epsilon_0)\|v_t\|^2 - \left[\frac{C^2}{4\epsilon_0}(1+\lambda_m) + \frac{m-2}{m}\lambda\right] e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx + 2\alpha\lambda\tilde{G}(t,v), \tag{11}$$

where

$$I(t) = -\frac{1}{2}\|v_t\|^2 - \frac{\lambda^2}{2}\|v\|^2 - \frac{d_0}{2}\|\nabla v\|^2 - \frac{e^{(m-2)\lambda t}}{m}\int_\Omega |\nabla v|^m\,dx + \tilde{G}(t,v).$$

Equation (11) implies the following inequality:

$$\begin{aligned}\frac{d}{dt}I(t) \geq 2\alpha\lambda \left[-\frac{1}{2}\|v_t\|^2 - \frac{\lambda^2}{2}\|v\|^2 - \frac{d_0}{2}\|\nabla v\|^2 - \frac{e^{(m-2)\lambda t}}{m}\int_\Omega |\nabla v|^m\,dx + \tilde{G}(t,v)\right] \\ + [\alpha\lambda + 2(\lambda - \epsilon_0)]\|v_t\|^2 + \alpha\lambda d_0\|\nabla v\|^2 \\ + \left[\frac{2\alpha\lambda}{m} - \frac{\lambda(m-2)}{m} - \frac{C^2}{4\epsilon_0}(1+\lambda_m)\right] e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx.\end{aligned} \tag{12}$$

Choosing in (12),

$$\epsilon_0 = \frac{\lambda(\alpha+2)}{2} \qquad \text{and} \qquad \lambda = \sqrt{\frac{C^2(1+\lambda_m)m}{2(2+\alpha)(2\alpha - m + 2)}},$$

we obtain $\frac{d}{dt}I(t) \geq 2\alpha\lambda I(t)$, that is, $\frac{d}{dt}\{e^{-2\alpha\lambda t}I(t)\} \geq 0$. Thus, we have $I(t) \geq e^{2\alpha\lambda t}I(0)$. Since $I(0) \geq 0$, we see that $I(t) \geq 0$, $\forall\, t > 0$.

Let us multiply equation (4) in $L_2(\Omega)$ with v,

$$(v_{tt},v) + 2\lambda(v_t,v) + \lambda^2(v,v) - d_0(\Delta v, v) - e^{(m-2)\lambda t}\left(\operatorname{div}|\nabla v|^{m-2}\nabla v, v\right) + \tilde{h}((t,v,\nabla v),v) = \tilde{f}((t,v),v).$$

By using the equalities

$$\int_\Omega \operatorname{div}\left(|\nabla v|^{m-2}\nabla v\right) v\,dx = -\int_\Omega |\nabla v|^m\,dx, \qquad \int_\Omega v\Delta v\,dx = -\int_\Omega |\nabla v|^2\,dx,$$

and conditions (5),(6), we can easily get

$$\begin{aligned}(v_{tt},v) \geq -\lambda\frac{d}{dt}\|v\|^2 - \lambda^2\|v\|^2 - d_0\|\nabla v\|^2 - e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx \\ - Ce^{((m-2)\lambda t)/2}\int_\Omega |\nabla v|^{m/2} v\,dx - Ce^{((m-2)\lambda t)/2}\int_\Omega |v|^{m/2} v\,dx + \left(\tilde{f}(t,v),v\right),\end{aligned}$$

or

$$\begin{aligned}(v_{tt},v) \geq -\lambda\frac{d}{dt}\|v\|^2 - \lambda^2\|v\|^2 - d_0\|\nabla v\|^2 - e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx \\ -\frac{C^2}{4\epsilon_1}e^{(m-2)\lambda t}\int_\Omega |v|^m\,dx - 2\epsilon_1\|v\|^2 - \frac{C^2e^{(m-2)\lambda t}}{4\epsilon_1}\int_\Omega |\nabla v|^m\,dx + 2(\alpha+1)\tilde{G}(t,v).\end{aligned} \tag{13}$$

By using the Poincaré inequality, we obtain from (13), the following inequality:

$$\begin{aligned}(v_{tt},v) \geq -\lambda\frac{d}{dt}\|v\|^2 - \lambda^2\|v\|^2 - d_0\|\nabla v\|^2 \\ -\left[1 + (1+\lambda_m)\frac{C^2}{4\epsilon_1}\right] e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx - 2\epsilon_1\|v\|^2 + 2(\alpha+1)\tilde{G}(t,v),\end{aligned}$$

$$\begin{aligned}(v_{tt},v) - (\alpha+1)\|v_t\|^2 \geq 2(\alpha+1)\left[-\frac{1}{2}\|v_t\|^2 - \frac{\lambda^2}{2}\|v\|^2 - \frac{d_0}{2}\|\nabla v\|^2\right. \\ \left.-\frac{e^{(m-2)\lambda t}}{m}\int_\Omega |\nabla v|^m\,dx + \tilde{G}(t,v)\right] + \left(\alpha\lambda^2 - 2\epsilon_1\right)\|v\|^2 + \alpha\, d_0\|\nabla v\|^2 \\ + \left[\frac{2(\alpha+1)}{m} - (1+\lambda_m)\frac{C^2}{4\epsilon_1} - 1\right] e^{(m-2)\lambda t}\int_\Omega |\nabla v|^m\,dx - \lambda\frac{d}{dt}\|v\|^2,\end{aligned}$$

$$(v_{tt},v) - (\alpha+1)\|v_t\|^2 \geq 2(\alpha+1)I(t) - 2\lambda^2\|v\|^2 - \lambda\frac{d}{dt}\|v\|^2,$$

where $\epsilon_1 = m(1+\lambda_m)C^2/(8(\alpha+1)-4m)$. We know that $I(t) \geq 0$. So we have the following inequality:

$$(v_{tt}, v) - (\alpha+1)\|v_t\|^2 \geq -2\lambda^2\|v\|^2 - \lambda\frac{d}{dt}\|v\|^2. \tag{14}$$

Let us show that the function $\Psi(t) = \|v(.,t)\|^2$ satisfies all hypotheses of Lemma 1 with $\alpha_1 = \alpha/2$. Since $\Psi'(t) = 2(v, v_t)$ and $\Psi''(t) = 2\|v_t\|^2 + 2(v_{tt}, v)$, we have

$$\begin{aligned}\Psi''(t)\Psi(t) - (\alpha_1+1)(\Psi'(t))^2 &= \left[2\|v_t\|^2 + 2(v_{tt}, v)\right]\|v\|^2 - 4(\alpha_1+1)(v_t, v)^2 \\ &= 4(\alpha_1+1)\left[\|v_t\|^2\|v\|^2 - (v_t, v)^2\right] + 2\Psi(t)\left[(v_{tt}, v) - (2\alpha_1+1)\|v_t\|^2\right].\end{aligned}$$

Due to (14) and the Schwarz inequality, we get

$$\Psi''(t)\Psi(t) - (1+\alpha_1)(\Psi'(t))^2 \geq -4\lambda^2\Psi^2(t) - 2\lambda\Psi'(t)\Psi(t).$$

$\Psi'(0) > -\gamma_2\alpha_1^{-1}\Psi(0)$ is also true, since $(u_1, u_0) > (\sqrt{\alpha+2}+1+\alpha)\lambda(\|u_0\|^2/\alpha)$, that is, conditions of Lemma 1 are satisfied. So $\Psi(t)$ tends to infinity as t tends to t_1.

PARABOLIC EQUATION

Consider the problem

$$u_t - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left(d_0 + |\nabla u|^{m-2}\right)\frac{\partial u}{\partial x_i}\right) + h(u, \nabla u) = f(u), \tag{15}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{16}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{17}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$.

THEOREM 2. *Suppose that conditions (F) and (G) are satisfied for the functions f. Let u be the solution of problem (15)–(17). Assume that the following conditions are valid:*

$$\alpha > \frac{m-2}{2}, \qquad m \geq 2, \qquad d_0 > 0,$$

$$I(0) \equiv -\frac{d_0}{2}\|u_0\|^2 - \frac{1}{m}\int_\Omega |\nabla u_0|^m\,dx - \frac{\lambda}{2}\|u_0\|^2 + G(u_0) > 0,$$

$$\lambda = \frac{C^2(1+\lambda_m)(1+\alpha)m}{2(\alpha-\beta)(2\alpha+2-m)}, \qquad \beta \in (0,\alpha).$$

Then $\int_0^t \|u(.,s)\|_{L_2(\Omega)}\,ds \to \infty$ as

$$t \to t_1 \leq t_2 = \frac{2\alpha+2-m}{(1+\lambda m)mC^2}\ln\frac{K_1}{K_2},$$

where

$$K_1 = 4(2\alpha+2-m)\left(\sqrt{1+\beta}-1\right)^2(\alpha+1)I(0),$$

$$K_2 = K_1 - (1+\beta)(1+\lambda_m)C^2m\|u_0\|^2 > 0, \qquad \|u_0\| > 0.$$

PROOF. The proof of this theorem is similar to the proof of Theorem 1. Here we prove that if u is a solution of problem (15)–(17) then the function $\Psi(t) = \int_0^t \|u(.,s)\|^2\,ds + c_1$, with suitable chosen c_1 satisfies the conditions of Lemma 1.

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