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Modified Mixed Tsirelson Spaces

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We study the modified and boundedly modified mixed Tsirelson spaces $T_M[(\mathscr{F}_{k_n}, \theta_n)_{n=1}^{\infty}]$ and $T_{M(s)}[(\mathscr{F}_{k_n}, \theta_n)_{n=1}^{\infty}]$, respectively, defined by a subsequence $(\mathscr{F}_{k_n})$ of the sequence of Schreier families $(\mathscr{F}_n)$. These are reflexive asymptotic ℓ_1 spaces with an unconditional basis $(e_i)_i$ having the property that every sequence $\{x_i\}_{i=1}^{n}$ of normalized disjointly supported vectors contained in $\langle e_i \rangle_{i=n}^{\infty}$ is equivalent to the basis of ℓ_1^n. We show that if $\lim \theta_n^{1/n} = 1$ then the space $T[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$ and its modified variations $T_M[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$ or $T_{M(s)}[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$ are totally incomparable by proving that c_0 is finitely disjointly representable in every block subspace of $T[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$. Next, we present an example of a boundedly modified mixed Tsirelson space $X_{M(1),u} = T_{M(1)}[(\mathscr{F}_{k_n}, \theta_n)_{n=1}^{\infty}]$ which is arbitrarily distortable. Finally, we construct a variation of the space $X_{M(1),u}$ which is hereditarily indecomposable. © 1998 Academic Press

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INTRODUCTION

Given a sequence $(\mathcal{M}_k)_{k=1}^{\infty}$ of compact families of finite subsets of $\mathbb{N}$ and a sequence $(\theta_k)_{k=1}^{\infty}$ of reals converging to zero, the mixed Tsirelson space $T[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ is defined as follows.

$T[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ is the completion of the linear space c_{00} of the sequences which are eventually zero under the norm $\|\cdot\|$ defined by the following implicit formula: For $x \in c_{00}$,

$$\|x\| = \left\{ \|x\|_{\infty}, \sup_k \theta_k \sup \left\{ \sum_{i=1}^{n} \|E_i x\| : n \in \mathbb{N}, (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-admissible} \right\} \right\}. \tag{1}$$

Here, for $E \subset \mathbb{N}$, $\|Ex\|$ is the restriction of the vector x on the set E and, for a family $\mathcal{M}$ of subsets of $\mathbb{N}$, an *$\mathcal{M}$-admissible* sequence is a sequence $(E_i)_{i=1}^{n}$ of *successive* subsets of $\mathbb{N}$ such that the set $\{\min E_1, ..., \min E_n\}$ belongs to $\mathcal{M}$. Mixed Tsirelson spaces were introduced in [3]. However, this class includes the previously constructed Schlumprecht's space [16] which initiated a series of results answering fundamental and longstanding problems of the theory of Banach spaces. The remarkable nonlinear transfer by Odell and Schlumprecht [13] of the biorthogonal asymptotic sets from Schlumprecht's space to ℓ_p, $1 < p < \infty$, which settled the distortion problem, indicates the impact of the new spaces on the understanding of the classical Banach spaces. On the other hand, these new norms led to the discovery of the class of hereditarily indecomposable (H.I.) spaces [9], that is, spaces with the property that no subspace can be written as a topological direct sum of two infinite dimensional closed subspaces. As it was proved by Gowers [8], the H.I. property is a consequence of the absence of unconditionality in the sense that every Banach space which does not contain any unconditional basic sequence has an H.I. subspace. Gowers and Maurey [9] have proved that the H.I. spaces have small spaces of operators; it is a fundamental open problem whether there exists such a space with the property that every bounded linear operator $T: X \to X$ is of the form $T = \lambda I + K$ where K is a compact operator. On the other hand, a recent result of Argyros and Felouzis [4] shows that a large class of Banach spaces that includes ℓ_p, $1 < p < \infty$, are quotients of H.I. spaces.

In the present paper we study variations of mixed Tsirelson spaces which we call modified mixed Tsirelson spaces. Given a family $\mathcal{M}$ of finite subsets of $\mathbb{N}$, a sequence $(E_i)_{i=1}^{n}$ of subsets of $\mathbb{N}$ is called *$\mathcal{M}$-allowable* if the sets E_i are *disjoint* and the set $\{\min E_1, ..., \min E_n\}$ belongs to $\mathcal{M}$. The *modified mixed Tsirelson space* X_M corresponding to the mixed Tsirelson space

$X=T[(\mathcal{M}_k,\theta_k)_{k=1}^{\infty}]$ is the Banach space whose norm $\|\cdot\|$ satisfies the implicit equation

$$\|x\|=\max\left\{\|x\|_{\infty},\sup_k \theta_k \sup\left\{\sum_{i=1}^{n}\|E_i x\| : n\in\mathbb{N}, (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-allowable}\right\}\right\}. \tag{2}$$

We also consider *boundedly modified mixed Tsirelson spaces* that lie between X and X_M. Such a space is denoted by $X_{M(s)}$, for some $s\in\mathbb{N}$, and its norm is given by an implicit formula analogous to (1) or (2) where the inner "sup" is taken over all $\mathcal{M}_k$-allowable families for $1\leqslant k\leqslant s$ and over all $\mathcal{M}_k$-admissible families for $k\geqslant s+1$. It is clear that the modified and boundedly modified mixed Tsirelson spaces which are defined by a subsequence $\mathcal{M}_k=\mathcal{F}_{n_k}$ of the sequence of Schreier families $(\mathcal{F}_n)_n$ have the property that, for every n, every normalized sequence $(x_i)_{i=1}^n$ of n disjointly supported vectors with supports contained in $[n,\infty)$ is θ_1-equivalent to the basis of ℓ_1^n.

The modified Tsirelson space T_M was introduced by W. B. Johnson [10] shortly after Tsirelson's discovery [19]. Later, P. Casazza and E. Odell [6] proved that the modified Tsirelson space is isomorphic to the original one. The use of the modified version of the norm in the 2-convexification of T is crucial for the proof of the fact that it is a weak Hilbert space. The relation between modified mixed Tsirelson norms and the corresponding mixed Tsirelson norms is in general quite different from the one between T and T_M. To explain the situation we restrict our attention to the two main examples of mixed Tsirelson norms.

The first is Schlumprecht's space S [16] defined by $\mathcal{M}_k=\mathcal{A}_k=\{A\subset\mathbb{N}\colon \#A\leqslant k\}$, and $\theta_k=1/\log_2(k+1)$. The second is the space X introduced by Argyros and Deliyanni in [3], defined by a certain subsequence $(\mathcal{F}_{n_k})_{k\in\mathbb{N}}$ of the sequence of Schreier families $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and an appropriate sequence $(\theta_k)_{k\in\mathbb{N}}$. It is known that c_0 is finitely representable in every infinite dimensional subspace of S and we show here that the same holds true for X. From this we easily see that the modified versions S_M, X_M are totally incomparable to S and X, respectively. Schlumprecht observed further that although his space S is reflexive, the space S_M contains ℓ_1 [17]. On the other hand, as we show here, the space X_M remains reflexive and contains no ℓ_p. This is the first property where we do not have an analogy between S and X. The result is somehow unexpected since X_M, being an asymptotic ℓ_1 space, has richer ℓ_1 structure than S_M. These results raise naturally certain questions related to the structure of S_M and X_M. For example, it is not known if S_M is ℓ_1-saturated or if X_M is arbitrarily distortable.

The results mentioned above are presented in Section 1. More precisely, we prove that if $\lim \theta_n^{1/n}=1$, then c_0 is finitely representable in every finite dimensional subspace of the space $T[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$. Next, for an arbitrary null sequence $(\theta_n)_n$, we show that the modified mixed Tsirelson space $T_M[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$ is reflexive. As a consequence we get that the 2-convexifications of such spaces yield weak Hilbert spaces not containing ℓ_2 and totally incomparable to $T^{(2)}$.

In Section 2 we consider a boundedly modified mixed Tsirelson space of the form $X_{M(1),u} = T_{M(1)}[(\mathcal{F}_{k_n}, \theta_n)_{n=1}^{\infty}]$ for a suitable choice of $(\mathcal{F}_{k_n})$ and (θ_n). We show that this space is arbitrarily distortable. This result is related to the question: Does there exist a distortable Banach space of bounded distortion? By [11, 12, 18] such a space must contain an asymptotic ℓ_p subspace with an unconditional basis which contains ℓ_1^n's uniformly; so the search turns to asymptotic ℓ_1 spaces with an unconditional basis. By [3] (also [2]), the class of spaces $T[(\mathcal{F}_n, \theta_n)_n]$ provides examples of such spaces which are arbitrarily distortable. However, it is not known whether the original representative of this class, Tsirelson's space T, is arbitrarily distortable, or whether it contains an arbitrarily distortable subspace. The space $X_{M(1),u}$ constructed here is closer to T than $T[(\mathcal{F}_n, \theta_n)_n]$, in the sense that it has more homogeneous ℓ_1 structure.

In Section 3 we construct a space X based on $X_{M(1),u}$ which is hereditarily indecomposable. The basic idea for the definition of X comes from [9].

The strategy in proving these results is similar to the one followed in [3]. We briefly explain the idea. In order to prove that $X_{M(1),u}$ is arbitrarily distortable, we start with a set $K = \bigcup_{j=1}^{\infty} \mathcal{A}_j$ of functionals which define the norm of the space. Each set $\mathcal{A}_j$ contains functionals of the form $\theta_j \sum_{l=1}^{n} f_l$ where the $\{f_l\}_{l=1}^{n}$ are disjointly supported functionals in the dual ball and the family $\{\operatorname{supp} f_l\}_{l=1}^{n}$ is $\mathcal{F}_{k_j}$-allowable if $j=1$ or $\mathcal{F}_{k_j}$-admissible if $j>1$. Our goal is to show the following.

There exists $c>0$ such that for every block subspace Y of $X_{M(1),u}$ and for large j there exists $y_j \in Y$ with $\|y_j\| = 1$ satisfying

$$\|y_j\| \approx \sup\{f(y_j) : f \in \mathcal{A}_j\}, \tag{3}$$

$$|f(y_j)| \leqslant c\theta_i \qquad \text{for all} \quad i<j, \quad f \in \mathcal{A}_i. \tag{4}$$

These two conditions imply that $X_{M(1),u}$ is an arbitrarily distortable space.

The fundamental objects that we use in order to find such vectors y_j are the (ε, j)-*basic special convex combinations*. The (ε, j)-basic s.c.c. are convex combinations of the basis $(e_n)_{n\in\mathbb{N}}$ of the space $X_{M(1),u}$ whose normalizations satisfy conditions (3) and (4) if ε is small enough. The choice of $(\theta_n)_n$, $(\mathcal{F}_{k_n})_n$ ensures that for every $j \geqslant 2$ and for every infinite $D \subseteq \mathbb{N}$, there exists an (ε, j)-basic special convex combination supported in D.

Next we show that in every block subspace Y of $X_{M(1),u}$ and for every $j \geqslant 2$ we can choose a normalized vector y_j in Y with the following: for every i and every $f \in \mathscr{A}_i$, there exist an (ε, j)-basic special convex combination x_f and a functional $g_f \in \mathscr{A}_i$ such that

$$|f(y_j)| \leqslant C g_f(x_f)$$

for some constant C. Thus, we reduce the estimation of the action of $\mathscr{A}_i$ on y_j to the estimation of the action of $\mathscr{A}_i$ on basic special convex combinations. Our basic tool for this proof is the *analysis* of a functional $f \in \bigcup_{i=1}^{\infty} \mathscr{A}_i$ which is the array of functionals used for the inductive construction of f.

In the case of the space X with no unconditional basic sequence which is constructed in the third section, the scheme of ideas is similar with some additional difficulties coming from the existence of the dependent chains of functionals.

1. MIXED TSIRELSON SPACES AND THEIR MODIFIED VERSIONS

A. Preliminaries

Notation. Let $(e_i)_{i=1}^{\infty}$ be the standard basis of the linear space c_{00} of finitely supported sequences. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$, the *support* of x is the set $\operatorname{supp} x = \{i \in \mathbb{N}: a_i \neq 0\}$. For E, F finite subsets of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or either E or F is empty. For $n \in \mathbb{N}$, $E \subset \mathbb{N}$, $n < E$ (resp. $E < n$) means $n < \min E$ (resp. $\max E < n$). For x, y in c_{00}, $x < y$ means $\operatorname{supp} x < \operatorname{supp} y$. For $n \in \mathbb{N}$, $x \in c_{00}$ we write $n < x$ (resp. $x < n$) if $n < \operatorname{supp} x$ (resp. $\operatorname{supp} x < n$). We say that the sets $E_i \subset \mathbb{N}$, $i = 1, ..., n$ are *successive* if $E_1 < E_2 \cdots < E_n$. Similarly, the vectors x_i, $i = 1, ...n$ are *successive* if $x_1 < x_2 < \cdots < x_n$. For $x = \sum_{i=1}^{\infty} a_i e_i$ and E a subset of $\mathbb{N}$, we denote by Ex the vector $Ex = \sum_{i \in E} a_i e_i$.

The Schreier Families $\mathscr{F}_\alpha$. Let $\mathscr{M}$ be a family of finite subsets of $\mathbb{N}$. We say that $\mathscr{M}$ is *compact* if it is closed in the topology of pointwise convergence in $2^{\mathbb{N}}$. $\mathscr{M}$ is *hereditary* if whenever $B \subset A$ and $A \in \mathscr{M}$ then $B \in \mathscr{M}$. $\mathscr{M}$ is *spreading* if whenever $A = \{m_1, ..., m_k\} \in \mathscr{M}$ and $B = \{n_1, ..., n_k\}$ is such that $m_i \leqslant n_i$, $i = 1, ..., k$, then $B \in \mathscr{M}$.

Notation. Let $\mathscr{M}$, $\mathscr{N}$ be families of finite subsets of $\mathbb{N}$. We denote by $\mathscr{M}[\mathscr{N}]$ the family

$$\mathscr{M}[\mathscr{N}] = \left\{ \bigcup_{i=1}^{n} A_i : n \in \mathbb{N}, A_i \in \mathscr{N}, A_1 < A_2 < \cdots < A_n \quad \text{and} \right.$$

$$\left. \{\min A_1, ..., \min A_n\} \in \mathscr{M} \right\}.$$

The Schreier family $\mathcal{S}$ is defined as

$$\mathcal{S}=\{A\subset\mathbb{N}:\ \#A\leqslant\min A\}.$$

The *generalized Schreier families* $\mathcal{F}_\alpha$, $\alpha<\omega_1$, were introduced in [1]:

1.1. Definition.

$$\mathcal{F}_0=\{\emptyset\}\cup\{\{n\}: n\in\mathbb{N}\}$$

$$\mathcal{F}_{\alpha+1}=\{\emptyset\}\cup\left\{\bigcup_{i=1}^{n} A_i: n\in\mathbb{N},\ A_i\in\mathcal{F}_\alpha,\ n\leqslant A_1<A_2<\cdots<A_n\right\}$$

and for a limit ordinal α we choose a sequence $(\alpha_n)_n$, $\alpha_n\uparrow\alpha$ and set

$$\mathcal{F}_\alpha=\{\emptyset\}\cup\{A\text{: there exists } n\in\mathbb{N} \text{ such that } A\in\mathcal{F}_{\alpha_n} \text{ and } n\leqslant A\}.$$

Notice that $\mathcal{F}_1=\mathcal{S}$. Also, for $n, m<\omega$, $\mathcal{F}_n[\mathcal{F}_m]=\mathcal{F}_{n+m}$.
It is easy to see that each $\mathcal{F}_\alpha$ is a compact, hereditary, and spreading family.

1.2. Lemma. *For $n<\omega$ define the family $\mathcal{F}_n^M$ inductively as follows*:

$\mathcal{F}_0^M=\mathcal{F}_0$.

$\mathcal{F}_{n+1}^M=\{\bigcup_{i=1}^k A_i: k\in\mathbb{N},\ A_i\in\mathcal{F}_n^M$ *for* $i=1, ..., k$, $A_i\cap A_j=\emptyset$ *for* $i\neq j$ *and* $k\leqslant\min A_1<\min A_2<\cdots<\min A_k\}$.

Then, for all n, $\mathcal{F}_n^M=\mathcal{F}_n$.

Proof. The proof is an immediate consequence of the following.

Claim. *Let $n\in\mathbb{N}$ and let $A_i\in\mathcal{F}_n$, $i=1, ..., k$ be such that $A_i\cap A_j=\emptyset$ for $i\neq j$ and $\min A_1<\min A_2<\cdots<\min A_k$. Then, there exist sets $A_i'\in\mathcal{F}_n$, $i=1, ..., k$ such that $A_1'<A_1'<\cdots<A_k'$, $\min A_i\leqslant\min A_i'$ for $i=1, ..., k$, and $\bigcup_{i=1}^k A_i'=\bigcup_{i=1}^k A_i$.*

Proof of the Claim. It is done by induction on *n*. For $n=0$ it is trivial. Suppose it is true for *n*.

Let A_i, $i=1, ..., k$ be sets in $\mathcal{F}_{n+1}$ such that $A_i\cap A_j=\emptyset$ for $i\neq j$ and $\min A_1<\min A_2<\cdots<\min A_k$. Each A_i is of the form $A_i=\bigcup_{j=1}^{m_i} B_j^i$ where $B_j^i\in\mathcal{F}_n$ and, for each *i*, $m_i\leqslant B_1^i<B_2^i<\cdots<B_{m_i}^i$. Let $\{B_j\}_{j=1}^{m_1+\cdots+m_k}$ be a rearrangement of the family $\{B_j^i\colon\ i=1, ..., k,\ j=1, ..., m_i\}$, which satisfies $\min B_1<\min B_2<\cdots<\min B_{m_1+\cdots+m_k}$. It is easy to see that, for each *i*,

$$\min A_i=\min B_1^i\leqslant\min B_{m_1+\cdots+m_{i-1}+1}. \tag{$*$}$$

By the inductive assumption, there exist sets B'_j, $j=1, ..., m_1+\cdots+m_k$, with $B'_j \in \mathcal{F}_n$, $\bigcup_{j=1}^{m_1+\cdots+m_k} B'_j = \bigcup_{j=1}^{m_1+\cdots+m_k} B_j$ and such that $B'_1 < B'_2 < \cdots < B'_{m_1+\cdots+m_k}$ and $\min B_j \leqslant \min B'_j$ for all $j=1, ..., m_1+\cdots+m_k$. For $i=1, ..., k$, we set

$$A'_i = \bigcup_{j=m_1+\cdots+m_{i-1}+1}^{m_1+\cdots+m_i} B'_j.$$

Then, $A'_1 < A'_2 < \cdots < A'_k$, $\bigcup_{i=1}^k A'_i = \bigcup_{i=1}^k A'_i$, and for each $i=1, ..., k$ we have by $(*)$

$$m_i \leqslant \min B_{m_1+\cdots+m_{i-1}+1} \leqslant \min B'_{m_1+\cdots+m_{i-1}+1}$$

so $A'_i \in \mathcal{F}_{n+1}$. Moreover, using $(*)$ again, we see that

$$\min A_i \leqslant \min B'_{m_1+\cdots+m_{i-1}+1} = \min A'_i.$$

This completes the proof of the Claim. The lemma follows. ∎

Distortion. Let $\lambda > 1$. A Banach space X is *λ-distortable* if there exists an equivalent norm $|\cdot|$ on X such that, for every infinite dimensional subspace Y of X,

$$\sup\left\{\frac{|y|}{|z|}: y, z \in Y, \|y\| = \|z\| = 1\right\} \geqslant \lambda.$$

X is *arbitrarily distortable* if it is λ-distortable for every $\lambda > 1$.

B. *Mixed Tsirelson Spaces*

A Banach space X with a basis $(e_i)_{i=1}^\infty$ is an *asymptotic ℓ_1* space if there exists a constant C such that, for all n and all block sequences $(x_i)_{i=1}^n$ in X with $n \leqslant x_1 < x_2 < \cdots < x_n$,

$$\frac{1}{C}\sum_{i=1}^n \|x_i\| \leqslant \left\|\sum_{i=1}^n x_i\right\|.$$

The first example of an asymptotic ℓ_1 space not containing ℓ_1 was constructed by Tsirelson [19]. Tsirelson's space is the completion of the vector space c_{00} of all eventually zero sequences under the norm $\|\cdot\|_T$ defined implicitly as

$$\|x\|_T = \max\left\{\|x\|_\infty, \sup\left\{\tfrac{1}{2}\sum_{i=1}^n \|E_i x\|_T : n \in \mathbb{N} \quad \text{and} \quad n \leqslant E_1 < E_2 < \cdots < E_n\right\}\right\}.$$

A sequence $(E_i)_{i=1}^n$ of finite subsets of $\mathbb{N}$ with $n \leqslant E_1 < E_2 < \cdots < E_n$ is called *Schreier admissible* (or $\mathcal{S}$-admissible). In other words, a sequence $(E_i)_{i=1}^n$ is Schreier admissible if the E_i's are successive and $\{\min E_1, ..., \min E_n\} \in \mathcal{S}$. More generally, we give the following definition.

1.3. DEFINITION. Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$.

(a) A finite sequence $(E_i)_{i=1}^n$ of subsets of $\mathbb{N}$ is *$\mathcal{M}$-admissible* if $E_1 < E_2 < \cdots < E_n$ and $\{\min E_1, ..., \min E_n\} \in \mathcal{M}$.

(b) A finite sequence $(x_i)_{i=1}^n$ of vectors in c_{00} is *$\mathcal{M}$-admissible* if the sequence $(\operatorname{supp} x_i)_{i=1}^n$ is $\mathcal{M}$-admissible.

The mixed Tsirelson spaces are defined as follows:

1.4. DEFINITION. Let $\{\mathcal{M}_n\}_{n=1}^\infty$ be a sequence of compact families of finite subsets of $\mathbb{N}$ and let $(\theta_n)_{n=1}^\infty$ be a sequence of numbers in $(0, 1)$ with $\theta_n \to 0$. The *mixed Tsirelson space* $T[(\mathcal{M}_n, \theta_n)_{n=1}^\infty]$ is the completion of c_{00} under the norm $\|\cdot\|$ defined implicitly by

$$\|x\| = \max\left\{\|x\|_\infty, \sup_k \sup\left\{\theta_k \sum_{i=1}^n \|E_i x\| : n \in \mathbb{N} \quad \text{and} \quad (E_i)_{i=1}^n \text{ is } \mathcal{M}_k\text{-admissible}\right\}\right\}.$$

The mixed Tsirelson spaces $T[(\mathcal{M}_n, \theta_n)_{n=1}^\infty]$ where $(\mathcal{M}_n)_n$ is a subsequence of the sequence of Schreier families $(\mathcal{F}_j)_{j=1}^\infty$ were introduced in [3] and further studied in [2, 14]. Every such space is a reflexive asymptotic ℓ_1 Banach space and the natural basis $(e_i)_i$ is a 1-unconditional basis for it. The first example of an arbitrarily distortable asymptotic ℓ_1 Banach space was a space of this type [3]. More generally, Androulakis and Odell have proved the following:

1.5. THEOREM [2]. *Suppose that the sequence $(\theta_n)_n$ satisfies $\theta_{n+m} \geqslant \theta_n \theta_m$ for all n, m and let $\theta = \lim \theta_n^{1/n}$. If $\theta_n / \theta^n \to 0$ then the space $T[(\mathcal{F}_n, , \theta_n)_{n=1}^\infty]$ is arbitrarily distortable.*

In particular, this is the case if $\lim \theta_n^{1/n} = 1$. The first result of this section concerns mixed Tsirelson spaces $T[(\mathcal{F}_n, \theta_n)_n]$ corresponding to such sequences $(\theta_n)_n$. Following [2] we call a sequence $(\theta_n)_n$ *regular*, if $\theta_n \in (0, 1)$ for all n, $\theta_n \downarrow 0$ and $\theta_{n+m} \geqslant \theta_n \theta_m$ for all $n, m \in \mathbb{N}$.

1.6. THEOREM. *Let $(\theta_n)_{n=1}^{\infty}$ be a regular sequence with $\lim \theta_n^{1/n} = 1$. Let $X = T[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$. For every $\varepsilon > 0$, every infinite dimensional block subspace Y of X contains for every n a sequence of disjointly supported vectors $(y_i)_{i=1}^n$ which is $(1+\varepsilon)$-equivalent to the canonical basis of ℓ_∞^n.*

Given a block subspace Y of X and $n \in \mathbb{N}$ we shall construct a sequence $(x_i)_{i=1}^n$ if disjointly supported normalized vectors in Y such that $\|\sum_{i=1}^n x_i\| \leqslant 36$. Since the basis $(e_n)_n$ of X is 1-unconditional this implies that $(x_i)_{i=1}^n$ is 36-equivalent to the canonical basis of ℓ_∞^n. From this the theorem follows by a standard argument due to R. C. James. The building blocks of our construction are the (ε, j)-*rapidly increasing special convex combinations*, the prototypes of which were used in [3]. Before proceeding to the construction we need to establish some preliminary results most of which also have their analogues in [3].

Notation. Let $X = T[(\mathscr{F}_n, \theta_n)_{n=1}^{\infty}]$.

(A) Inductively, we define a subset $K = \bigcup_{n=0}^{\infty} K^n$ of B_{X^*} as follows: For $j = 1, 2, ...,$

$$K_j^0 = \{\pm e_n : n \in \mathbb{N}\}.$$

Assume that K_j^n, $j = 1, 2, ...$ have been defined. We set $K^n = \bigcup_{j=1}^{\infty} K_j^n$ and, for $j = 1, 2, ...,$ we set

$$K_j^{n+1} = K_j^n \cup \{\theta_j(f_1 + \cdots f_d): d \in \mathbb{N}, f_i \in K^n, i = 1, ..., n,$$
$$\operatorname{supp} f_1 < \cdots < \operatorname{supp} f_d \text{ and } (f_i)_{i=1}^d \text{ is } \mathscr{F}_j\text{-admissible}\}.$$

Let $K = \bigcup_{n=0}^{\infty} K^n$.
Then K is a norming set for X, that is, for $x \in X$

$$\|x\| = \sup\{f(x): f \in K\}.$$

(B) For $j = 1, 2, ...,$ we denote by $\mathscr{A}_j$ the set $\mathscr{A}_j = \bigcup_{n=1}^{\infty} (K_j^n \setminus K^0)$.

(C) Let $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$. An *analysis* of φ is a family $\{K^s(\varphi)\}_{s=0}^m$ of subsets of K such that

(1) For every $s \leqslant m$, $K^s(\varphi) \subset K^s$, the elements of $K^s(\varphi)$ are disjointly supported and $\bigcup_{f \in K^s(\varphi)} \operatorname{supp} f = \operatorname{supp} \varphi$.

(2) If f belongs to $K^{s+1}(\varphi)$ then either $f \in K^s(\varphi)$ or, some $j \geqslant 1$, there exists a $\mathscr{F}_j$-admissible family $(f_i)_{i=1}^d$ in $K^s(\varphi)$ such that $f = \theta_j(f_1 + \cdots + f_d)$.

(3) $K^m(\varphi) = \{\varphi\}$.

It is easy to see that every $\varphi \in K$ as an analysis.

1.7. Definition. Let $n \geqslant 1$, $\varepsilon > 0$, and $F \subseteq \mathbb{N}$, $F \in \mathscr{F}_n$. A convex combination $\sum_{k \in F} a_k e_k$ is called an *(ε, n)-basic special convex combination* (basic s.c.c.) if, for every $G \in \mathscr{F}_{n-1}$, $\sum_{k \in G} a_k < \varepsilon$.

1.8. Proposition. *Let D be an infinite subset of $\mathbb{N}$. Then, for every $n \geqslant 1$ and $\varepsilon > 0$, there exists an (ε, n)-basic special convex combination $x = \sum_{k \in F} a_k e_k$ with $F = \operatorname{supp}(x) \subset D$.*

Proof. For $n = 1$, we choose $m_0 > 1/\varepsilon$ and $A \subset D$ with $m_0 < A$ and $|A| = m_0$. Then, $x = (1/m_0) \sum_{k \in A} e_k$ is an $(\varepsilon, 1)$-basic s.c.c.

For $n > 1$ the proof is by induction based on the following:

1.9. Lemma. *Let $n \geqslant 1$ and suppose that the integers $m_0, m_1, ..., m_{m_0}$ and the block vectors $x_1, x_2, ..., x_{m_0}$ satisfy the following: For every $k = 1, 2, ..., m_0 - 1$,*

(a) $2m_{k-1} < m_k$.

(b) $\operatorname{supp}(x_k) \subset (m_{k-1}, m_k]$.

(c) *x_k is a $(1/2m_{k-1}, n)$-basic s.c.c.*

Then, the vector $x = (1/m_0) \sum_{k=1}^{m_0} x_k$ is a $(2/m_0, n+1)$-basic s.c.c.

Proof. The proof is straightforward (see also Lemma 1.6 of [3]). ∎

1.10. Definition. Let $\varepsilon > 0$, $j \in \mathbb{N}$, and suppose that $\{z_k\}_{k=1}^n$ is a finite block sequence with the property that there exist integers $\{l_k\}_{k=1}^n$ with $2 < z_1 \leqslant l_1 < z_2 \leqslant l_2 < \cdots \leqslant l_{n-1} < z_n \leqslant l_n$, and such that a convex combination $\sum_{k=1}^n a_k e_{l_k}$ is an (ε, j)-basic s.c.c. Then, the corresponding convex combination of the z_k's, $x = \sum_{k=1}^n a_k z_k$, is called an *(ε, j)-s.c.c. of $\{z_k\}_{k=1}^n$*.

An (ε, j)-s.c.c. $x = \sum_{k=1}^n a_k z_k$ of *unit* vectors $\{z_k\}_{k=1}^n$ is said to be *seminormalized* if $\|x\| \geqslant \frac{1}{2}$.

Remark. It is easy to see that if $x = \sum_{k=1}^n a_k z_k$ is an (ε, j)-s.c.c. and $\|z_k\| = 1$, $k = 1, ..., n$, then $\|x\| \geqslant \theta_{j+1}$. Indeed, if $f_k \in B_{X^*}$ are chosen so that $f_k(z_k) = \|z_k\| = 1$, $\operatorname{supp}(f_1) \subset (2, l_1]$, and $\operatorname{supp} f_k \subset (l_{k-1}, l_k]$ for $k = 2, ..., n$, then the family $\{f_k\}_k$ is $\mathscr{F}_{j+1}$-admissible. This implies that the functional $\varphi = \theta_{j+1} \sum f_k$ belongs to B_{X^*}, hence $\|x\| \geqslant \varphi(x) \geqslant \theta_{j+1}$.

The following lemma states that every block subspace Y of X contains for any ε and j a seminormalized (ε, j)-s.c.c. The condition $\lim \theta_j^{1/j} = 1$ is essential at this point.

1.11. Lemma. *Let $j \in \mathbb{N}$, $\varepsilon > 0$, and let $\{z_k\}_{k=1}^\infty$ be a block sequence in X. There exists $n \in \mathbb{N}$ and normalized blocks y_k, $k = 1, ..., n$ of the sequence*

$\{z_k\}_{k=1}^{\infty}$ such that a convex combination $x=\sum_{k=1}^{n} a_k y_k$ is a seminormalized (ε, j)-s.c.c.

Proof. We may assume that the vectors z_k, $k=1, 2, \ldots$ are normalized. Choose an infinite block sequence $\{x_l^1\}_{l=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ such that, for each l, $x_l^1=\sum_{k\in A_l} a_k z_k$ is an (ε, j)-s.c.c. of $\{z_k\}_{k\in A_l}$.

If for some l, $\|x_l^1\|\geqslant\frac{1}{2}$, then we are done. If not, we set $y_l^1=x_l^1/\|x_l^1\|$ and, as before, choose an infinite sequence $\{x_l^2\}_l$ of (ε, j)-s.c.c. of $\{y_l^1\}_{l=1}^{\infty}$.

Notice that, for each l, the family $\{z_k\colon \operatorname{supp}(z_k)\subset\operatorname{supp}(x_l^2)\}$ is $\mathcal{F}_{2j+2}$-admissible (since $\mathcal{F}_{2j+2}=\mathcal{F}_{j+1}[\mathcal{F}_{j+1}]$), and so x_l^2 is a combination of the form $x_l^2=\sum b_k(\lambda_k z_k)$ where $\sum b_k=1$, $\lambda_k\geqslant 2$, and $\{z_k\}$ is an $\mathcal{F}_{2j+2}$-admissible family. This gives that $\|x_l^2\|\geqslant 2\theta_{2j+2}$.

If, for some l, $\|x_l^2\|\geqslant\frac{1}{2}$ then we are done. If not, then we set $y_l^2=x_l^2/\|x_l^2\|$ and continue as before.

Continuing in this manner, if we never get some (ε, j)-s.c.c. x_l^k with $\|x_l^k\|\geqslant\frac{1}{2}$, then we can repeat the same procedure for as many steps s as we wish and always get $1\geqslant\|x_l^s\|\geqslant 2^{s-1}\theta_{s(j+1)}$.

But the assumption that $\lim_n \theta_n^{1/n}=1$ implies that $\lim_{s\to\infty} 2^{s-1}\theta_{s(j+1)}=\infty$. This leads to a contradiction which completes the proof. ∎

1.12. Lemma. *Let $x=\sum_{l\in F} a_l e_l$, where $F\in\mathcal{F}_j$, be an (ε, j)-basic s.c.c. Then, $\theta_j\leqslant\|x\|<\theta_j+\varepsilon$.*

Proof. It is obvious that $\varphi=\theta_j(\sum_{l\in F} e_l^*)$ belongs to B_{X^*} and $\varphi(x)=\theta_j$. This yields the lower estimate for $\|x\|$.

It remains to prove that, for all $\psi\in K$, $|\psi(x)|<\theta_j+\varepsilon$. Let $\psi\in K$; we may assume that ψ is positive. Set

$$J=\{l\in F\colon \psi(e_l)\leqslant\theta_j\}.$$

and

$$L=F\backslash J=\{l\in F\colon \psi(e_l)>\theta_j\}.$$

We shall prove that $L\in\mathcal{F}_{j-1}$ and so $\sum_{k\in L} a_k<\varepsilon$. This is a consequence of the following:

Claim. *Let $r=1, 2, \ldots$, $f\in K$ and suppose that $f(e_k)>\theta_r$ for all $k\in\operatorname{supp}(f)$. Then, $\operatorname{supp}(f)\in\mathcal{F}_{r-1}$.*

Proof of the Claim. The proof is by induction on s, for $f\in K^s$, $s=1, 2, \ldots$.

For $s=1$, let $f\in K^1$, with $f=\theta_i\sum_{k\in A} e_k^*$, $A\in\mathcal{F}_i$. Since $\theta_i>\theta_r$, we get $i\leqslant r-1$ and so $A=\operatorname{supp}(f)\in\mathcal{F}_{r-1}$.

Suppose that the claim is true for all $g \in K^s$ and let $f \in K^{s+1}$. Then, $f = \theta_i(\sum_{l=1}^m f_l)$ where the set $(f_l)_{l=1}^m$ is $\mathcal{F}_i$-admissible and, for each l, $f_l \in K^s$. Suppose that $f(e_k) > \theta_r$ for all $k \in \operatorname{supp}(f)$. Then, $r > i$ and, for each $l = 1, ..., m$, $f_l(e_k) > \theta_r/\theta_i \geqslant \theta_{r-i}$. It follows from the inductive hypothesis that $\operatorname{supp}(f_l) \in \mathcal{F}_{r-i-1}$, $l = 1, ..., m$. So, $\operatorname{supp}(f) \in \mathcal{F}_i[\mathcal{F}_{r-i-1}] = \mathcal{F}_{r-1}$. This completes the proof of the claim.

We conclude that $L \in \mathcal{F}_{j-1}$ and so

$$\left|\psi\left(\sum_{l \in F} a_l e_l\right)\right| \leqslant \psi\left(\sum_{l \in J} a_l e_l\right) + \sum_{l \in L} a_l < \theta_j + \varepsilon. \quad \blacksquare$$

1.13. Lemma. *Let $x = \sum_{k=1}^n a_k y_k$ be an (ε, j)-s.c.c. of $\{y_k\}_{k=1}^n$, where $\varepsilon < \theta_j$. Let $i < j$ and suppose that $(E_r)_{r=1}^s$ is an $\mathcal{F}_i$-admissible family of intervals. Then,*

$$\sum_{r=1}^s \|E_r x\| \leqslant \left(1 + \frac{\varepsilon}{\theta_i}\right) \max_{1 \leqslant k \leqslant n} \|y_k\| \leqslant 2 \max_{1 \leqslant k \leqslant n} \|y_k\|.$$

Proof. We can assume that the E_r's are adjacent intervals. Set

$$L = \{k: k = 1, ..., n \text{ and } \operatorname{supp}(y_k) \text{ is intersected by at least two different } E_r\text{'s}\}.$$

For each $r = 1, ..., s$, define

$$B_r = \{k: k = 1, ..., n \text{ and } \operatorname{supp}(y_k) \subset E_r\}.$$

The sets B_r are mutually disjoint and $\{1, 2, ..., n\} = (\bigcup_{r=1}^s B_r) \cup L$. So,

$$\sum_{r=1}^s \|E_r x\| \leqslant \sum_{r=1}^s \left\|E_r\left(\sum_{k \in B_r} a_k y_k\right)\right\| + \sum_{k \in L} a_k \sum_{r=1}^s \|E_r y_k\|$$
$$\leqslant \sum_{k=1}^n a_k \|y_k\| + \sum_{k \in L} a_k \frac{\|y_k\|}{\theta_i}.$$

Suppose now that $2 < y_1 \leqslant l_1 < y_2 \leqslant \cdots \leqslant l_{k-1} < y_k \leqslant l_k$ and $\sum_{k=1}^n a_k e_{l_k}$ is the basic s.c.c. which defines the s.c.c. $x = \sum_{k=1}^n a_k y_k$. We shall show that $\{l_k: k \in L\} \in \mathcal{F}_i \subset \mathcal{F}_{j-1}$. This will imply that $\sum_{k \in L} a_k < \varepsilon$ and hence complete the proof.

To see that $\{l_k: k \in L\} \in \mathcal{F}_i$, for each $k \in L$ let $r_k = \min\{r: E_r \text{ intersects } \operatorname{supp}(y_k)\}$. The map $k \to r_k$ from L to $\{1, 2, ..., s\}$ is one to one. This gives that $\#L \leqslant s$. Consider now, for each $k \in L$, $m_{r_k} = \min E_{r_k}$. Then, $m_{r_k} \leqslant l_k$, $k \in L$. Since the set $\{m_{r_k}: k \in L\}$ belongs to $\mathcal{F}_i$, we conclude (by the spreading property of $\mathcal{F}_i$) that $\{l_k: k \in L\} \in \mathcal{F}_i$ as well. $\blacksquare$

1.14. DEFINITION. (A) A finite or infinite sequence $\{z_k\}_k$ is called a *rapidly increasing sequence* if there exists an increasing sequence of positive integers $\{t_k\}_k$ such that the following are satisfied:

(a) The sequence $\{\theta_{t_k}/\theta_{t_{k+1}}\}_k$ is increasing, $2<\theta_{t_k}/\theta_{t_{k+1}}$ for each k, and $\lim_{k\to\infty}(\theta_{t_k}/\theta_{t_{k+1}})=\infty$ if the sequence is infinite.

(b) Each z_k is a semi-normalized $(\theta^2_{t_k}, t_k)$-s.c.c.

(c) For each k, $\|z_k\|_{\ell_1}\leqslant\theta_{t_k}/\theta_{t_{k+1}}$.

(B) Let $k\in\mathbb{N}$, $\varepsilon>0$. Let $\{z_k\}_{k=1}^n$ be a rapidly increasing sequence, where each z_k is a semi-normalized $(\theta^2_{t_k}, t_k)$-s.c.c. and $2<\theta_{j+1}/\theta_{t_1}<\theta_{t_1}/\theta_{t_2}$. Suppose also that there exist coefficients $\{a_k\}_{k=1}^n$ such that the vector $x=\sum_{k=1}^n a_k z_k$ is an (ε, j)-s.c.c. of $\{z_k\}_{k=1}^n$. Then x is called an (ε, j)-*rapidly increasing special convex combination* ((ε, j)-R.I.s.c.c.).

1.15. PROPOSITION. *Let $j\in\mathbb{N}$, $0<\varepsilon<\theta_j^2$, and let $x=\sum_{k=1}^n a_k z_k$ be an (ε, j)-R.I.s.c.c. of the z_k's where each z_k is a seminormalized $(\theta^2_{t_k}, t_k)$-s.c.c. Let t_0 be any integer such that $j+1\leqslant t_0<t_1$ and $2<\theta_{t_0}/\theta_{t_1}$.*

Then, for every φ in the norming set K of X, we have the following estimates:

(i) $|\varphi(x)|\leqslant 8\theta_j$, *if* $\varphi\in\mathscr{A}_i$, $i<j$.

(ii) $|\varphi(x)|\leqslant 4\theta_i$, *if* $\varphi\in\mathscr{A}_i$, $j\leqslant i<t_1$.

(iii) $|\varphi(x)|\leqslant 4(\theta_{t_{p-1}}+a_{t_p})$, *if* $\varphi\in\mathscr{A}_i$, $t_p\leqslant i<t_{p+1}$, $p\geqslant 1$.

In particular, $\theta_{j+1}/2\leqslant\|x\|\leqslant 8\theta_j$.

Proof. The lower estimate for $\|x\|$ follows by the remark after Definition 1.10 and the fact that $\|z_k\|\geqslant\frac{1}{2}$. The upper estimate follows from the first part of the proposition. The proof of this is similar to the one of Proposition 2.12 in [3]. Let $\{l_k\}_{k=1}^n$ be such that $2<z_1\leqslant l_1<\cdots\leqslant l_{n-1}<z_n\leqslant l_n$ and $\sum_{k=1}^n a_k e_{l_k}$ is an (ε, j)-basic s.c.c.

Given $\varphi\in K$, we shall construct $\psi\in\mathrm{co}(K)$ such that

(a) $\varphi(\sum_{k=1}^n a_k z_k)\leqslant 4\psi(\sum_{k=1}^n a_k e_{l_k})$.

(b) If $\varphi\in\mathscr{A}_i$, $i<t_1$, then $\psi\in\mathrm{co}(\mathscr{A}_i)$.

(c) If $\varphi\in\mathscr{A}_i$, $t_p\leqslant i<t_{p+1}$ for some $p\geqslant 1$, then $\psi=\frac{1}{2}(\psi_1+e^*_{l_p})$, where $\psi_1\in\mathrm{co}(\mathscr{A}_{t_{p-1}})$.

Since, for $\psi\in\mathrm{co}(\mathscr{A}_i)$ we have $\psi(\sum z_k e_{l_k})\leqslant\theta_i$, estimates (ii) and (iii) will follow immediately. For (i) we apply Lemma 1.12.

We consider an analysis $\{K^s(\varphi)\}_{s=1}^m$ of φ, and we cut each z_k into two parts, z'_k and z''_k, with the following property:

($*$) *For each level* $K^s(\varphi)$ *of the analysis of* φ, *and for each* z'_k, *either there exists a unique* $f \in K^s(\varphi)$ *with* $\operatorname{supp}(z'_k) \cap \operatorname{supp}(f) \neq \emptyset$ *or there exists* $f \in K^s(\varphi)$ *such that* $\max \operatorname{supp}(z'_{k-1}) < \operatorname{supp}(f) < \min \operatorname{supp}(z'_{k+1})$.

The same is true for z''_k. This partition of the z_k's is possible, as done in [3, Definition 2.4].

We shall see that using property ($*$) we can build ψ' and ψ'' such that $|\varphi(z'_k)| \leqslant \psi'(e_{l_k})$ and $|\varphi(z''_k)| \leqslant \psi''(e_{l_k})$ for all k. So we may assume that the z_k's have property ($*$) and then multiply our estimate by 2.

For each $f \in \bigcup_{s=0}^m K^s(\varphi)$ we set

$$D_f = \{k\colon \operatorname{supp}(\varphi) \cap \operatorname{supp}(z_k) = \operatorname{supp}(f) \cap \operatorname{supp}(z_k) \neq \emptyset\}.$$

By induction on $s = 0, ..., m$ we shall define a function $g_f \in \operatorname{co}(K)$, supported on $\{l_k\colon k \in D_f\}$ and such that:

(a) $|f(z_k)| \leqslant 2 g_f(e_{l_k})$ for all $k \in D_f$.

(b) If $f \in \mathscr{A}_q$, $q < t_1$, then $g_f \in \operatorname{co}(\mathscr{A}_q)$. If $f \in \mathscr{A}_q$, $t_p \leqslant q < t_{p+1}$, then $g_f = \frac{1}{2}(g_f^1 + e^*_{l_p})$, where $g_f^1 \in \operatorname{co}(\mathscr{A}_{t_{p-1}})$.

For $s = 0$, $f = e^*_r$, if $D_f = \{k\}$ we set $g_f = e^*_{l_k}$.

Let $s > 0$. Suppose that g_f has been defined for all $f \in \bigcup_{t=0}^{s-1} K^t(\varphi)$. Let

$$f = \theta_q(f_1 + \cdots + f_d) \in K^s(\varphi) \backslash K^{s-1}(\varphi).$$

We set $I = \{i\colon 1 \leqslant i \leqslant d,\ D_{f_i} \neq \emptyset\}$ and $T = D_f \backslash \bigcup_{i \in I} D_{f_i}$.

Case 1. $q < t_1$. Then, we set

$$g_f = \theta_q \left(\sum_{i \in I} g_{f_i} + \sum_{k \in T} e^*_{l_k} \right).$$

Property (a) for the case $k \in \bigcup_{i \in I} D_{f_i}$ follows from the inductive assumption. For $k \in T$ we get, by Lemma 1.13, since $q < t_k$, that

$$|f(z_k)| \leqslant \theta_q \sum_{i=1}^d |f_i(z_k)| \leqslant 2\theta_q = 2 g_f(e_{l_k}).$$

To prove that $g_f \in \operatorname{co}(\mathscr{A}_q)$ we need to show that the set $\{g_{f_i}\colon i \in I\} \cup \{l_k\colon k \in T\}$ is $\mathscr{F}_q$-admissible.

Here we use property ($*$). According to ($*$), for each $k \in T$ there exists an $i_k \in \{1, ..., d\}$ such that $\max \operatorname{supp}(z_{k-1}) < \operatorname{supp}(f_{i_k}) < \min \operatorname{supp}(z_{k+1})$.

This means that $i_k \neq i_l$ for $k \neq l \in T$ and $i_k \notin I$. It follows that $|T| + |I| \leqslant d$. Since also, for each $k \in T$, $\min \operatorname{supp}(f_{i_k}) \leqslant l_k$, by the spreading property of $\mathscr{F}_q$ we get that

$$\{\min \operatorname{supp}(f_i) : i \in I\} \cup \{l_k : k \in T\} \in \mathscr{F}_q,$$

hence the family $\{g_{f_i}\}_{i \in I} \cup \{e^*_{l_k}\}_{k \in T}$ is $\mathscr{F}_q$-admissible.

Case 2. $q \geqslant t_1$. Suppose that $t_p \leqslant q < t_{p+1}$. If $p \notin D_f$ or $p \in \bigcup_{i \in I} D_{f_i}$, then we set

$$g_f = \theta_{t_{p-1}} \left(\sum_{i \in I} g_{f_i} + \sum_{k \in T} e^*_{l_k} \right).$$

Since $\operatorname{supp}(g_f) \subset \{l_k : k = 1, ..., n\} \in \mathscr{F}_j$ and $j < t_{p-1}$, it is clear that $g_f \in \operatorname{co}(\mathscr{A}_{t_{p-1}})$.

For $k \in \bigcup_{i \in I} D_{f_i}$ we get

$$|f(z_k)| = \theta_q \, |f_i(z_k)| < 2\theta_q \, g_{f_i}(e_{l_k}) < \theta_{t_{p-1}} \, g_{f_i}(e_{l_k}) = g_f(e_{l_k})$$

by the inductive assumption and the fact that $2\theta_{t_p} < \theta_{t_{p-1}}$.

For $k \in T$, $k < p$, we have

$$|f(z_k)| \leqslant \theta_q \sum_{i=1}^{d} |f_i(z_k)| \leqslant \theta_q \, \|z_k\|_{\ell_1} \leqslant \theta_q \frac{\theta_{t_k}}{\theta_{t_{k+1}}}$$

$$\leqslant \theta_{t_p} \frac{\theta_{t_{p-1}}}{\theta_{t_p}} = \theta_{t_{p-1}} = g_f(e_{l_k})$$

by the property of the R.I.S. $\{z_k\}_k$.

For $k \in T$, $k > p$, we have $q < t_{p+1} \leqslant t_k$, so

$$|f(z_k)| = \theta_q \sum_{i=1}^{d} |f_i(z_k)| \leqslant 2\theta_q < \theta_{t_{p-1}} = g_f(e_{l_k})$$

by Lemma 1.13.

Suppose now that $p \in T$. Then we set

$$g_f = \tfrac{1}{2} \left[\theta_{t_{p-1}} \left(\sum_{i \in I} g_{f_i} + \sum_{k \in T \setminus \{p\}} e^*_{l_k} \right) + e^*_{l_p} \right].$$

As before, we get

$$|f(z_k)| < 2 g_f(e_{l_k})$$

for $k \neq p$, and

$$|f(z_p)| \leqslant 1 = 2g_f(e_{l_p}).$$

This completes the inductive step of the construction and the proof of the proposition. ∎

In what follows, a finite tree of sequences $\mathscr{T}$ will be a finite set of finite sequences of positive integers, partially ordered by the relation $\alpha \prec \beta$ iff α is an initial part of β, and satisfying the following properties:

(a) For each $\alpha \in \mathscr{T}$, the set $\{\beta: \beta$ is an initial part of $\alpha\}$ is a subset of $\mathscr{T}$.

(b) If $\alpha = (k_1, ..., k_{m-1}, k_m) \in \mathscr{T}$ and $1 \leqslant l \leqslant k_m$, then $(k_1, ..., k_{m-1}, l) \in \mathscr{T}$.

(c) The maximal (under $\prec$) elements of $\mathscr{T}$ are all of the same length.

It follows that $\mathscr{T}$ has a unique *root*, the empty sequence which we denote by 0. The length of the sequence α is denoted by $|\alpha|$. The *height* of $\mathscr{T}$ is the length of the maximal elements of $\mathscr{T}$. For each $\alpha \in \mathscr{T}$ which is not maximal we set $S_\alpha = \{\beta \in \mathscr{T}: \alpha \prec \beta$ and $|\beta| = |\alpha| + 1\}$. We also consider the lexicographic order, denoted by $<$, on $\mathscr{T}$. For $\alpha = (k_1, ..., k_{m-1}, k_m) \in \mathscr{T}$ we denote by α^+ the sequence $\alpha^+ = (k_1, ..., k_{m-1}, k_m + 1)$.

1.16. Definition. Let $r \in \mathbb{N}$. Let $j_1, ..., j_r$ be positive integers, and $\varepsilon > 0$. An $(\varepsilon, (j_1, ..., j_r))$-*tree in* X is a set of vectors $\mathscr{T}^X = \{u_\gamma\}_{\gamma \in \mathscr{T}}$ indexed by a finite tree $\mathscr{T}$ of height r, and satisfying the following properties:

(a) The terminal nodes $\{u_\alpha\}_{|\alpha| = r}$ of the tree are elements of the basis $\{e_n\}_{n=1}^\infty$, i.e., for $|\alpha| = r$, $\alpha \in \mathscr{T}$, $u_\alpha = e_{l_\alpha}$. Moreover, for α, $\beta \in \mathscr{T}$ with $|\alpha| = |\beta| = r$, if $\alpha < \beta$ (in the lexicographic order), then $l_\alpha < l_\beta$.

(b) There exist positive coefficients $\{a_\beta\}_{\beta \in \mathscr{T} \setminus \{0\}}$ such that, for each $\gamma \in \mathscr{T}$, $|\gamma| = t < r$, we have $\sum_{\beta \in S_\gamma} a_\beta = 1$ and $u_\gamma = \sum_{\alpha \in \mathscr{T}, |\alpha| = r, \gamma \prec \alpha} (\prod_{\gamma \prec \beta \preccurlyeq \alpha} a_\beta)\, e_{l_\alpha}$ is an $(\varepsilon, j_{t+1} + j_{t+2} + \cdots + j_r)$-basic s.c.c. of $\{e_{l_\alpha}\}_{\alpha \in \mathscr{T}, |\alpha| = r}$.

It is clear that, given an infinite subset L of $\mathbb{N}$, $j_1, ..., j_r$ positive integers, and $\varepsilon > 0$, one can construct an $(\varepsilon, (j_1, ..., j_r))$-tree in X, supported in L, by repeatedly applying Lemma 1.9. It is also not hard to see in the same manner that the following construction is possible.

1.17. Lemma. *Let L be an infinite subset of $\mathbb{N}$, $n \in \mathbb{N}$, $\varepsilon > 0$, and $j_1, ..., j_n$ be positive integers. There exist a tree of sequences $\mathscr{T}$, subsets $\mathscr{T}_1^X, ..., \mathscr{T}_n^X$ of X, and positive coefficients $\{a_\beta\}_{\beta \in \mathscr{T} \setminus \{0\}}$ such that*:

(a) *For* $r \leqslant n$, *set* $\mathcal{T}_r = \{\alpha \in \mathcal{T} : |\alpha| \leqslant r\}$. *Then*, $\mathcal{T}_r^X = \{u_\alpha^r\}_{\alpha \in \mathcal{T}_r}$ *is an* $(\varepsilon, (j_1, ..., j_r))$*-tree in* X *with coefficients* $\{a_\beta\}_{\beta \in \mathcal{T}_r \setminus \{0\}}$, *supported in* L.

(b) *Let* $\{e_{l_\alpha^r}, \alpha \in \mathcal{T}, |\alpha| = r\}$ *be the terminal nodes of the tree* $\mathcal{T}_r^X$. *Then, if* $\alpha, \beta \in \mathcal{T}$, $|\alpha| = r < n$, *and* $\beta \in S_\alpha$, *we have* $l_\alpha^r < l_\beta^{r+1} < l_{\alpha^+}^r$.

1.18. Definition. A finite family $\mathcal{T}_1^X, ..., \mathcal{T}_n^X$ as described in Lemma 1.17 is called an $(\varepsilon, (j_1, ..., j_n))$ *family of nested trees* in X.

Proof of Theorem 1.6. Given $n \in \mathbb{N}$, and a block subspace Y of X we shall construct a sequence $x_1, ..., x_n$ of disjointly supported unit vectors in Y which is 36-equivalent to the canonical basis of ℓ_∞^n.

The construction is as follows.

First, choose $\eta > 0$ with $\eta < 1/60n$. Choose j_0 such that $64\theta_{j_0} < \eta$. Let $s_0 \in \mathbb{N}$ be such that $\theta_1^{s_0} < \eta$. Choose j_1 such that

$$s_0 j_0 < j_1 \qquad \text{and} \qquad \frac{\theta_{j_1+1}}{\theta_{j_1}} \geqslant \frac{1}{1+\eta}.$$

Inductively, choose $j_2, ..., j_n$ so that, for each $k = 2, ..., n$,

$$j_1 + \cdots + j_{k-1} < j_k, \qquad \frac{8\theta_{j_k}}{\theta_{j_1 + \cdots + j_{k-1}+1}} < \eta, \qquad \text{and} \qquad \frac{\theta_{j_1 + \cdots + j_k + 1}}{\theta_{j_k}} \geqslant \frac{1}{1+\eta}.$$

The latter is possible, since $\lim_{n \to \infty} \theta_n^{1/n} = 1$.

Next, we choose an infinite R.I.S. $\{z_i\}_{i=1}^\infty$ in Y where each z_i is a $(\theta_{t_i}^2, t_i)$-seminormalized s.c.c. For each i, let $l_i = \max(\operatorname{supp} z_i)$. Let i_0 be such that

$$t_{i_0} > j_1 + \cdots + j_n + 1 \qquad \text{and} \qquad \frac{\theta_{t_{i_0}}}{\theta_{j_1 + \cdots + j_n + 1}} < \frac{\eta}{16}.$$

We set $L_0 = \{l_i\}_{i > i_0}$.

Now let $0 < \varepsilon < \min\{\theta_{j_1 + \cdots + j_n + 1}^2, \eta(1 - \theta_1)\}$.

We choose an $(\varepsilon, (j_1, ..., j_n))$-family of nested trees $(\mathcal{T}_1^X, ..., \mathcal{T}_n^X)$ in X, indexed by a tree $\mathcal{T}$, supported in L_0. Let $\{a_\beta\}_{\beta \in \mathcal{T}}$ be the corresponding coefficients. Then, for each $r \leqslant n$, there exists a set $\{l_\alpha^r\}_{\alpha \in \mathcal{T}, |\alpha| = r}$, contained in L_0, and such that for all $t < r$ and $\gamma \in \mathcal{T}$ with $|\gamma| = t$,

$$u_\gamma^r = \sum_{\gamma \prec \alpha, |\alpha| = r} \left(\prod_{\gamma \prec \beta \preccurlyeq \alpha} a_\beta \right) e_{l_\alpha^r}$$

is an $(\varepsilon, j_{t+1} + \cdots + j_r)$-basic s.c.c. of $\{e_{l_\alpha^r}\}_{\alpha \in \mathcal{T}, |\alpha| = r}$.

For each $\alpha \in \mathcal{T}$ with $|\alpha| = r$, denote by z^r_α the element of $\{z_i\}_{i \in \mathbb{N}}$ with $\max \operatorname{supp}(z^r_\alpha) = l^r_\alpha$. Then, for $\gamma \in \mathcal{T}$ with $|\gamma| = t < r$, the vector

$$y^r_\gamma = \sum_{\gamma \prec \alpha,\ |\alpha| = r} \left(\prod_{\gamma \prec \beta \preccurlyeq \alpha} a_\beta \right) z^r_\alpha$$

is an $(\varepsilon, j_{t+1} + \cdots + j_r)$-R.I.s.c.c.

For each $r = 1, ..., n$, we set $x^r = y^r_0 / \|y^r_0\|$. If $r \geqslant 2$ then for each $\alpha \in \mathcal{T}$, $1 \leqslant |\alpha| \leqslant r-1$, we set $x^r_\alpha = (1/\|y^r_0\|)\, y^r_\alpha$, so that, for each $t \leqslant r-1$,

$$x^r = \frac{1}{\|y^r_0\|} \sum_{\alpha \in \mathcal{T},\ |\alpha| = r} \left(\prod_{0 \prec \beta \preccurlyeq \alpha} a_\beta \right) z^r_\alpha = \sum_{\alpha \in \mathcal{T},\ |\alpha| = t} \left(\prod_{0 \prec \beta \preccurlyeq \alpha} a_\beta \right) x^r_\alpha.$$

1.19. Lemma. *For each $r \leqslant n$, $t < r$, and $\alpha \in \mathcal{T}$ with $|\alpha| = t$,*

$$\frac{1}{16} \leqslant \|x^r_\alpha\| \leqslant 16(1+\eta).$$

Proof. By the construction, for each $t \leqslant r-1$ and $\alpha \in \mathcal{T}$ with $|\alpha| = t$, y^r_α is an $(\varepsilon, j_{t+1} + \cdots + j_r)$-R.I.s.c.c. It follows from Proposition 1.15 that

$$\frac{\theta_{j_{t+1} + \cdots + j_r + 1}}{2} \leqslant \|y^r_\alpha\| \leqslant 8\theta_{j_{t+1} + \cdots + j_r}.$$

Hence, for $0 < |\alpha| = t$,

$$\frac{1}{16} \leqslant \frac{1}{16} \frac{\theta_{j_{t+1} + \cdots + j_r + 1}}{\theta_{j_1 + \cdots + j_r}} \leqslant \|x^r_\alpha\| = \frac{\|y^r_\alpha\|}{\|y^r_0\|} \leqslant \frac{16\theta_{j_{t+1} + \cdots + j_r}}{\theta_{j_1 + \cdots + j_r + 1}} \leqslant 16(1+\eta). \quad \blacksquare$$

1.20. Lemma. *Let $r \geqslant 2$ and $\alpha \in \mathcal{T}$ with $|\alpha| = t < r-1$. If $i < j_{t+1} + \cdots + j_{r-1}$ and $(E_p)_{p=1}^k$ is an $\mathcal{F}_i$-admissible family of sets, then*

$$\sum_{p=1}^k \|E_p x^r_\alpha\| \leqslant 32(1+\eta).$$

Proof. By the construction,

$$y^r_\alpha = \sum_{|\gamma| = r-1,\ \alpha \prec \gamma} \left(\prod_{\alpha \prec \beta \preccurlyeq \gamma} a_\beta \right) y^r_\gamma,$$

where $l^{r-1}_\gamma < y^r_\gamma < l^{r-1}_{\gamma^+}$ for every $\gamma \in \mathcal{T}$ with $|\gamma| = r-1$ and $\alpha \prec \gamma$. (Recall that y^r_γ is a convex combination of $(z^r_\beta)_{|\beta| = r}$ and that $\max \operatorname{supp}(z^r_\beta) = l^r_\beta$. By the definition of $(\mathcal{T}^X_1, ..., \mathcal{T}^X_n)$, we have $l^{r-1}_\gamma < l^r_\beta < l^{r-1}_{\gamma^+}$.)

Also, the corresponding basic convex combination

$$u_\alpha^{r-1}=\sum_{|\gamma|=r-1,\, \alpha\prec\gamma}\left(\prod_{\alpha\prec\beta\preccurlyeq\gamma} a_\beta\right) e_{l_\gamma^{r-1}}$$

is an $(\varepsilon, j_{t+1}+\cdots+j_{r_1})$-basic s.c.c.

An argument similar to the one in Lemma 1.13 yields

$$\sum_{p=1}^{k}\|E_p y_\alpha^r\|\leqslant 2\max_{|\gamma|=r-1,\,\alpha\prec\gamma}\|y_\gamma^r\|.$$

Dividing by $\|y_0^r\|$ we obtain the conclusion. ∎

1.21. PROPOSITION. *The sequence* $\{x^r\}_{r=1}^n$ *is* 36-*equivalent to the standard basis of* ℓ_∞^n.

Proof. We need to prove that

$$\left\|\sum_{r=1}^{n} x^r\right\|\leqslant 36.$$

To do this we estimate $\varphi(\sum_{r=1}^n x^r)$ for $\varphi\in K$, distinguishing two cases for φ:

Case I. $\varphi\in\mathscr{A}_i$, $i\geqslant j_0$. Let $r_0\in\{0, ..., n\}$ be such that

$$j_{r_0}\leqslant i<j_{r_0+1}.$$

Then

(a) For $r\geqslant r_0+2$ we get $i<j_{r_1}<j_1+\cdots+j_{r-1}$. Using Lemma 1.20, we see that

$$|\varphi(x^r)|\leqslant 32\theta_i(1+\eta)\leqslant 64\theta_{j_0}<\eta.$$

(b) Let now $1\leqslant r\leqslant r_0-1$. We know that y_0^r is an $(\varepsilon, j_1+j_2+\cdots+j_r)$-R.I.s.c.c. of the z_i's. Also, $\varphi\in\mathscr{A}_i$, where $j_1+j_2+\cdots+j_r<j_{r+1}\leqslant i$.

Let $z_{i_1}, ..., z_{i_k}$ be the semi-normalized s.c.c.'s which compose y_0^r where, for $p=1, ..., k$, z_{i_p} is a $(\theta_{t_p^r}^2, t_p^r)$-seminormalized s.c.c. Set $t_0^r=t_{i_0}$ where by construction t_{i_0} is such that $\theta_{t_{i_0}}/\theta_{j_1+\cdots+j_n+1}<\eta/16$ and $t_{i_0}=t_0^r<t_p^r$ for all $p=1, ..., k$.

From Proposition 1.15 we get

$$|\varphi(y_0^r)|\leqslant 4\theta_i\leqslant 4\theta_{j_{r+1}}\qquad \text{if}\quad i<t_1^r,$$

and

$$|\varphi(y_0^r)| \leqslant 4(\theta_{t_0^r} + \varepsilon) \qquad \text{if} \quad i \geqslant t_1^r.$$

Dividing by $\|y_0^r\|$ and by the choice of the j_k's we obtain

$$|\varphi(x^r)| \leqslant \frac{8\theta_{j_{r+1}}}{\theta_{j_1+\cdots+j_r+1}} < \eta \qquad \text{if} \quad i < t_1^r,$$

and

$$|\varphi(x^r)| \leqslant 8\,\frac{\theta_{t_{i_0}}}{\theta_{j_1+\cdots+j_r+1}} + 8\,\frac{\theta^2_{j_1+\cdots+j_r+1}}{\theta_{j_1+\cdots+j_r+1}}$$

$$< \frac{\eta}{2} + 8\theta_{j_1+\cdots+j_r+1} < \eta \qquad \text{if} \quad i \geqslant t_1^r.$$

We conclude that, in this case,

$$\left|\varphi\left(\sum_{r=1}^{n} x^r\right)\right| \leqslant \left|\varphi\left(\sum_{r\neq r_0,\, r_0+1} x^r\right)\right| + |\varphi(x^{r_0})| + |\varphi(x^{r_0+1})| \leqslant n\eta + 2 < 3.$$

Case II. $\varphi \in \mathscr{A}_i$, $i < j_0$. Consider an analysis $\{K^s(\varphi)\}_{s=1}^q$ of φ. For $s \leqslant q$ and $f \in K^s(\varphi)$, let $f^+ \in K^s(\varphi)$ be the successor of f in $K^s(\varphi)$; that is, f^+ is such that $\operatorname{supp} f < \operatorname{supp} f^+$ and if $g \in K^s(\varphi)$ with $\operatorname{supp} f < \operatorname{supp} g$ then either $g = f^+$ or $\operatorname{supp} f^+ < \operatorname{supp} g$.

For $f \in \bigcup_s K^s(\varphi)$, we set

$$E^f = [\min(\operatorname{supp} f), \min(\operatorname{supp} f^+)) \subset \mathbb{N}$$

($E^f = [\min(\operatorname{supp} f), \max(\operatorname{supp} x^n)]$ if f does not have a successor).

Recall that $x^1 = \sum_{k=1}^m a_k z_k^1$ and, for $k = 1, ..., m$, $l_k^1 = \max(\operatorname{supp} z_k^1)$. We set

$$I_k = [l_k^1, l_{k+1}^1) \subset \mathbb{N}, \qquad k = 1, ..., m-1 \qquad \text{and}$$

$$I_m = [l_m^1, \max(\operatorname{supp} x^n)].$$

Notice that for $r \geqslant 2$ we have $\operatorname{supp}(x_k^r) \subset I_k$.

For $k = 1, ..., m$ and $f \in \bigcup_s K^s(\varphi)$, we say that f *covers* I_k if $I_k \subset E^f$.

We may assume without loss of generality that $\min(\operatorname{supp} \varphi) \leqslant l_1^1$. Therefore, for fixed s, any I_k is either covered by some f in $K^s(\varphi)$ or intersected by E^f for at least two different f's in $K^s(\varphi)$. Also, every I_k is covered by φ.

Set now

$$J_1 = \{k = 1, ..., m \colon I_k \text{ is covered by some functional in } \cup K^s(\varphi) \text{ belonging to some class } \mathcal{A}_l \text{ with } l \geqslant j_0\},$$

and

$$J_2 = \left\{ k = 1, ..., m \colon I_k \text{ is covered only by functionals in } \cup K^s(\varphi) \text{ which belong to } \bigcup_{l < j_0} \mathcal{A}_l \right\}.$$

Consider any $k \in J_1$. Let $f \in \cup K^s(\varphi)$ be a functional which covers I_k and such that $f \in \mathcal{A}_l$ for some $l \geqslant j_0$. Then, exactly as in Case I we can get

$$|\varphi(x_k^r)| \leqslant |f(x_k^r)| < \eta$$

for all but two $r \in \{2, ..., n\}$. This gives $|\varphi(\sum_{r=2}^n x_k^r)| \leqslant n\eta + 32(1+\eta) < 34$, and we conclude that

$$\left| \varphi \left(\sum_{r=2}^{n} \sum_{k \in J_1} a_k x_k^r \right) \right| < 34.$$

We turn now to J_2. Let $\varphi = \theta_i \sum_{p=1}^s f_p$ where $i < j_0$. Consider the set

$$R_1 = \{k \in J_2 \colon I_k \text{ is intersected by at least two } f_p\text{'s}\}.$$

Since the family $(f_p)_{p=1}^s$ is $\mathcal{F}_i$-admissible, the set $\{l_k^1 \colon k \in R_1 \setminus \{\min R_1\}\}$ belongs to $\mathcal{F}_i \subset \mathcal{F}_{j_0}$ and so, $\{l_k^1 \colon k \in R_1\} \in \mathcal{F}_{j_1 - 1}$. Therefore, $\sum_{k \in R_1} a_k < \varepsilon$.

Let $L_1 = J_2 \setminus R_1$ and, for $p = 1, ..., s$, let

$$L_1^p = \{k \in L_1 \colon I_k \subset E^{f_p}\}.$$

For any $r \geqslant 2$, we get

$$\begin{aligned} \left| \varphi \left(\sum_{k \in J_1} a_k x_k^r \right) \right| &\leqslant \theta_i \left(\sum_{p=1}^{s} \left| f_p \left(\sum_{k \in L_1^p} a_k x_k^r \right) \right| \right) + \left(\sum_{k \in R_1} a_k \right) \max_k \|x_k^r\| \\ &\leqslant \theta_1 \left(\sum_{p=1}^{s} \left| f_p \left(\sum_{k \in L_1^p} a_k x_k^r \right) \right| \right) + \varepsilon \max_k \|x_k^r\|. \end{aligned}$$

Consider now any p, $1 \leqslant p \leqslant s$, with $L_1^p \neq \emptyset$. By the definition of J_2 this implies that $f_p = \theta_{i_p} \sum_{t=1}^{l_p} g_t^p$ where $i_p < j_0$ and $(g_t^p)_{t=1}^{l_p}$ is $\mathcal{F}_{i_p}$-admissible. (It is clear that we cannot have $f_p \in K^0$ and $L_1^p \neq \emptyset$.) We will partition L_1^p in the same way that we partitioned J_2: We set

$$R_2^p = \{k \in L_1^p : I_k \text{ is intersected by at least two } g_t^p\text{'s}\}$$

and for each $t = 1, ..., l_p$,

$$L_2^t(p) = \{k \in L_1^p : I_k \subset E^{g_t^p}\}.$$

The family $\{g_t^p : p \text{ such that } L_1^p \neq \emptyset,\ t = 1, ...l_p\}$ is $\mathscr{F}_{i+j_0}$-admissible and so the set $\{l_k^1 = k \in \bigcup_{p=1}^s R_2^p\}$ belongs to $\mathscr{F}_{i+j_0+1} \subset \mathscr{F}_{2j_0} \subset \mathscr{F}_{j_1-1}$. We conclude that

$$\sum_{k \in \bigcup_p R_2^p} a_k < \varepsilon.$$

So, for each $r \geqslant 2$ we get the estimate

$$\begin{aligned}
&\left|\varphi\left(\sum_{k \in J_2} a_k x_k^r\right)\right| \\
&\quad \leqslant \theta_1 \sum_p \theta_{i_p} \sum_t \left|g_t^p\left(\sum_{k \in L_2^t(p)} a_k x_k^r\right)\right| + \theta_1 \sum_p f_p\left(\sum_{k \in R_2^p} a_k x_k^r\right) + \varepsilon \max_k \|x_k^r\| \\
&\quad \leqslant \theta_1^2 \sum_{p,\,t} \left|g_t^p\left(\sum_{k \in L_2^t(p)} a_k x_k^r\right)\right| + \theta_1\left(\sum_{k \in \bigcup_p R_2^p} a_k\right) \max_k \|x_k^r\| + \varepsilon \max_k \|x_k^r\| \\
&\quad \leqslant \theta_1^2 \sum_{p,\,t} \left|g_t^p\left(\sum_{k \in L_2^t(p)} a_k x_k^r\right)\right| + (\theta_1 + 1)\,\varepsilon.
\end{aligned}$$

We can now partition each $L_2^t(p)$ and continue in this manner for s_0 steps, where $\theta_1^{s_0} < \eta$. By the choice of $j_1, j_0 s_0 < j_1$. Recall that $\varphi \in K^q$. If $q > s_0$ then for $r \geqslant 2$,

$$\begin{aligned}
\left|\varphi\left(\sum_{k \in J_2} a_k x_k^r\right)\right| &\leqslant \theta_1^{s_0} \sum_{f \in K^{q-s_0}(\varphi)} f\left(\sum_{I_k \subset E^f} a_k x_k^r\right) \\
&\quad + (1 + \theta_1 + \cdots + \theta_1^{s_0-1})\,\varepsilon \max_k \|x_k^r\|.
\end{aligned}$$

Of course, if $q \leqslant s_0$ then we have only the second term at the right hand side. Finally, for $r \geqslant 2$, we get

$$\left|\varphi\left(\sum_{k \in J_2} a_k x_k^r\right)\right| < \max_k \|x_k^r\| \left(\eta + \frac{\varepsilon}{1-\theta_1}\right) < 60\eta.$$

We conclude that

$$\left|\varphi\left(\sum_{r=1}^{n} x^r\right)\right| \leqslant |\varphi(x^1)| + \left|\varphi\left(\sum_{r=2}^{n} \sum_{k\in J_1} a_k x_k^r\right)\right| + \sum_{r=2}^{n} \left|\varphi\left(\sum_{k\in J_2} a_k x_k^r\right)\right|$$

$$\leqslant 1 + 34 + 60n\eta < 36.$$

This completes the proof of the proposition. Theorem 1.6 now follows. ∎

C. Modified Mixed Tsirelson Spaces

The modified Tsirelson space T_M was introduced by W. B. Johnson in [10]. Later, P. Casazza and E. Odell [6] proved that T_M is naturally isomorphic to T. Analogously, given a sequence of compact families $\{\mathcal{M}_k\}_{k=1}^{\infty}$ in $[\mathbb{N}]^{<\omega}$ and a sequence of positive reals $\{\theta_k\}_{k=1}^{\infty}$, we define the *modified mixed Tsirelson space* $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$.

1.22. Definition. Let $\mathcal{M}$ be a family of finite subsets of $\mathbb{N}$.

(a) A finite sequence $(E_i)_{i=1}^{k}$ of finite non-empty subsets of $\mathbb{N}$ is said to be *$\mathcal{M}$-allowable* if the set $\{\min E_1, \min E_2, ..., \min E_k\}$ belongs to $\mathcal{M}$ and $E_i \cap E_j = \emptyset$ for all $i, j = 1, ..., k$, $i \neq j$.

(b) A finite sequence $(x_i)_{i=1}^{k}$ of vectors in c_{00} is *$\mathcal{M}$-allowable* if the sequence $(\mathrm{supp}(x_i))_{i=1}^{k}$ is $\mathcal{M}$-allowable.

1.23. Definition of the Space $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$. Let $(\mathcal{M}_k)_k$ be a sequence of compact, hereditary and spreading families of finite subsets of $\mathbb{N}$ and let $(\theta_k)_k$ be a sequence of positive reals with $\theta_k < 1$ for every k and $\lim_k \theta_k = 0$. Inductively, we define a subset K of B_{ℓ_∞} as follows.

We set $K^0 = \{\pm e_n : n \in \mathbb{N}\}$.

For $s \geqslant 0$, given K^s we define for each $k \geqslant 1$,

$$K_k^{s+1} = \left\{\theta_k\left(\sum_{i=1}^{n} f_i\right) : n \in \mathbb{N},\ f_i \in K^s,\ i \leqslant n,\right.$$
$$\left.\text{and the sequence } (f_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-allowable}\right\}.$$

We set

$$K^{s+1} = K^s \cup \left(\bigcup_{k=1}^{\infty} K_k^{s+1}\right)$$

Finally, we define

$$K=\bigcup_{s=0}^{\infty} K^s.$$

Note that K is the smallest subset of B_{ℓ_∞} which contains $\pm e_n$ for all $n\in\mathbb{N}$ and has the property that $\theta_k(f_1+\cdots+f_n)$ is in K whenever $f_1, ..., f_n\in K$ and the sequence $(f_i)_{i=1}^n$ is $\mathcal{M}_k$-allowable.

We now define a norm on c_{00} by

$$\|x\|=\sup_{f\in K}\langle x, f\rangle \qquad \text{for all} \quad x\in c_{00}.$$

The space $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$ is the completion of $(c_{00}, \|\cdot\|)$. We call K the *norming set* of $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$.

The following proposition is an easy consequence of the definition:

1.24. PROPOSITION. *Let $X=T_M[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$.*

(a) *The norm of X satisfies the following implicit equation: For all $x\in X$,*

$$\|x\|=\max\left\{\|x\|_\infty, \sup_k \theta_k \sup\left\{\sum_{i=1}^n \|E_i x\| : (E_i)_{i=1}^n \text{ is } \mathcal{M}_k\text{-allowable}\right\}\right\}.$$

(b) *The sequence $(e_n)_{n=1}^\infty$ is a* 1*-unconditional basis for X.*

We also consider *boundedly modified mixed Tsirelson spaces* denoted by

$$T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^\infty],$$

for some $m\in\mathbb{N}$. The definition of $T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$ is similar to that of $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^\infty]$, the only difference being that at the inductive step $s+1$ we set

$$K_k^{s+1}=\left\{\theta_k\left(\sum_{i=1}^n f_i\right) : n\in\mathbb{N}, f_i\in K^s, i\leqslant n, \right.$$
$$\left.\text{and the sequence } (f_i)_{i=1}^n \text{ is } \mathcal{M}_k\text{-allowable}\right\}.$$

for $k \leqslant m$, while

$$K_k^{s+1} = \left\{ \theta_k \left(\sum_{i=1}^{n} f_i \right) : n \in \mathbb{N},\ f_i \in K^s,\ i \leqslant n, \right.$$
$$\left. \text{and the sequence } (f_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-admissible} \right\}.$$

for $k \geqslant m+1$.

1.25. Proposition. *Let* $Y = T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$.

(a) *The norm* $\|\cdot\|$ *of* Y *satisfies the following implicit equation*:

$$\|x\| = \max \left\{ \|x\|_\infty,\ \max_{k \leqslant m} \theta_k \sup \left\{ \sum_{i=1}^{n} \|E_i x\| : (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-allowable} \right\}, \right.$$
$$\left. \sup_{k \geqslant m+1} \theta_k \sup \left\{ \sum_{i=1}^{n} \|E_i x\| : (E_i)_{i=1}^{n} \text{ is } \mathcal{M}_k\text{-admissible} \right\} \right\}.$$

(b) *The sequence* $(e_n)_n$ *is a* 1-*unconditional basis for* Y.

In the sequel we consider spaces $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ or $T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ where $(\mathcal{M}_k)_k$ is a subsequence of the Schreier sequence $(\mathcal{F}_n)_{n=1}^{\infty}$. In this case, by Proposition 1.24(a) (resp. Proposition 1.25(a)) we have that for all sequences $(x_i)_{i=1}^{n}$ of disjointly supported vectors with $\operatorname{supp} x_i \subset [n, \infty)$,

$$\left\| \sum_{i=1}^{n} x_i \right\| \geqslant \theta_1 \sum_{i=1}^{n} \|x_i\|$$

in $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ (resp. $T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$). It is clear from this inequality that c_0 is not finitely disjointly representable in any block subspace of $T_M[[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ or $T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$. Combining this with Theorem 1.6 we get the following.

1.26. Corollary. *Let* $(\theta_n)_{n=1}^{\infty}$ *be a regular sequence with* $\lim \theta_n^{1/n} = 1$. *Let* $X = T_M[(\mathcal{F}_k, \theta_k)_{k=1}^{\infty}]$ *or* $X = T_{M(m)}[(\mathcal{F}_k, \theta_k)_{k=1}^{\infty}]$. *Then the spaces* X *and* $T[(\mathcal{F}_k, \theta_k)_{k=1}^{\infty}]$ *are totally incomparable.*

1.27. Theorem. *Suppose that the sequence* $(\theta_k)_k$ *decreases to* 0 *and that the Schreier family* $\mathcal{S}$ *is contained in* $\mathcal{M}_1$. *Then, the spaces* $T_M[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$ *and* $T_{M(m)}[(\mathcal{M}_k, \theta_k)_{k=1}^{\infty}]$, $m = 1, 2, \ldots$ *are reflexive.*

Proof. Let $X = T_M[(\mathscr{M}_k, \theta_k)_{k=1}^{\infty}]$. The proof for $T_{M(m)}[(\mathscr{M}_k, \theta_k)_{k=1}^{\infty}]$ is the same. We shall prove that the basis $(e_n)_{n=1}^{\infty}$ is boundedly complete and shrinking in X.

(a) $(e_n)_{n=1}^{\infty}$ *is boundedly complete.* Suppose on the contrary there exist $\varepsilon > 0$ and a block sequence $\{x_i\}_{i=1}^{\infty}$ of $\{e_n\}_{n=1}^{\infty}$ such that $\sup_n \|\sum_{i=1}^{n} x_i\| \leqslant 1$ while $\|x_i\| \geqslant \varepsilon$ for $i = 1, 2, ...$

Choose $n_0 \in \mathbb{N}$ such that $n_0 \theta_1 > \varepsilon$. Then, the finite sequence $(x_i)_{i=n_0+1}^{2n_0}$ s $\mathscr{S}$-allowable and since $\mathscr{S} \subseteq \mathscr{M}_1$ it is $\mathscr{M}_1$-allowable. Using Proposi tion 1.24(a) (resp. Proposition 1.25(a)) we get

$$\left\| \sum_{i=n_0+1}^{2n_0} x_i \right\| \geqslant \theta_1 \sum_{i=n_0+1}^{2n_0} \|x_i\| \geqslant n_0 \theta_1 \varepsilon > 1,$$

a contradiction which completes the proof.

(b) $(e_n)_{n=1}^{\infty}$ *is a shrinking basis.* For $f \in X^*$, $m \in \mathbb{N}$, we denote by $Q_m(f)$ the restriction of f to the space spanned by $(e_k)_{k \geqslant m}$. We need to prove that, for every $f \in B_{X^*}$, $Q_m(f) \to 0$ as $m \to \infty$.

Let K be the norming set of X. Then $B_{X^*} = \overline{\text{co}(K)}$ where the closure is in the topology of pointwise convergence. We shall show that for all $f \in B_{X^*}$ there is $l \in \mathbb{N}$ such that $Q_l(f) \in \theta_1 B_{X^*}$. By standard arguments it suffices to prove this for $f \in \bar{K}$.

Let $f \in \bar{K}$. Let $(f^n)_{n=1}^{\infty}$ be a sequence in K converging pointwise to f. If $f^n \in K^0$ for an infinite number of n, then there is nothing to prove. So, suppose that for every n there are $k_n \in \mathbb{N}$, a set $M_n = \{m_1^n, ..., m_{d_n}^n\} \in \mathscr{M}_{k_n}$ and vectors $f_i^n \in K$, $i = 1, ..., d_n$ such that $f^n = \theta_{k_n} \sum_{i=1}^{d_n} f_i^n$, $m_i^n = \min \text{supp}(f_i^n)$, $i = 1, ..., d_n$ and $\text{supp}(f_i^n) \cap \text{supp}(f_j^n) = \varnothing$ for $i \neq j$. If there is a subsequence of $(\theta_{k_n})_n$ converging to 0, then $f = 0$. So we may assume that there is a k such that $k_n = k$ for all n, that is, $\theta_{k_n} = \theta_k$ and $M_n = \{m_1^n, ..., m_{d_n}^n\} \in \mathscr{M}_k$.

Since $\mathscr{M}_k$ is compact, substituting $\{f^n\}$ with a subsequence we get that there is a set $M = \{m_1, ..., m_d\} \in \mathscr{M}_k$ such that the sequence of indicator functions of M_n converges to the indicator function of M. So, for large n, $m_i^n = m_i$, $i = 1, 2, ..., d$ and $m_{d+1}^n \to \infty$ as $n \to \infty$. Since $\min \text{supp}\, f_{d+1}^n = m_{d+1}^n \to \infty$, the sequence $\tilde{f}^n = \theta_k \sum_{i=1}^{d} f_i^n$ tends to f pointwise and we may assume that $f^n = \theta_k \sum_{i=1}^{d} f_i^n$. Passing again to a subsequence of $\{f^n\}$ we have that, for each $i = 1, ..., d$ there exists $f_i \in \bar{K}$ with $f_i^n \to f_i$ pointwise and $f = \theta_k(f_1 + \cdots + f_d)$.

Now, for each $i = 1, ..., d$, either $f_i^n = e_{m_i}^*$ for all n (eventually) or

$$f_i^n = \theta_{k_i^n} \sum_{m=1}^{l_i^n} g_m^{n,i}, \qquad i = 1, ..., d,$$

where for every $n \in \mathbb{N}$ and $m = 1, ..., l_i$, $g_m^{n,i} \in K$ and the family $\{g_m^{n,i}\}_{m=1}^{l_i^n}$ is $\mathscr{M}_{k_i^n}$-allowable. Let $A \subset \{1, ..., d\}$ be the set of indices i for which f_i^n is of the second type for all n. As before, forgetting those i's for which $f_i^n \to 0$, we may assume that, for each $i \in A$, there is k_i such that $k_i^n = k_i$ and a set $M_i = \{m_1^i, ..., m_{l_i}^i\}$ such that $m_r^i = \min \operatorname{supp}(g_r^{n,i})$ for all $n = 1, 2, ...,$ $r = 1, ..., l_i$, and $\min \operatorname{supp}(g_{l_i+1}^{n,i}) \to \infty$ as $n \to \infty$. So, for $i \in A$, the sequence $\tilde{f}_i^n = \theta_{k_i} \sum_{m=1}^{l_i} g_m^{n,i}$ tends to f_i pointwise.

Let $l = \max(\{\sum_{i \in A} l_i\} \cup \{m_i : i = 1 \cdots d\})$ and $h_m^{n,i} = Q_l(g_m^{n,i}) \in K$, $i \in A$, $m = 1, ..., l_i$, $n = 1, 2, ...$ Then, the sequence $\theta_k \sum_{i \in A} \theta_{k_i} \sum_{m=1}^{l_i} h_m^{n,i} = Q_l(\theta_k \sum_{i=1}^d \tilde{f}_i^n)$ tends to $Q_l(f)$ as $n \to \infty$.

On the other hand, since, for each n, $\#\{h_m^{n,i},\ i \in A,\ m = 1, ..., l_i\} \leqslant l$, $l \leqslant \min \operatorname{supp}(h_m^{n,i})$ for every i and m, and the sets $\operatorname{supp}(h_m^{n,i})$, $i \in A$, $m = 1, ..., l_i$ are mutually disjoint, we get that the family $\{h_m^{n,i}\}_{i,m}$ is Schreier-allowable. Since the Schreier family $\mathscr{S}$ is contained in $\mathscr{M}_1$, $0 < \theta_{k_i}/\theta_1 \leqslant 1$, $\{h_m^{n,i}\}_{i,m}$ is $\mathscr{S}$-allowable for every n and $h_m^{n,i} \in K$, it is easy to see that $(1/\theta_k)\, Q_l(\theta_k \sum_{i=1}^d \tilde{f}_i^n) = \theta_1(\sum_{i \in A} (\theta_{k_i}/\theta_1) \sum_{m=1}^{l_i} h_m^{n,i}) \in \operatorname{co}(K)$ for all n. We conclude that $Q_l(\theta_k \sum_{i=1}^d \tilde{f}_i^n) \in \theta_k \operatorname{co}(K)$, and so, $Q_l(f) \in \theta_k \overline{\operatorname{co}(K)} \subseteq \theta_1 \overline{\operatorname{co}(K)}$. ∎

We note that the 2-convexifications $T_M^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ and $T_{M(m)}^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ of $T_M[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ and $T_{M(m)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ are weak Hilbert spaces. The proof of this is similar to the proof of the analogous statement for the 2-convexifications $T_\delta^{(2)}$ of the Tsirelson spaces T_δ as presented in [15, Lemma 13.5]. It is an immediate consequence of Theorem 1.27 that $T_M^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ (and $T_{M(m)}^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$) does not contain ℓ_2. Moreover, we can show that for sequences $(\theta_n)_n$ with $\lim_n \theta_n^{1/n} = 1$, no subspace of $T_M^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ (or $T_{M(m)}^{(2)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$) can be isomorphic to a subspace of $T_\delta^{(2)}$. It suffices to prove the following.

1.28. PROPOSITION. *Let $0 < \delta < 1$ and let $(\theta_n)_n$ be a regular sequence with $\lim \theta_n^{1/n} = 1$. Let $X = T_M[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ or $X = T_{M(m)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$. Then the spaces X and T_δ are totally incomparable.*

Proof. Let $X = T_M[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$ or $X = T_{M(m)}[(\mathscr{F}_k, \theta_k)_{k=1}^\infty]$. Suppose on the contrary that there exist normalized block sequences $\{x_i\}_i$ in X and $\{y_i\}_i$ in T_δ which are equivalent as basic sequences. Let $l_i = \min \operatorname{supp} y_i$, $i = 1, 2, ...$ From [5, Theorem 13] we get that $\{x_i\}_X$ is equivalent to $\{e_{l_i}\}_{T_\delta}$. Let $m_i = \min \operatorname{supp} x_i$, $i = 1, 2, ...$ We choose a subsequence $\{i_k\}_k$ of indices such that either $l_{i_1} \leqslant m_{i_1} < l_{i_2} \leqslant m_{i_2} < \cdots$ or $m_{i_1} < l_{i_1} < m_{i_2} < l_{i_2} < \cdots$. In either case, using Theorem 13 of [5] once more, we get that the basic sequences $\{e_{l_{i_k}}\}$ and $\{e_{m_{i_k}}\}$ are equivalent in T_δ. We conclude that $\{e_{m_{i_k}}\}_{T_\delta}$ is equivalent to $\{x_{i_k}\}_X$.

Let now $j \in \mathbb{N}$ and let $\sum_{k \in A} a_k e_{m_{i_k}}$ be a (θ_j^j, j)-special convex combination. As in Lemma 1.12 we get that $\|\sum_{k \in A} a_k e_{m_{i_k}}\|_{T_\delta} \leqslant \delta^j + \theta_j^j$. On the other hand, since the sequence $(x_{i_k})_{k \in A}$ is $\mathscr{F}_j$-admissible, we have that

$\|\sum_{k\in A} a_k x_{i_k}\|_X \geqslant \theta_j$. But the assumption $\lim \theta_j^{1/j}=1$ leads to a contradiction which completes the proof. ▮

2. THE SPACE $X_{M(1),u}$

We give an example of a boundedly modified mixed Tsirelson space of the form $T_{M(1)}[(\mathcal{F}_{k_j}, \theta_j)_{j=1}^{\infty}]$ which is arbitrarily distortable.

Definition of $X_{M(1),u}$. We choose a sequence of integers $(m_j)_{j=1}^{\infty}$ such that $m_1=2$ and for $j=2, 3, ..., m_j > m_{j-1}^{m_{j-1}}$.

We choose inductively a subsequence $(\mathcal{F}_{k_j})_{j=0}^{\infty}$ of $(\mathcal{F}_n)_n$.

We set $k_1=1$. Suppose that k_j, $j=1, ..., n-1$ have been chosen. Let t_n be such that $2^{t_n} \geqslant m_n^2$. We set $k_n = t_n(k_{n-1}+1)+1$.

For $j=0, 1, ...,$ we set $\mathcal{M}_j = \mathcal{F}_{k_j}$. We define

$$X_{M(1),u} = T_{M(1)}\left[\left(\mathcal{M}_j, \frac{1}{m_j}\right)_{j=1}^{\infty}\right].$$

Notation. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. We set

$$\mathcal{F}' = \{A \cup B : A \in \mathcal{F}, B \in \mathcal{F}, A \cap B = \varnothing\}.$$

2.1. Definition. Given $\varepsilon > 0$ and $j=2, 3, ...,$ an *(ε, j)-basic special convex combination* (*(ε, j)-basic* s.c.c.) *relative to* $X_{M(1),u}$ is a vector of the form $\sum_{k\in F} a_k e_k$ such that $F \in \mathcal{M}_j$, $a_k \geqslant 0$, $\sum_{k\in F} a_k = 1$, $\{a_k\}_{k\in F}$ is decreasing, and, for every $G \in \mathcal{F}'_{t_j(k_{j-1}+1)}$, $\sum_{k\in G} a_k < \varepsilon$.

2.2. Lemma. *Let $j \geqslant 2$, $\varepsilon > 0$, D be an infinite subset of $\mathbb{N}$. There exists an (ε, j)-basic special convex combination relative to $X_{M(1),u}$, $x = \sum_{k\in F} a_k e_k$, with $F = \operatorname{supp} x \subset D$.*

Proof. Since $\mathcal{M}_j = \mathcal{F}_{t_j(k_{j-1}+1)+1}$, by Proposition 1.8 there exists a convex combination $x = \sum_{k\in F} a_k e_k$ with $F \in \mathcal{M}_j$, $F \subset D$ and such that $\sum_{k\in G} a_k < \varepsilon/2$ for all $G \in \mathcal{F}_{t_j(k_{j-1}+1)}$. It is clear that this x is an (ε, j)-basic s.c.c. relative to $X_{M(1),u}$. ▮

In the sequel, when we refer to (ε, j)-special convex combinations we always imply "relative to $X_{M(1),u}$."

Notation. Let $X'_{(n)} = T_{M(1)}[(\mathcal{M}'_l, 1/m_l)_{l=1}^n]$ and let $K'(n)$ be the norming set of $X'_{(n)}$. We denote by $|\cdot|_n$ the norm of $X_{(n)}$ and by $|\cdot|_n^*$ the corresponding dual norm.

We set

$$\mathcal{G}_{(n)} = \left\{ \text{supp } f \colon f \in K'(n) \text{ and for every } k \in \text{supp } f,\ f(e_k) > \frac{1}{m_{n+1}^2} \right\}.$$

Remark. Using lemma 1.2 it is easy to see that $\mathcal{G}_{(n-1)} \subset \mathcal{F}_{t_n(k_{n-1}+1)}$. It follows that if $x = \sum_{k \in F} a_k e_k$ is an (ε, n)-basic s.c.c. then, for all $G \in \mathcal{G}'_{(n-1)}$, $\sum_{k \in G} a_k < \varepsilon$.

We give the definition of the set K of functionals that define the norm of the space $X_{M(1),u}$.

We set $K_j^0 = \{\pm e_n \colon n \in \mathbb{N}\}$ for $j = 1, 2, \ldots$.

Assume that the $\{K_j^n\}_{j=1}^\infty$ have been defined. Then, we set $K^n = \bigcup_{j=1}^\infty K_j^n$, and for $j = 2, 3, \ldots$ we set

$$K_j^{n+1} = K^n \cup \left\{ \frac{1}{m_j}(f_1 + \cdots + f_d) \colon \text{supp } f_1 < \cdots < \text{supp } f_d, \right.$$
$$\left. (f_i)_{i=1}^d \text{ is } \mathcal{M}_j\text{-admissible and } f_1, \ldots, f_d \text{ belong to } K^n \right\},$$

while for $j = 1$, we set

$$K_1^{n+1} = K_1^n \cup \{ \tfrac{1}{2}(f_1 + \cdots + f_d) \colon f_i \in K^n, d \in \mathbb{N},$$
$$d \leqslant \min \text{supp } f_1 < \cdots < \min \text{supp } f_d, \text{ and for } i \neq j,$$
$$\text{supp } f_i \cap \text{supp } f_j = \varnothing \}.$$

Set $K = \bigcup_{n=0}^\infty K^n$. Then, the norm $\|\cdot\|$ of $X_{M(1),u}$ is

$$\|x\| = \sup \{ f(x) \colon f \in K \}.$$

Notation. For $j = 1, 2, \ldots$, we denote by $\mathcal{A}_j$ the set $\mathcal{A}_j = \bigcup_{n=1}^\infty (K_j^n \setminus K^0)$. Then, $K = K^0 \cup (\bigcup_{j=1}^\infty \mathcal{A}_j)$.

We will also consider the space $T_{M(1)}[(\mathcal{M}'_j, 1/m_j)_{j=1}^\infty]$. We denote by K' the norming set of this space and by K'^n, $K_j'^n$, $\mathcal{A}'_j$ the subsets of K' corresponding to K^n, K_j^n, and $\mathcal{A}_j$, respectively.

2.3. Definition. (A) Let $m \in \mathbb{N}$, $\varphi \in K^m \setminus K^{m-1}$. An *analysis* of φ is a sequence $\{K^s(\varphi)\}_{s=0}^m$ of subsets of K such that:

(1) For every s, $K^s(\varphi)$ consists of disjointly supported elements of K^s, and $\bigcup_{f \in K^s(\varphi)} \text{supp } f = \text{supp } \varphi$.

(2) If f belongs to $K^{s+1}(\varphi)$, then either $f \in K^s(\varphi)$ or there exists an $\mathcal{S}$-allowable family $(f_i)_{i=1}^d$ in $K^s(\varphi)$ such that $f = \frac{1}{2}(f_1 + \cdots + f_d)$, or, for some $j \geqslant 2$, there exists an $\mathcal{M}_j$-admissible family $(f_i)_{i=1}^d$ in $K^s(\varphi)$ such that $f = (1/m_j)(f_1 + \cdots + f_d)$.

(3) $K^m(\varphi) = \{\varphi\}$.

(B) For $g \in K^{s+1}(\varphi) \backslash K^0(\varphi)$; the set of functionals $\{f_1, ..., f_l\} \subset K^s(\varphi)$ such that $g = (1/m_j)(\sum_{i=1}^l f_i)$ is called the *decomposition* of g.

2.4. LEMMA. *Let $j \geqslant 2$, $0 < \varepsilon \leqslant 1/m_j^2$, $M > 0$, and let $x = \sum_{k=1}^m b_k e_{n_k}$ be an (ε, j)-basic s.c.c.*

Suppose that the vectors $x_k = \sum_{i=1}^{l_k} a_{i,k} e_{n_{i,k}}$ are such that $a_{i,k} \geqslant 0$ for all i, k, $\sum_{i=1}^{l_k} a_{i,k} \leqslant M$, $k = 1, 2, ..., m$, and $n_1 \leqslant n_{1,1} < n_{2,1} < \cdots < n_{l_1,1} < n_2 \leqslant n_{1,2} < n_{2,2} < \cdots < n_3 \leqslant \cdots < n_{l_m,m}$. Then

(a) *For $\varphi \in \bigcup_{s=1}^\infty \mathcal{A}'_s$,*

$$\left|\varphi\left(\sum_{k=1}^m b_k x_k\right)\right| \leqslant \frac{M}{m_s}, \qquad \textit{if} \quad \varphi \in \mathcal{A}'_s, \quad s \geqslant j$$

$$\left|\varphi\left(\sum_{k=1}^m b_k x_k\right)\right| \leqslant \frac{2M}{m_s m_j}, \qquad \textit{if} \quad \varphi \in \mathcal{A}'_s, \quad s < j.$$

(b) *If φ belongs to the norming set $K'(j-1)$ of $T_{M(1)}[(\mathcal{M}'_l, 1/m_l)_{l=1}^{j-1}]$, then*

$$\left|\varphi\left(\sum b_k x_k\right)\right| \leqslant \frac{2M}{m_j^2}.$$

Proof. (1) If $s \geqslant j$, then the estimate is obvious.

Let $s < j$ and $\varphi = (1/m_s) \sum_{l=1}^d f_l$. Without loss of generality we assume that $\varphi(e_{n_{i,k}}) \geqslant 0$ for all $n_{i,k}$. We set

$$D = \left\{n_{i,k} : \sum_{l=1}^d f_l(e_{n_{i,k}}) > \frac{1}{m_j}\right\}.$$

We set $g_l = f_l|_D$. Then, $(1/m_s) \sum_{l=1}^d g_l \in K'(j-1)$, and for every $k \in \operatorname{supp}((1/m_s) \sum_{l=1}^d g_l)$ we have $(1/m_s) \sum_{l=1}^d g_l(e_k) > 1/m_s m_j > 1/m_j^2$. Therefore, $D = \operatorname{supp}((1/m_s) \sum_{l=1}^d g_l) \in \mathcal{G}_{j-1})$. Let $B = \{k$: there exists i with $n_{i,k} \in D\}$. Then $B \in \mathcal{G}'_{(j-1)}$ and so, by the Remark after Lemma 2.2, $\sum_{k \in B} b_k < \varepsilon \leqslant 1/m_j^2$. We get

$$\frac{1}{m_s} \sum_{l=1}^d g_l\left(\sum_{k=1}^m b_k x_k\right) \leqslant \sum_{k \in B} b_k \left(\sum_{i=1}^{l_k} a_{i,k}\right) \leqslant M \sum_{k \in B} b_k \leqslant \frac{M}{m_j^2}.$$

On the other hand,

$$\left(\frac{1}{m_s}\sum_{l=1}^{d} f_{l|_{D^c}}\right)\left(\sum b_k x_k\right) \leqslant \frac{M}{m_s m_j}.$$

Hence,

$$\varphi\left(\sum b_k x_k\right) \leqslant \frac{M}{m_s m_j} + \frac{M}{m_j^2} \leqslant \frac{2M}{m_s m_j}.$$

(b) We assume again that φ is positive. We set $L = \{n_{i,k}: \varphi(e_{n_{i,k}}) > 1/m_j^2\}$. Then,

$$\varphi|_{L^c}\left(\sum b_k x_k\right) \leqslant \frac{M}{m_j^2}.$$

On the other hand, $\operatorname{supp}(\varphi|_L) \in \mathscr{G}_{(j-1)}$ and as before we get $\varphi|_L(\sum b_k x_k) \leqslant M/m_j^2$. Hence,

$$\left|\varphi\left(\sum b_k x_k\right)\right| \leqslant \frac{2M}{m_j^2}. \quad \blacksquare$$

2.5. Definition. (a) Given a block sequence $(x_k)_{k\in\mathbb{N}}$ in $X_{M(1),u}$ and $j \geqslant 2$, a convex combination $\sum_{i=1}^n a_i x_{k_i}$ is said to be an (ε, j)-*special combination* of $(x_k)_{k\in\mathbb{N}}$ ((ε, j)-s.c.c.), if there exist $l_1 < l_2 < \cdots < l_n$ such that $2 < \operatorname{supp} x_{k_1} \leqslant l_1 < \operatorname{supp} x_{k_2} \leqslant l_2 < \cdots < \operatorname{supp} x_{k_n} \leqslant l_n$, and $\sum_{i=1}^n a_i e_{l_i}$ is an (ε, j)-basic s.c.c.

(b) An (ε, j)-s.c.c. $\sum_{i=1}^n a_i x_{k_i}$ is called *seminormalized* if $\|x_{k_i}\| = 1$, $i = 1, ..., n$ and

$$\left\|\sum_{i=1}^{n} a_i x_{k_i}\right\| \geqslant \tfrac{1}{2}.$$

2.6. Lemma. *Let $(x_k)_{k=1}^{\infty}$ be a block sequence in $X_{M(1),u}$ and $j = 2, 3, ...,$ $\varepsilon > 0$. Then, there exists a normalized finite block sequence $\{y_k\}_{k=1}^n$ of $\{x_k\}_{k=1}^{\infty}$ and a convex combination $\sum_{k=1}^n a_k y_k$ which is a seminormalized (ε, j)-s.c.c.*

Proof. Using that $\mathscr{M}_j = \mathscr{F}_{t_j(k_{j-1}+1)+1}$ where $2^{t_j} \geqslant m_j^2$, the proof is similar to the proof of Lemma 1.11. $\blacksquare$

2.7. Lemma. *Let $j \geqslant 3$ and let $x = \sum_{k=1}^{n} a_k x_k$ be a $(1/m_j^4, j)$-s.c.c. where $\|x_k\| \leqslant 1$, $k = 1, ..., n$. Suppose $\varphi = (1/m_r) \sum_{i=1}^{d} f_i \in \mathscr{A}_r$, $2 \leqslant r < j$. Let*

$$L = \{k \in \{1, 2, ..., n\}: \textit{there exist at least two } i_1 \neq i_2 \in \{1, ..., d\} \textit{ with } \operatorname{supp} f_{i_l} \cap \operatorname{supp} x_k \neq \varnothing, l = 1, 2\}.$$

Then,

(a) $|\varphi(\sum_{k \in L} a_k x_k)| \leqslant 1/m_j^4$.

(b) $|\varphi(\sum_{k=1}^{n} a_k x_k)| \leqslant 2/m_r$.

Proof. (a) Let $\{l_1, ..., l_n\} \in \mathscr{M}_j$ be such that $2 \leqslant x_1 < l_1 < x_2 \leqslant l_2 < \cdots \leqslant l_n$. Let $n_i = \min \operatorname{supp} f_i$, $i = 1, ..., d$. Then $\{n_i: i = 1, ..., d\} \in \mathscr{M}_r$. For each $k \in L$, let $i_k = \min\{i: \operatorname{supp} f_i \text{ intersects } \operatorname{supp} x_k\}$. The map $k \to n_{i_k}$ from L to $\{n_i: i = 1, ..., d\}$ is $1-1$, so $\# L \leqslant d$. Moreover, $n_{i_k} \leqslant l_k$ for each $k \in L$, so $\{l_k: k \in L\}$ belongs to $\mathscr{M}_r$. It follows that $\sum_{k \in L} a_k < 1/m^4$, and so,

$$\left|\varphi\left(\sum_{k \in L} a_k x_k\right)\right| \leqslant \sum_{k \in L} a_k \|x_k\| < \frac{1}{m_j^4}.$$

(b) Let $P = \{1, ..., n\} \setminus L$ and, for each $i = 1, ..., d$, let $P_i = \{k \in P: \operatorname{supp} x_k \cap \operatorname{supp} f_i \neq \varnothing\}$. Then

$$\left|\varphi\left(\sum_{k=1}^{n} a_k x_k\right)\right| \leqslant \frac{1}{m_r} \sum_{i=1}^{d} \left|f_i\left(\sum_{k \in P_i} a_k x_k\right)\right| + \sum_{k \in L} a_k \|x_k\|$$
$$< \frac{1}{m_r} + \frac{1}{m_j^4} < \frac{2}{m_r}. \quad \blacksquare$$

In the sequel we shall write $\tilde{K} \prec K$ if $\tilde{K}$ is a subset of K satisfying the following.

(i) For every $f \in \tilde{K}$ there exists an analysis $\{K^s(f)\}$ such that $\bigcup K^s(f) \subset \tilde{K}$.

(ii) If $f \in K$ then $-f \in \tilde{K}$ and $f | [m, n] \in \tilde{K}$ for all $m < n \in \mathbb{N}$.

(iii) If $(f_i)_{i=1}^{d}$ is an $\mathscr{S}$-allowable family in $\tilde{K}$ then $\frac{1}{2} \sum_{i=1}^{d} f_i$ belongs to $\tilde{K}$.

(iv) For every $n \in \mathbb{N}$, $e_n \in \tilde{K}$.

For $\tilde{K} \prec K$ we denote by $\|\cdot\|_{\tilde{K}}$ the norm induced by $\tilde{K}$:

$$\|x\|_{\tilde{K}} = \sup\{f(x): f \in \tilde{K}\}.$$

The results that follow involve a subset $\tilde{K}$ of K having the properties mentioned above. For the purposes of this section we only need these results with $\tilde{K} = K$. However, we find it convenient to present them now in the more general formulation that we will need in Section 3.

2.8. Definition. Let $\tilde{K} \prec K$. A finite block sequence $(x_k)_{k=1}^n$ is said to be a rapidly increasing sequence (R.I.S.) with respect to $\tilde{K}$ if there exist integers $j_1, ..., j_n$ satisfying the following:

(i) $2 \leqslant j_1 < j_2 < \cdots < j_n$.

(ii) Each x_k is a seminormalized $(1/m_{j_k}^4, j_k)$-s.c.c. with respect to $\tilde{K}$. That is, x_k is a $(1/m_{j_k}^4, j_k)$-s.c.c. of the form $x_k = \sum_t a_{(k,t)} x_{(k,t)}$ where $\|x_{(k,t)}\|_{\tilde{K}} = 1$ for each t, and $\|x_k\|_{\tilde{K}} \geqslant \frac{1}{2}$.

(iii) For $k = 1, 2, ..., n$, let $l_k = \max \operatorname{supp} x_k$ and let $n_k \in \mathbb{N}$ be such that

$$\frac{l_k}{2^{n_k}} < \frac{1}{m_{j_k}}.$$

We set

$$O_{x_k} = \left\{ f \in K\colon \operatorname{supp} f \subset [1, l_k] \text{ and } |f(e_m)| > \frac{1}{2^{n_k}} \text{ for every } m \in \operatorname{supp} f \right\}.$$

Then j_{k+1} is such that $m_{j_{k+1}} > \# O_{x_k}$ and x_{k+1} satisfies $\min \operatorname{supp} x_{k+1} > \# O_{x_k}$.

(iv) $\|x_k\|_{\ell_1} \leqslant m_{j_{k+1}} / m_{j_{k+1}-1}$.

Notation. If $\varphi \in K \backslash K^0$ then φ is of the form $\varphi = (1/m_r) \sum_{i=1}^d f_i$, where either $r = 1$ and $(f_i)_{i=1}^d$ is an $\mathscr{S}$-allowable family of functionals in K, or $r \geqslant 2$ and $(f_i)_{i=1}^d$ is a $\mathscr{M}_r$-admissible family of functionals in K. In either case we set $w(\varphi) = (1/m_r)$ (the *weight* of φ). That is, $w(\varphi) = 1/m_r$ if and only if $\varphi \in \mathscr{A}_r$.

The following proposition is the central result of this section.

2.9. Proposition. *Let $\tilde{K} \prec K$. Let $(x_k)_{k=1}^n$ be a R.I.S. with respect to $\tilde{K}$ and let $\varphi \in \tilde{K}$. There exists a functional $\psi \in K'$ with $w(\varphi) = w(\psi)$ and vectors u_k, $k = 2, ..., n$, with $\|u_k\|_{\ell_1} \leqslant 16$ and* supp $u_k \subset$ supp x_k *for each k, such that*

$$\left| \varphi \left(\sum_{k=1}^n \lambda_k x_k \right) \right| \leqslant \max_{1 \leqslant k \leqslant n} |\lambda_k| + \psi \left(\sum_{k=2}^n |\lambda_k| \, u_k \right) + 6 \sum_{k=1}^n |\lambda_k| \frac{1}{m_{j_k}}$$

for every choice of coefficients $\lambda_1, ..., \lambda_n \in \mathbb{R}$.

As it follows from the above statement, we reduce the estimation of the action of φ on the R.I.S. $\{x_k\}_k$ to the estimation of the action of the functional ψ on a finite block sequence $\{u_k\}_k$ of subconvex combinations of the basic vectors. The construction of the functional ψ and the finite block sequence $\{u_k\}_k$ will be done in several steps. We describe this process briefly:

We fix an analysis $\{K^s(\varphi)\}$ of the functional φ. We first replace each vector x_k by its "essential part" relative to φ, denoted by $\bar{x}_k$. Next, for each $\bar{x}_k$ we consider certain families of functionals in $\bigcup K^s(\varphi)$ which fall under two types (families of type I and type II, Definition 2.11). These families yield a partition of the support of $\bar{x}_k$. The restriction from x_k to $\bar{x}_k$ gives us a control on the number of families of type I and type II which act on each $\bar{x}_k$ (Lemma 2.13). Fixing k, to each such family of functionals acting on $\bar{x}_k$, we correspond a subconvex combination of the basis and the sum of these combinations is the vector u_k. The functional ψ is defined inductively, following the analysis of the functional φ.

From now on we fix the R.I.S. $(x_k)_{k=1}^n$ and the functional φ of Proposition 2.9. We also fix an analysis $\{K^s(\varphi)\}$ of φ contained in $\widetilde{K}$. We first partition each vector x_k into three disjointly supported vectors x'_k, x''_k, and $\bar{x}_k$; this partition depends on the analysis $\{K^s(\varphi)\}$.

Definition of x'_k, x''_k, $\bar{x}_k$. Let

$$F_k = \{f \in \cup K^s(\varphi)\colon \operatorname{supp} f \cap \operatorname{supp} x_k \neq \emptyset,\ \operatorname{supp} f \cap \operatorname{supp} x_j \neq \emptyset$$
$$\text{for some } j > k \text{ and } w(f) \leqslant 1/m_{j_{k+1}}\}.$$

We set $A_k = \bigcup_{f \in F_k} \operatorname{supp} f$ and $x'_k = x_k \mid A_k$.

Let now

$$F'_k = \{f \in \cup K^s(\varphi)\colon |f(e_m)| \leqslant 1/2^{n_k} \text{ for every } m \in \operatorname{supp} f \cap \operatorname{supp}(x_k - x'_k)$$
$$\text{and } \operatorname{supp} f \cap \operatorname{supp}(x_j - x'_j) \neq \emptyset \text{ for some } j > k\}.$$

We set $A'_k = \bigcup_{f \in F'_k} \operatorname{supp} f$ and $x''_k = (x_k - x'_k) \mid A'_k$.
Finally, $\bar{x}_k = x_k - x'_k - x''_k$.

2.10. Lemma. *For $\varphi(x'_k)$ and $\varphi(x''_k)$ we have the following estimates*:

$$(1)\quad |\varphi(x'_k)| \leqslant \frac{1}{m_{j_{k+1}-1}} \qquad \text{and} \qquad (2)\quad |\varphi(x''_k)| < 1/m_{j_k}.$$

Proof. To see (1), let us call an $f \in F_k$ maximal if there is no $f' \neq f$ in F_k such that $\operatorname{supp} f \subset \operatorname{supp} f'$. The maximal elements of F_k have disjoint supports. So

$$|\varphi(x_k')| \leqslant \sum_{f \text{ maximal in } F_k} |f(x_k')| \leqslant \sum_f \frac{1}{m_{j_{k+1}}} \|x_k|_{\operatorname{supp} f}\|_{\ell_1}$$

$$\leqslant \frac{1}{m_{j_{k+1}}} \frac{m_{j_{k+1}}}{m_{j_{k+1}-1}} = \frac{1}{m_{j_{k+1}-1}},$$

by property (iv) of the R.I.S.

For (2), we notice that for every $n \in \operatorname{supp} x_k''$ we have $|\varphi(e_n)| \leqslant 1/2^{n_k}$. Also, since $\|x_k\|_\infty \leqslant 1$, we have $\|x_k\|_{\ell_1} \leqslant \max \operatorname{supp} x_k$. Hence

$$|\varphi(x_k'')| \leqslant \frac{\|x_k\|_{\ell_1}}{2^{n_k}} \leqslant \frac{\max \operatorname{supp} x_k}{2^{n_k}} < \frac{1}{m_{j_k}}.$$

Remarks. (1) By the definition of x_k' and x_k'' we have $x_n' = x_n'' = 0$, since x_n is the last element of $(x_k)_1^n$.

(2) If $f \in \cup K^s(\varphi)$ and $1 \leqslant k < l \leqslant n$ are such that $\operatorname{supp} f \cap \operatorname{supp} \bar{x}_k \neq \varnothing$ and $\operatorname{supp} f \cap \operatorname{supp} \bar{x}_l \neq \varnothing$ then $w(f) > 1/m_{j_{k+1}}$ and there exists $m \in \operatorname{supp} \bar{x}_k$ such that $|f(e_m)| > 1/2^{n_k}$.

2.11 Definition (Families of Type I and Type II w.r.t. $\bar{x}_k$).

Without loss of generality, we assume that $\operatorname{supp} \varphi \cap \operatorname{supp} \bar{x}_1 \neq \varnothing$. Let $k \in \{2, ..., n\}$ be fixed.

(A) A set of functionals $F = \{f_1, ..., f_l\}$ contained in some level $K^s(\varphi)$ of the analysis of φ is said to be a family of type I with respect to $\bar{x}_k$ if

(A1) $\operatorname{supp} f_i \cap \operatorname{supp} \bar{x}_k \neq \varnothing$ and $\operatorname{supp} f_i \cap \operatorname{supp} \bar{x}_j = \varnothing$ for every $j \neq k$ and every $i = 1, 2, ..., l$.

(A2) There exists $g \in K^{s+1}(\varphi)$ such that $f_1, ..., f_l$ belong to the decomposition of g and $\operatorname{supp} g \cap \operatorname{supp} \bar{x}_j \neq \varnothing$ for some $j < k$. Moreover, F is the maximal subset of the decomposition of g with property (A1); that is, $g = (1/m_r)(\sum_{i=1}^d h_i + \sum_{i=1}^l f_i)$, where, for each $i = 1, ..., d$, either $\operatorname{supp} h_i \cap \operatorname{supp} \bar{x}_k = \varnothing$ or $\operatorname{supp} h_i \cap \operatorname{supp} \bar{x}_j \neq \varnothing$ for some $j \neq k$.

(B) A set of functionals $F = \{f_1, ..., f_m\}$ contained in some level $K^s(\varphi)$ of the analysis of φ is said to be a family of type II with respect to $\bar{x}_k$ if

(B1) $\operatorname{supp} f_i \cap \operatorname{supp} \bar{x}_k \neq \emptyset$, $\operatorname{supp} f_i \cap \operatorname{supp} \bar{x}_j = \emptyset$ for every $j<k$ and every $i=1, 2, ..., m$, and for every $i=1, 2, ..., m$ we can find $j_i>k$ such that $\operatorname{supp} f_i \cap \operatorname{supp} \bar{x}_{j_i} \neq \emptyset$.

(B2) There exists $g \in K^{s+1}(\varphi)$ such that $f_1, ..., f_m$ belong to the decomposition of g and $\operatorname{supp} g \cap \operatorname{supp} \bar{x}_j \neq \emptyset$ for some $j<k$. Moreover, F is the maximal subset of the decomposition of g with property (B1); that is, $g=(1/m_r)(\sum_{i=1}^{d} h_i + \sum_{i=1}^{m} f_i)$, where, for each $i=1, ..., d$, either $\operatorname{supp} h_i \cap \operatorname{supp} \bar{x}_k = \emptyset$ or $\operatorname{supp} h_i \cap \operatorname{supp} \bar{x}_j \neq \emptyset$ for some $j<k$ or $\operatorname{supp} h_i \cap \operatorname{supp} \bar{x}_j = \emptyset$ for all $j \neq k$.

Remarks. (1) It is easy to see that for $k=2, 3, ..., n$,

$$\operatorname{supp} \bar{x}_k \cap \operatorname{supp} \varphi$$

$$= \operatorname{supp} \bar{x}_k \cap \bigcup \left\{ \bigcup_{f \in F} \operatorname{supp} f \colon F \text{ is a family of type I or type II w.r.t. } \bar{x}_k \right\}.$$

(2) Let k be fixed. If each of the families $\{f_1, ..., f_l\}$ and $\{f'_1, ..., f'_m\}$ is of type I or of type II w.r.t. $\bar{x}_k$ and they are not identical, then, for all $i \leqslant l$, $j \leqslant m$, $\operatorname{supp} f_i \cap \operatorname{supp} f'_j = \emptyset$.

(3) Let F be a family of type I or type II w.r.t. $\bar{x}_k$ and let g_F be the functional in $\bigcup K^s(\varphi)$ which contains F in its decomposition. Then g_F intersects $\bar{x}_j$ for some $j<k$. By Remark (2) after Lemma 2.10 this implies that $w(g_F) > 1/m_{j_k}$.

2.12. Lemma. *Let $2 \leqslant k \leqslant n$. If f is a member of a family of type I or type II with respect to $\bar{x}_k$, then there exist sets $A_{k,f}$, $A'_{k,f} \subset \operatorname{supp} f$ satisfying*

$$|f(x'_k)| \leqslant \frac{1}{m_{j_{k+1}}} \|x_k|_{A_{k,f}}\|_{\ell_1}$$

and

$$|f(x''_k)| \leqslant \frac{1}{2^{n_k}} \|x_k|_{A'_{k,f}}\|_{\ell_1}.$$

Moreover, if f and f' are two distinct such functionals then $A_{k,f} \cap A_{k,f'} = \emptyset$ and $A'_{k,f} \cap A'_{k,f'} = \emptyset$.

Proof. Let F_k be the subset of $\bigcup K^s(\phi)$ introduced in the definition of x'_k. If $f(x'_k) \neq 0$ then, by the definition of x'_k, either there exists $g \in F_k$ with $\operatorname{supp} f \subset \operatorname{supp} g$ or there exists $g \in F_k$ with $\operatorname{supp} g \subset \operatorname{supp} f$. But the first case is impossible because then we would have $\operatorname{supp} f \cap \operatorname{supp} x_k \subset \operatorname{supp} x'_k$ and so $\operatorname{supp} f \cap \operatorname{supp} \bar{x}_k = \varnothing$. So, if we set

$$A_{k,f} = \bigcup \{\operatorname{supp} g \cap \operatorname{supp} x_k : g \in F_k \text{ and } \operatorname{supp} g \subset \operatorname{supp} f\},$$

then $f(x'_k) = f(x_k \mid A_{k,f})$. This gives

$$|f(x'_k)| \leqslant \frac{1}{m_{j_{k+1}}} \|x_k \mid A_{k,f}\|_{\ell_1}.$$

In the same way, if $f(x''_k) \neq 0$ we set

$$A'_{k,f} = \bigcup \{\operatorname{supp} g \cap \operatorname{supp}(x_k - x'_k) : g \in F'_k \text{ and } \operatorname{supp} g \subset \operatorname{supp} f\}.$$

Then $f(x''_k) = f(x_k \mid A'_{k,f})$, so

$$|f(x''_k)| \leqslant \frac{1}{2^{n_k}} \|x_k \mid A'_{k,f}\|_{\ell_1}.$$

The disjointness follows from the preceding Remark (2). ∎

2.13. Lemma. *Let* $k = 2, 3, ..., n$. *Then*:

(a) *The number of families of type I w.r.t.* $\bar{x}_k$ *is less than* $\min \operatorname{supp} x_k$.

(b) *The number of families of type II w.r.t.* $\bar{x}_k$ *is less than* $\min \operatorname{supp} x_k$.

Proof. (a) For each family F of type I w.r.t. $\bar{x}_k$ let g_F be the (unique) functional in $\bigcup K^s(\varphi)$ which contains F in its decomposition.

By the maximality of F in the decomposition of g_F, it is clear that if $F \neq F'$ are two families of type I then $g_F \neq g_{F'}$. Since both g_F and $g_{F'}$ are elements of the analysis of φ, it follows that either $\operatorname{supp} g_F \subset \operatorname{supp} g_{F'}$ or $\operatorname{supp} g_{F'} \subset \operatorname{supp} g_F$ or $\operatorname{supp} g_F \cap \operatorname{supp} g_{F'} = \varnothing$. In either case $g_F(e_k) \neq g_{F'}(e_k)$ for all k. Moreover, for each F, g_F has the property that $\operatorname{supp} g_F \cap \operatorname{supp} \bar{x}_i \neq \varnothing$ for some $i < k$. Let $i_F = \min\{i : \operatorname{supp} g_F \cap \operatorname{supp} \bar{x}_i \neq \varnothing\}$. It follows from Remark 2 after Lemma 2.10 that there exists m_F in $\operatorname{supp} \bar{x}_{i_F}$ with $|g_F(e_{m_F})| > 1/2^{n_{i_F}} > 1/2^{n_{k-1}}$.

So, for each family F of type I w.r.t. $\bar{x}_k$, we set $h_F = g|_{\{m_F\}} \in K$. The map $F \to h_F$ is one to one; moreover, each h_F belongs to $O_{x_{k-1}}$ (see Definition 2.8).

It follows that

$$\#\{F : F \text{ is a family of type I w.r.t. } \bar{x}_k\} \leqslant \# O_{x_{k-1}} < \min \operatorname{supp} x_k.$$

(b) The proof is the same as that of part (a). ∎

Notation. For each $k=2, 3, ..., n$, we classify the families of type I and type II into four classes according to the weight $w(g_F)$ of the functional g_F which contains each family F in its decomposition. We set

$$A_{\bar{x}_k}=\{F: F \text{ is a family of type I w.r.t. } \bar{x}_k \text{ and } w(g_F)=\tfrac{1}{2}\},$$

$$B_{\bar{x}_k}=\{F: F \text{ is a family of type I w.r.t. } \bar{x}_k \text{ and } w(g_F)<\tfrac{1}{2}\},$$

$$C_{\bar{x}_k}=\{F: F \text{ is a family of type II w.r.t. } \bar{x}_k \text{ and } w(g_F)=\tfrac{1}{2}\},$$

$$D_{\bar{x}_k}=\{F: F \text{ is a family of type II w.r.t. } \bar{x}_k \text{ and } w(g_F)<\tfrac{1}{2}\}.$$

Remarks. (1) If $F\in D_{\bar{x}_k}$, then F is a singleton, i.e., $F=\{f\}$. Indeed, if $g_F=(1/m_s)(\sum h_i+\sum_{i=1}^m f_i)$ where $s>1$ and $F=\{f_1, ..., f_m\}$, then $f_1<f_2<\cdots<f_m$, and each $\operatorname{supp} f_i$ intersects $\operatorname{supp} \bar{x}_k$ and $\operatorname{supp} \bar{x}_{j_i}$, for some $j_i>k$. This is impossible unless $m=1$.

(2) If $f'<f<f''$ belong to $\bigcup K^s(\varphi)$ and there exists a family of type II w.r.t. $\bar{x}_k$ which is contained in the analysis of f, then $\operatorname{supp} f'\cap \operatorname{supp} \bar{x}_k=\varnothing$ and $\operatorname{supp} f''\cap \operatorname{supp} \bar{x}_k=\varnothing$.

Notation. (A) Each x_k is a seminormalized $(1/m_{j_k}^4, j_k)$-s.c.c. of the form

$$x_k=\sum_{t=1}^{r_k} a_{(k,t)}x_{(k,t)},$$

where $a_{(k,t)}\geqslant 0$, $\sum_t a_{(k,t)}=1$, and $\|x_{(k,t)}\|_{\tilde{K}}=1$.

For each $k=1, ..., n$, $t=1, ..., r_k$, we set

$$\bar{x}_{(k,t)}=x_{(k,t)}|_{\operatorname{supp} \bar{x}_k}.$$

(B) Fix $k\in\{2, ..., n\}$. If $f\in\cup K^s(\varphi)$ is a member of a family of type I or type II w.r.t. $\bar{x}_k$, we set

$$n_f=\min(\operatorname{supp} \bar{x}_k\cap \operatorname{supp} f) \qquad \text{and} \qquad e_f=e_{n_f}.$$

Also, if $F=\{f_1, ..., f_l\}$ is a family of type I or type II w.r.t. $\bar{x}_k$, then we set

$$n_F=\min\left(\operatorname{supp} \bar{x}_k\cap\left(\bigcup_{i=1}^{l} \operatorname{supp} f_i\right)\right) \qquad \text{and} \qquad e_F=e_{n_F}.$$

For $F=\{f_1, ..., f_l\}\in A_{\bar{x}_k}\cup C_{\bar{x}_k}$ we set

$$h_F=\tfrac{1}{2}(f_1+\cdots+f_l) \qquad \text{and} \qquad a_F=|2h_F(\bar{x}_k)|.$$

For $\{f\} \in D_{\bar{x}_k}$ we set

$$a_f = |f(\bar{x}_k)|.$$

Finally, if $F \in B_{\bar{x}_k}$, for every $f \in F$ we set

$$\Omega_f = \{t\colon \operatorname{supp} f \cap \operatorname{supp} \bar{x}_{(k,t)} \neq \emptyset \text{ and } \operatorname{supp} h \cap \operatorname{supp} \bar{x}_{(k,t)} = \emptyset \text{ for every } h \neq f \text{ in } F\}$$

and

$$a_f = \sum_{t \in \Omega_f} a_{(k,t)} \, |f(\bar{x}_{(k,t)})|.$$

(C) For each $k = 2, 3, ..., n$ we define

$$u_k = \sum_{\{f\} \in D_{\bar{x}_k}} a_f e_f + \sum_{F \in A_{\bar{x}_k} \cup C_{\bar{x}_k}} a_F e_F + \sum_{F \in B_{\bar{x}_k}} \sum_{f \in F} a_f e_f.$$

2.14. Lemma. *For* $k = 2, 3, ..., n$,

$$\|u_k\|_{\ell_1} = \sum_{\{f\} \in D_{\bar{x}_k}} a_f + \sum_{F \in A_{\bar{x}_k} \cup C_{\bar{x}_k}} a_F + \sum_{F \in B_{\bar{x}_k}} \sum_{f \in F} a_f \leqslant 16.$$

Proof. For each f with $\{f\} \in D_{\bar{x}_k}$, set $\varepsilon_f = \operatorname{sign}(f(\bar{x}_k))$. Then,

$$\begin{aligned}
\sum_{\{f\} \in D_{\bar{x}_k}} a_f &= \sum_{\{f\} \in D_{\bar{x}_k}} |f(\bar{x}_k)| = \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f(\bar{x}_k) \\
&= \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f(x_k) - \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f(x'_k) - \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f(x''_k) \\
&\leqslant \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_k f(x_k) + \sum_{\{f\} \in D_{\bar{x}_k}} |f(x'_k)| + \sum_{\{f\} \in D_{\bar{x}_k}} |f(x''_k)| \\
&\leqslant \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f(x_k) + \frac{1}{m_{j_{k+1}}} \sum_{\{f\} \in D_{\bar{x}_k}} \|x_k|_{A_{k,f}}\|_{\ell_1} \\
&\quad + \frac{1}{2^{n_k}} \sum_{\{f\} \in D_{\bar{x}_k}} \|x_k|_{A'_{k,f}}\|_{\ell_1},
\end{aligned}$$

where the last inequality follows from Lemma 2.12. From the same lemma and Definition 2.8 we get

$$\frac{1}{m_{j_{k+1}}} \sum_{\{f\} \in D_{\bar{x}_k}} \|x_k|_{A_{k,f}}\|_{\ell_1} \leqslant \frac{1}{m_{j_{k+1}}} \|x_k\|_{\ell_1} < \frac{1}{m_{j_k}}$$

and

$$\frac{1}{2^{n_k}} \sum_{\{f\} \in D_{\bar{x}_k}} \|x_k|_{A'_{k,f}}\|_{\ell_1} \leqslant \frac{1}{2^{n_k}} \|x_k\|_{\ell_1} \leqslant \frac{l_k}{2^{n_k}} < \frac{1}{m_{j_k}}.$$

For every $f \in \tilde{K}$ we have that $\varepsilon_f f|_{[\min \operatorname{supp} x_k, \infty)} \in \tilde{K}$. Also, by Remark (2) following Definition 2.11, we have that if $f \neq f'$ and both $\{f\}$ and $\{f'\}$ are families of type II w.r.t. $\bar{x}_k$, then $\operatorname{supp} f \cap \operatorname{supp} f' = \varnothing$. By Lemma 2.13 we have $\# D_{\bar{x}_k} < \min \operatorname{supp} x_k$. It follows that the set

$$\{\varepsilon_f f|_{[\min(\operatorname{supp} x_k), \infty)} : \{f\} \in D_{\bar{x}_k}\}$$

is $\mathcal{S}$-allowable, and so the functional $\frac{1}{2} \sum_{\{f\} \in D_{\bar{x}_k}} \varepsilon_f f|_{[\min(\operatorname{supp} x_k), \infty)}$ belongs to $\tilde{K}$. We conclude that $|\frac{1}{2} \sum \varepsilon_f f(x_k)| \leqslant \|x_k\|_{\tilde{K}} \leqslant 1$, and so,

$$\sum_{\{f\} \in D_{\bar{x}_k}} a_f \leqslant 2 + \frac{2}{m_{j_k}} < 3. \tag{1}$$

For $F \in C_{\bar{x}_k}$ we set $\varepsilon_F = \operatorname{sign} h_F(\bar{x}_k)$. Then,

$$\begin{aligned}
\sum_{F \in C_{\bar{x}_k}} a_F &= \sum_{F \in C_{\bar{x}_k}} |2h_F(\bar{x}_k)| = 2 \sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F(\bar{x}_k) \\
&= 2\left(\sum \varepsilon_F h_F(x_k) - \sum \varepsilon_F h_F(x'_k) - \sum \varepsilon_F h_F(x''_K)\right) \\
&\leqslant 2 \sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F(x_k) + 2 \sum_{F \in C_{\bar{x}_k}} \sum_{f \in F} |f(x'_k)| + 2 \sum_{F \in C_{\bar{x}_k}} \sum_{f \in F} |f(x''_k)| \\
&\leqslant 2 \sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F(x_k) + \frac{2}{m_{j_{k+1}}} \sum_{F \in C_{\bar{x}_k}} \sum_{f \in F} \|x_k|_{A_{k,f}}\|_{\ell_1} \\
&\quad + \frac{2}{2^{n_k}} \sum_{F \in C_{\bar{x}_k}} \|x_k|_{A'_{k,f}}\|_{\ell_1} \\
&\leqslant 2 \sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F(x_k) + \frac{4}{m_{j_k}},
\end{aligned}$$

again by Lemma 2.12. On the other hand, for $F = \{f_1, ..., f_l\} \in C_{\bar{x}_k}$, $h_F = \frac{1}{2}(f_1 + \cdots + f_l) \in \tilde{K}$ and $\varepsilon_F h_F \in \tilde{K}$. By Lemma 2.13 we have that $\# C_{\bar{x}_k} < \min \operatorname{supp} x_k$ and by Remark (2) after 2.11 we have that the functionals h_F, $F \in C_{\bar{x}_k}$, are disjointly supported. We conclude that the set $\{h_F|_{[\min \operatorname{supp} x_k, \infty)} : F \in C_{\bar{x}_k}\}$ is $\mathcal{S}$-allowable and so, the functional $\frac{1}{2} \sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F|_{[\min \operatorname{supp} x_k, \infty)}$ belongs to $\tilde{K}$ and

$$\left|\sum_{F \in C_{\bar{x}_k}} \varepsilon_F h_F(x_k)\right| \leqslant 2 \|x_k\| \leqslant 2.$$

We conclude that

$$\sum_{F\in C_{\bar{x}_k}} a_F \leqslant 4+\frac{4}{m_{j_k}}<5. \tag{2}$$

In the same way we get

$$\sum_{F\in A_{\bar{x}_k}} a_F<5. \tag{3}$$

Finally, we have

$$\begin{aligned}\sum_{F\in B_{\bar{x}_k}}\sum_{f\in F} a_f &= \sum_{F\in B_{\bar{x}_k}}\sum_{f\in F}\sum_{t\in\Omega_f} a_{(k,t)}\,|f(\bar{x}_{(k,t)})|\\ &\leqslant \sum_{F\in B_{\bar{x}_k}}\sum_{f\in F}\sum_{t\in\Omega_f} a_{(k,t)}(|f(x_{(k,t)})|+|f(x'_{(k,t)})|+|f(x''_{(k,t)})|).\end{aligned}$$

For each $F\in B_{\bar{x}_k}$ and $f\in F$ we have

$$\sum_{t\in\Omega_f} a_{(k,t)}\,|f(x'_{(k,t)})|\leqslant\frac{1}{m_{j_{k+1}}}\,\|a_k|_{A_{k,f}}\|_{\ell_1}$$

and

$$\sum_{t\in\Omega_f} a_{(k,t)}\,|f(x''_{(k,t)})|\leqslant\frac{1}{2^{n_k}}\,\|x_k|_{A'_{k,f}}\|_{\ell_1}.$$

Since the sets $A_{k,f}$, $f\in\bigcup_{F\in B_{\bar{x}_k}} F$ are disjoint, we get

$$\sum_{F\in B_{\bar{x}_k}}\sum_{f\in F}\sum_{t\in\Omega_f} a_{(k,t)}\,|f(x'_{(k,t)})|\leqslant\frac{1}{m_{j_{k+1}}}\,\|x_k\|_{\ell_1}<\frac{1}{m_{j_k}}. \tag{i}$$

In a similar way,

$$\sum_{F\in B_{\bar{x}_k}}\sum_{f\in F}\sum_{t\in\Omega_f} a_{(k,t)}\,|f(x''_{(k,t)})|\leqslant\frac{1}{2^{n_k}}\,\|x_k\|_{\ell_1}<\frac{1}{m_{j_k}}. \tag{ii}$$

It remains to estimate

$$\sum_{F\in B_{\bar{x}_k}}\sum_{f\in F}\sum_{t\in\Omega_f} a_{(k,t)}\,|f(x_{(k,t)})|.$$

For each $F \in B_{\bar{x}_k}$ and $t \in \bigcup_{f \in F} \Omega_f$, let f_t^F be the unique element of F with $f_t^F(\bar{x}_{(k,t)}) \neq 0$. Let also, $\Omega_F = \bigcup_{f \in F} \Omega_f$ and $\Omega = \bigcup_{F \in B_{\bar{x}_k}} \Omega_F$. Then,

$$\sum_{F \in B_{\bar{x}_k}} \sum_{f \in F} \sum_{t \in \Omega_f} a_{(k,t)} \, |f(x_{(k,t)})| = \sum_{F \in B_{\bar{x}_k}} \sum_{t \in \Omega_F} a_{(k,t)} \, |f_t^F(x_{(k,t)})|$$

$$= \sum_{t \in \Omega} a_{(k,t)} \sum_{F \in B_{\bar{x}_k}} |f_t^F(x_{(k,t)})|.$$

Fix $t \in \Omega$. For each $F \in B_{\bar{x}_k}$, we set $\varepsilon_F = \operatorname{sign} f_t^F(x_{(k,t)})$. Since $\# B_{\bar{x}_k} <$ min supp x_k, the functional

$$h = \tfrac{1}{2} \sum_{F \in B_{\bar{x}_k}} \varepsilon_F f_t^F |_{[\text{min supp } x_k, \infty)}$$

belongs to $\tilde{K}$. So, we get

$$\sum_{F \in B_{\bar{x}_k}} |f_t^F(x_{(k,t)})| = 2h(x_{(k,t)}) < 2 \, \|x_{(k,t)}\| = 2.$$

We conclude that

$$\sum_{t \in \Omega} a_{(k,t)} \sum_{F \in B_{\bar{x}_k}} |f_t^F(x_{(k,t)})| \leqslant 2 \sum_{t \in \Omega} a_{(k,t)} \leqslant 2. \qquad \text{(iii)}$$

Finally, by (i), (ii), and (iii),

$$\sum_{F \in B_{\bar{x}_k}} \sum_{f \in F} a_f \leqslant 2 + \frac{2}{m_{j_k}} < 3. \qquad (4)$$

Combining (1), (2), (3), (4) we get the desired estimate for $\|u_k\|_{\ell_1}$. ∎

2.15. LEMMA. *There exists a functional* $\psi \in K'$ *with* $w(\psi) = w(\varphi)$ *and such that, for* $k = 2, ..., n$,

$$|\varphi(\bar{x}_k)| \leqslant \psi(u_k) + \frac{2}{m_{j_k}}.$$

Proof. We build the functional ψ inductively, following the way φ is built by the analysis $\bigcup K^s(\varphi)$.

We first introduce some more notation: For $f \in \cup K^s(\varphi)$, we set

$$K(f) = \{f' \in \cup K^s(\varphi) : \operatorname{supp} f' \subset \operatorname{supp} f\},$$

that is, $K(f)$ is the analysis of f induced by $\bigcup K^s(\varphi)$.

For $f = (1/m_s) \sum_{i=1}^{d} f_i$ and each $k = 2, ..., n$, we set

$$I_k^f = \{i \in \{1, ..., d\} : f_i \text{ is an element of a family of type I w.r.t. } \bar{x}_k\},$$

$$J_k^f = \{i \in \{1, ..., d\} : f_i \text{ is an element of a family of type II w.r.t. } \bar{x}_k\},$$

and

$$\Lambda_k^f = \{i \in \{1, ..., d\} : K(f_i) \text{ contains a family of type I or type II w.r.t. } \bar{x}_k\}.$$

We also set

$$I^f = \bigcup_{k=2}^{n} I_k^f, \qquad J^f = \bigcup_{k=2}^{n} J_k^f, \qquad \Lambda^f = \bigcup_{k=2}^{n} \Lambda_k^f$$

and

$$D_f = \bigcup_{k=2}^{n} \bigcup \left\{ \bigcup_{f' \in F} \operatorname{supp} f' : F \text{ is a family of type I or type II w.r.t. } \bar{x}_k \text{ and } F \subset K(f) \right\}.$$

Let $k = 2, ..., n$ and let F be a family in $B_{\bar{x}_k}$. We set

$$L_F = \{t : \text{there exist at least two functionals } h, h' \in F \text{ such that } \operatorname{supp} h \cap \operatorname{supp} \bar{x}_{(k,t)} \neq \emptyset \text{ and } \operatorname{supp} h' \cap \operatorname{supp} \bar{x}_{(k,t)} \neq \emptyset\}.$$

Let g_F be the functional in $\bigcup K^s(\varphi)$ which contains the family F in its decomposition. We set

$$C_F = w(g_F) \left| \sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(\bar{x}_{(k,t)}) \right|.$$

Finally, for $f \in \bigcup K^s(\varphi)$ we set $B_k(f) = \{F \in B_{\bar{x}_k} : F \subset K(f)\}$.

By induction on $s = 0, ..., m$, for every $f \in K^s(\varphi)$ we shall construct a functional $\psi_f \in K'$ such that:

If $D_f = \emptyset$, then $\psi_f = 0$.

If $D_f \neq \emptyset$, then ψ_f has the following properties:

(a) $\operatorname{supp} \psi_f \subset D_f \subset \operatorname{supp} f$.

(b) For each $k=2, ..., n$,

$$|f(\bar{x}_k|_{D_f})| \leqslant \psi_f(u_k) + \sum_{F \in B_k(f)} C_F.$$

(c) $w(\psi_f) = w(f)$.

Suppose that ψ_f has been defined for all $f \in \bigcup_{t=1}^{s-1} K^t(\varphi)$. Let $f = (1/m_q) \sum_{i=1}^{d} f_i \in K^s(\varphi) \setminus K^{s-1}(\varphi)$ be such that $D_f \neq \emptyset$.

Case 1. $w(f) = 1/m_q < \frac{1}{2}$. Then we set

$$\psi_f = \frac{1}{m_q} \left(\sum_{i \in \Lambda^f} \psi_{f_i} + \sum_{i \in I^f} e^*_{f_i} + \sum_{i \in J^f} e^*_{f_i} \right).$$

By the inductive assumption, property (a) is satisfied.

We note that the sets Λ^f and J^f are not disjoint. If $i \in J^f_k$ then $i \in \Lambda^f_m$ for some $m > k$. In this case, $\operatorname{supp} \psi_{f_i} \subset D_{f_i} \subset [\min \operatorname{supp} \bar{x}_{k+1}, \infty)$, while $\operatorname{supp} e^*_{f_i} = \{n_{f_i}\} \subset \operatorname{supp} \bar{x}_k$. It follows that $e^*_{f_i} < \psi_{f_i}$.

Fix now $k \in \{2, ..., n\}$. Since $w(f) < \frac{1}{2}$, we have $f_1 < f_2 < \cdots < f_d$, so each of the sets J^f_k and Λ^f_k is either empty or a singleton. Suppose that $\Lambda^f_k = \{i_1\}$ and $J^f_k = \{i_2\}$. Then,

$$|f(\bar{x}_k|_{D_f})| = \frac{1}{m_q} \left| f_{i_1}(\bar{x}_k|_{D_f}) + \sum_{i \in I^f_k} f_i(\bar{x}_k) + f_{i_2}(\bar{x}_k) \right|$$

$$\leqslant \frac{1}{m_q} |f_{i_1}(\bar{x}_k|_{D_f})| + \frac{1}{m_q} \left| \sum_{i \in I^f_k} f_i(\bar{x}_k) \right| + \frac{1}{m_q} |f_{i_2}(\bar{x}_k)|.$$

We have

$$\frac{1}{m_q} |f_{i_1}(\bar{x}_k|_{D_f})| \leqslant \frac{1}{m_q} \left(\psi_{f_{i_1}}(u_k) + \sum_{F \in B_k(f_{i_1})} C_F \right) \tag{1}$$

by the inductive assumption. Also,

$$\frac{1}{m_q} |f_{i_2}(\bar{x}_k)| = \frac{1}{m_q} |f_{i_2}(\bar{x}_k)| \, e^*_{f_{i_2}}(a_{f_{i_2}} e_{f_{i_2}}) = \frac{1}{m_q} e^*_{f_{i_2}}(u_k). \tag{2}$$

Finally, let $G = \{f_i : i \in I^f_k\}$ be the family of type I w.r.t. $\bar{x}_k$ contained in the decomposition of f. Then,

$$\begin{aligned}
\frac{1}{m_q}&\Big|\sum_{i\in I_k^f} f_i(\bar{x}_k)\Big| \\
&=\frac{1}{m_q}\Big|\sum_{i\in I_k^f} f_i\Big(\sum_t a_{(k,t)}\bar{x}_{(k,t)}\Big)\Big| \\
&=\frac{1}{m_q}\Big|\sum_{f_i\in G}\sum_{t\in\Omega_{f_i}} a_{(k,t)}f_i(\bar{x}_{(k,t)})+\sum_{t\in L_G} a_{(k,t)}\Big(\sum_{f_i\in G} f_i\Big)(\bar{x}_{(k,t)})\Big| \\
&\leqslant\frac{1}{m_q}\sum_{f_i\in G}\sum_{t\in\Omega_{f_i}} a_{(k,t)}\,|f_i(\bar{x}_{(k,t)})|+\frac{1}{m_q}\Big|\sum_{t\in L_G} a_{(k,t)}\Big(\sum_{f_i\in G} f_i(\bar{x}_{(k,t)})\Big)\Big| \\
&=\frac{1}{m_q}\sum_{i\in I_k^f} a_{f_i}+C_G=\frac{1}{m_q}\sum_{i\in I_k^f} a_{f_i}e^*_{f_i}(e_{f_i})+C_G=\frac{1}{m_q}\sum_{i\in I_k^f} e^*_{f_i}(u_k)+C_G.
\end{aligned}$$

So,

$$\frac{1}{m_q}\Big|\sum_{i\in I_k^f} f_i(\bar{x}_k)\Big|\leqslant\frac{1}{m_q}\sum_{i\in I_k^f} e^*_{f_i}(u_k)+C_G. \tag{3}$$

From (1), (2), and (3) we conclude that property (b) holds for ψ_f, that is,

$$|f(\bar{x}_k|_{D_f})|\leqslant\psi_f(u_k)+\sum_{F\in B_k(f)} C_F.$$

It remains to show that $\psi_f\in K'$. We have to show that the set

$$\{\psi_{f_i}: i\in\Lambda^f\}\cup\{e^*_{f_i}: i\in I^f\cup J^f\}$$

is $\mathcal{M}'_q$-admissible. For $i=1, ..., d$, let $r_i=\min(\operatorname{supp} f_i)$. Then, $\{r_i: i=1, ..., d\}\in\mathcal{M}_q$.

To each $i\in I^f$ corresponds the vector $e^*_{f_i}$ with $r_i\leqslant e^*_{f_i}<r_{i+1}$.

If $i\in J^f$, then $i\in\Lambda^f$ also, so to it correspond two vectors $e^*_{f_i}$ and ψ_{f_i} with $r_i\leqslant e^*_{f_i}<\psi_{f_i}<r_{i+1}$.

Finally, if $i\in\Lambda^f\backslash J^f$, then to it corresponds the vectors ψ_{f_i} with $r_i\leqslant\psi_{f_i}<r_{i+1}$.

It follows from these relations that the family

$$\{\psi_{f_i}: i\in\Lambda^f\}\cup\{e^*_{f_i}: i\in I^f\cup J^f\}$$

is $\mathcal{M}'_q$-admissible, and since $\psi_{f_i}, e^*_{f_i}\in K'$, we get $\psi_f\in\mathcal{A}'_q$.

Case 2. $w(f)=1/m_q=\frac{1}{2}$. For each $k=2, ..., n$, let $F_1^k=\{f_i: i\in I_k^f\}$ be the family of type I w.r.t. $\bar{x}_k$ contained in the decomposition of f, and let

$F_2^k = \{f_i\colon i \in J_k^f\}$ be the family of type II w.r.t. $\bar{x}_k$ contained in the decomposition of f. We set

$$\psi_f = \tfrac{1}{2}\left(\sum_{i \in \Lambda^f} \psi_{f_i} + \sum_{k=2}^{n} (e^*_{F_1^k} + e^*_{F_2^k}) \right).$$

Then, for each k,

$$|f(\bar{x}_k|_{D_f})| = \tfrac{1}{2}\left| \sum_{i \in \Lambda_k^f} f_i(\bar{x}_k|_{D_f}) + \sum_{i \in I_k^f} f_i(\bar{x}_k) + \sum_{i \in J_k^f} f_i(\bar{x}_k) \right|.$$

We have

$$\tfrac{1}{2}\left| \sum_{i \in \Lambda_k^f} f_i(\bar{x}_k|_{D_f}) \right| \leqslant \tfrac{1}{2} \sum_{i \in \Lambda_k^f} |f_i(\bar{x}_k|_{D_f})| \leqslant \tfrac{1}{2} \sum_{i \in \Lambda_k^f} \psi_{f_i}(u_k) + \sum_{i \in \Lambda_k^f} \sum_{F \in B_k(f_i)} C_F.$$

Also,

$$\tfrac{1}{2}\left| \sum_{i \in I_k^f} f_i(\bar{x}_k) \right| = |h_{F_1^k}(\bar{x}_k)|\; e^*_{F_1^k}(e_{F_1^k}) = \tfrac{1}{2} e^*_{F_1^k}(a_{F_1^k} e_{F_1^k}) = \tfrac{1}{2} e^*_{F_1^k}(u_k),$$

and

$$\tfrac{1}{2}\left| \sum_{i \in J_k^f} f_i(\bar{x}_k) \right| = |h_{F_2^k}(\bar{x}_k)|\; e^*_{F_2^k}(e_{F_k}) = \tfrac{1}{2} e_{F_2^k}(u_k).$$

We conclude that

$$\begin{aligned} |f(\bar{x}_k|_{D_f})| &\leqslant \tfrac{1}{2}\left[\sum_{i \in \Lambda_k^f} \psi_{f_i}(u_k) + e^*_{F_1^k}(u_k) + e_{F_2^k}(u_k) \right] + \sum_{F \in B_k(f)} C_F \\ &= \psi_f(u_k) + \sum_{F \in B_k(f)} C_F. \end{aligned}$$

It remains to show that ψ_f belongs to K'. We need to show that the family

$$B = \{\psi_{f_i}\colon i \in \Lambda^f\} \cup \{e^*_{F_1^k}\colon k = 2, ..., n\} \cup \{e^*_{F_2^k}\colon k = 2, ..., n\}$$

is $\mathcal{S}'$-allowable.

We have $\operatorname{supp} \psi_{f_i} \subset D_{f_i} \subset \operatorname{supp} f_i$ for each $i \in \Lambda^f$ and $\operatorname{supp} e^*_{F_1^k} = \{n_{F_1^k}\} \subset \cup \{\operatorname{supp} f_i\colon f_i \in F_1^k\} \cap \operatorname{supp} \bar{x}_k$ and the same is true for $e^*_{F_k^2}$. Also, if f_i belongs to a family F_2^k, then $D_{f_i} \cap \operatorname{supp} \bar{x}_k = \emptyset$, while $n_{F_2^k} \in \operatorname{supp} \bar{x}_k$. Finally, we clearly have $e^*_{F_1^k} \neq e^*_{F_2^k}$.

The above remarks imply that the functionals in B are disjointly supported. Moreover, it is easy to see that

$$\# B \leqslant 2d = 2(\#\{f_i : i = 1, ..., d\}).$$

We conclude that the family B is $\mathscr{S}'$-allowable, and thus $\psi_f \in K'$.

This completes the inductive step. We set $\psi = \psi_\varphi$.

Then, $D_\varphi = \operatorname{supp} \varphi \cap (\bigcup_{k=2}^n \operatorname{supp} \bar{x}_k)$ (see Remark (1) following Definition 2.11), and by the inductive assumption (b) we get that for each $k = 2, ..., n$,

$$|\varphi(\bar{x}_k)| \leqslant \psi(u_k) + \sum_{F \in B_{\bar{x}_k}} C_F.$$

To complete the proof of the lemma it remains to show that, for each $k = 2, ..., n$,

$$\sum_{F \in B_{\bar{x}_k}} C_F < \frac{2}{m_{j_k}}.$$

For each $F \in B_{\bar{x}_k}$, setting $x'_{(k,t)} = x_{(k,t)} \,|\, \operatorname{supp} x'_k$ and $x''_{(k,t)} = x_{(k,t)} \,|\, \operatorname{supp} x''_k$, we have

$$\begin{aligned}\Big|\sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(\bar{x}_{(k,t)})\Big| &\leqslant \Big|\sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(x_{k,t})\Big| + \sum_{t \in L_F} \sum_{f \in F} |f(a_{(k,t)} x'_{(k,t)})| \\ &\quad + \sum_{t \in L_F} \sum_{f \in F} |f(a_{(k,t)} x''_{(k,t)})|.\end{aligned}$$

Using Lemma 2.12 we get

$$\begin{aligned}\Big|\sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(\bar{x}_{(k,t)})\Big| &\leqslant \Big|\sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(x_{(k,t)})\Big| + \sum_{t \in L_F} \sum_{f \in F} \frac{1}{m_{j_{k+1}}} \|a_{(k,t)} x_{(k,t)} |_{A_{k,f}}\|_{\ell_1} \\ &\quad + \sum_{t \in L_F} \sum_{f \in kF} \frac{1}{2^{n_k}} \|a_{(k,t)} x_{(k,t)} |_{A'_{k,f}}\|_{\ell_1} \\ &\leqslant \Big|\sum_{t \in L_F} a_{(k,t)} \sum_{f \in F} f(x_{(k,t)})\Big| + \frac{1}{m_{j_{k+1}}} \sum_{f \in F} \|x_k |_{A_{k,f}}\|_{\ell_1} + \frac{1}{2^{n_k}} \sum_{f \in F} \|x_k |_{A'_{k,f}}\|_{\ell_1}.\end{aligned}$$

To estimate

$$w(g_F)\left|\sum_{t\in L_F} a_{(k,t)} \sum_{f\in F} f(x_{(k,t)})\right|,$$

we use Remark (3) after 2.11. According to this remark, $w(g_F) < 1/m_{j_k}$ and so, $g_F \in \mathscr{A}_r$ for some $1 \leqslant r < j_k$. Let $g_F = w(g_F) \sum_{i=1}^{l} f_i$ where $f_1 < f_2 < \cdots < f_l$ and suppose $i_1 = \min\{i\colon f_i \in F\}$ and $i_2 = \max\{i\colon f_i \in F\}$. We set $\tilde{F} = \{f_i\colon i_1 \leqslant i \leqslant i_2\}$. The family $\tilde{F}$ contains F but might also contain some functionals f_i with $f_i(x_k) \neq 0$ but $f_i(\bar{x}_k) = 0$. Since $\tilde{K}$ is closed under projections onto intervals, the functional $w(g_F) \sum_{f\in\tilde{F}} f$ belongs to $\mathscr{A}_r \cap \tilde{K}$. Applying Lemma 2.7(a) (in fact, since our assumption is $\|x_{(k,t)}\|_{\tilde{K}} \leqslant 1$, we use the analogue of this lemma for the space with norm $\|\cdot\|_{\tilde{K}}$) we get that

$$w(g_F)\left|\sum_{t\in L_F} a_{(k,t)} \sum_{f\in\tilde{F}} f(x_{(k,t)})\right| \leqslant \frac{1}{m_{j_k}^4}.$$

Notice that $C_F := w(g_F)\,|\sum_{t\in L_F} a_{(k,t)} \sum_{f\in F} f(\bar{x}_{(k,t)})| = w(g_F)\,|\sum_{t\in L_F} a_{(k,t)} \sum_{f\in\tilde{F}} f(\bar{x}_{(k,t)})|$ and also that Lemma 2.12 remains true for $f \in \tilde{F}$.

We conclude that for each $F \in B_{\bar{x}_k}$,

$$C_F = w(g_F)\left|\sum_{t\in L_F} a_{(k,t)} \sum_{f\in\tilde{F}} f(\bar{x}_{(k,t)})\right|$$

$$\leqslant \frac{1}{m_{j_k}^4} + \frac{1}{m_{j_{k+1}}} \sum_{f\in\tilde{F}} \|x_k|_{A_{k,f}}\|_{\ell_1} + \frac{1}{2^{n_k}} \sum_{f\in\tilde{F}} \|x_k|_{A'_{k,f}}\|_{\ell_1}.$$

Now, we add over all $F \in B_{\bar{x}_k}$. By Lemma 2.13, $\# B_{\bar{x}_k} < m_{j_k}$. Also, by Lemma 2.12 we have that the sets $A_{k,f}$, for $f \in \bigcup_{F\in B_{\bar{x}_k}} \tilde{F}$, are mutually disjoint, and the same is true for the sets $A'_{k,f}$. We conclude that

$$\sum_{F\in B_{\bar{x}_k}} C_F \leqslant \frac{m_{j_k}}{m_{j_k}^4} + \frac{1}{m_{j_{k+1}}} \|x_k\|_{\ell_1} + \frac{1}{2^{n_k}} \|x_k\|_{\ell_1}$$

$$\leqslant \frac{1}{m_{j_k}^3} + \frac{1}{m_{j_{k+1}-1}} + \frac{1}{m_{j_k}} < \frac{2}{m_{j_k}}$$

by Definition 2.8. This completes the proof of the lemma. ∎

Proof of Proposition 2.9. Recall (Definition 2.11) that for our intermediate lemmas we have assumed that $\operatorname{supp} \varphi \cap \operatorname{supp} \bar{x}_1 \neq \varnothing$. If this is not true, then we can set $k_0 = \min\{k\colon \operatorname{supp} \varphi \cap \operatorname{supp} \bar{x}_k \neq \varnothing\}$ and construct in the same way u_k's, $k=k_0+1, ..., n$, and ψ supported on $\bigcup_{k=k_0+1}^n \operatorname{supp} u_k$, such that

$$|\varphi(\bar{x}_k)| \leqslant \psi(u_k) + \frac{2}{m_{j_k}}, \qquad k=k_0+1, ..., n.$$

Setting $u_k=0$, for $k=2, ..., k_0$ we have

$$\left|\varphi\left(\sum_{k=1}^n \lambda_k \bar{x}_k\right)\right| \leqslant |\lambda_{k_0}|\,|\varphi(\bar{x}_{k_0})| + \psi\left(\sum_{k=2}^n |\lambda_k|\, u_k\right) + \sum_{k=2}^n |\lambda_k|\,\frac{2}{m_{j_k}}$$

for any choice of coefficients $(\lambda_k)_{k=1}^n$.

For $\varphi(\sum_{k=1}^n \lambda_k x_k)$ we have

$$\left|\varphi\left(\sum_{k=1}^n \lambda_k x_k\right)\right| \leqslant \left|\varphi\left(\sum_{k=1}^n \lambda_k \bar{x}_k\right)\right| + \sum_{k=1}^n |\lambda_k|\,(|\varphi(x'_k)| + |\varphi(x''_k)|).$$

Using the previous estimate and Lemma 2.10 we get

$$\begin{aligned}
&\left|\varphi\left(\sum_{k=1}^n \lambda_k x_k\right)\right| \\
&\quad\leqslant |\lambda_{k_0}|\,|\varphi(\bar{x}_{k_0})| + \psi\left(\sum_{k=2}^n |\lambda_k|\, u_k\right) + 4\sum_{k=1}^n |\lambda_k|\,\frac{1}{m_{j_k}} \\
&\quad\leqslant |\lambda_{k_0}|\,(|\varphi(x_{k_0})| + |\varphi(x'_{k_0})| + |\varphi(x''_{k_0})|) + \psi\left(\sum_{k=2}^n |\lambda_k|\, u_k\right) \\
&\qquad + 4\sum_{k=1}^n |\lambda_k|\,\frac{1}{m_{j_k}} \\
&\quad\leqslant |\lambda_{k_0}|\,\|x_{k_0}\|_{\tilde{K}} + \psi\left(\sum_{k=2}^n |\lambda_k|\, u_k\right) + 6\sum_{k=1}^n |\lambda_k|\,\frac{1}{m_{j_k}} \\
&\quad\leqslant \max_{1\leqslant k\leqslant n} |\lambda_k| + \psi\left(\sum_{k=2}^n |\lambda_k|\, u_k\right) + 6\sum_{k=1}^n |\lambda_k|\,\frac{1}{m_{j_k}}. \qquad \blacksquare
\end{aligned}$$

2.16. Definition. Let $j\geqslant 2$, $\varepsilon>0$. An (ε, j)-special convex combination $\sum_{k=1}^n b_k x_k$ is called an (ε, j)-R.I.s.c.c. w.r.t. $\tilde{K}$ if the sequence $(x_k)_{k=1}^n$ is a R.I.S. w.r.t. $\tilde{K}$ and the corresponding integers $(j_k)_{k=1}^n$ satisfy $j+2<j_1<\cdots<j_n$.

2.17. Corollary. *If $\sum_{k=1}^n b_k x_k$ is a $(1/m_j^2, j)$-R.I.s.c.c. w.r.t. $\tilde{K}$ and $\varphi\in\tilde{K}$ with $w(\varphi)=1/m_s$, then*

$$\left|\varphi\left(\sum_{k=1}^{n} b_k x_k\right)\right| \leqslant 2b_1 + \frac{16}{m_s}, \qquad \text{if} \quad s \geqslant j. \tag{a}$$

$$\left|\varphi\left(\sum_{k=1}^{n} b_k x_k\right)\right| \leqslant \frac{33}{m_s m_j}, \qquad \text{if} \quad s < j.$$

$$\frac{1}{4m_j} \leqslant \left\| \sum_{k=1}^{n} b_k x_k \right\|_{\tilde{K}} \leqslant \frac{17}{m_j}. \tag{b}$$

Proof. (a) Recall that the sequence $(b_k)_{k=1}^n$ is decreasing. By Proposition 2.9,

$$\left|\varphi\left(\sum_{k=1}^{n} b_k x_k\right)\right| \leqslant b_1 + \psi\left(\sum_{k=2}^{n} b_k u_k\right) + 6 \sum_{k=1}^{n} \frac{b_k}{m_{j_k}},$$

where $\psi \in K'$ with $w(\psi) = w(\varphi) = s$ and $\|u_k\|_{\ell_1} \leqslant 16$. By Lemma 2.4 we get

$$\left|\varphi\left(\sum_{k=1}^{n} b_k x_k\right)\right| \leqslant 2b_1 + \frac{16}{m_s}$$

for $s \geqslant j$, and

$$\left|\varphi\left(\sum_{k=1}^{n} b_k x_k\right)\right| \leqslant 2b_1 + \frac{32}{m_s m_j} < \frac{33}{m_s m_j}$$

for $s < j$.

(b) The upper estimate follows from (a). The lower estimate is a consequence of the fact that $\|x_k\|_{\tilde{K}} \geqslant \frac{1}{2}$ and the sequence $(x_k)_{k=2}^n$ is $\mathscr{M}_j$-admissible. ∎

2.18. THEOREM. *The space $X_{M(1),u}$ is arbitrarily distortable.*

Proof. It follows from Lemmas 2.2 and 2.6 that for every $j \geqslant 2$ every block subspace Y contains a $(1/m_j^2, j)$-R.I.s.c.c. w.r.t. K.

Fix $i_0 \in \mathbb{N}$ large and define an equivalent norm $|||\cdot|||$ on $X_{M(1),u}$ by

$$|||x||| = \frac{1}{m_{i_0}} \|x\| + \sup\{\varphi(x) \colon \varphi \in \mathscr{A}_{i_0}\}.$$

Let Y be a block subspace and let $y = \sum a_k y_k \in Y$ be a $(1/m_j^2, j)$-R.I.s.c.c. for some $j > i_0$, and $z = \sum b_l z_l \in Y$ be a $(1/m_{i_0}^2, i_0)$-R.I.s.c.c. Then, by Corollary 2.17,

$$|||m_j y||| \leqslant \frac{17}{m_{i_0}} + \frac{33}{m_{i_0}} = \frac{50}{m_{i_0}} \qquad \text{and} \qquad \|m_j y\| \geqslant \frac{1}{4}.$$

On the other hand,

$$|||m_{i_0} z||| \geqslant \frac{1}{4} \qquad \text{and} \qquad \|m_{i_0} z\| \leqslant 17.$$

This shows that $|||\cdot|||$ is a $(1/10^3)\, m_{i_0}$-distortion. Since i_0 was arbitrary, this completes the proof. ∎

The following remarks on the proof of Proposition 2.9 will be used in the next section.

2.19. *Remarks.* Let φ, $\bar{x}_k$, ψ, u_k be as in Proposition 2.9. It follows from the proof of Lemma 2.15 that the functional ψ which is constructed inductively following the analysis $\{K^s(\varphi)\}$ of φ satisfies the following properties.

(a) There exists an analysis $\{K^s(\psi)\}$ of ψ contained in K' such that, for every $g \in \cup K^s(\psi)$ there exists a unique $f \in \cup K^s(\varphi)$ with $g = \psi_f$; moreover, if $g \notin K^0$ then $w(f) = w(g)$.

(b) The functional ψ is supported in the set

$$L = \{e_f \colon f \in \cup \{F \colon F \text{ is a family of type I or II w.r.t. some } \bar{x}_k\}\}.$$

Moreover, for $k = 2, ..., n$ and for every family F of type I or II w.r.t. $\bar{x}_k$, if we set $V_F = \bigcup_{f \in F} \operatorname{supp} f$ and $W_F = \{e_f \colon f \in F\}$ we have

$$|\varphi|_{V_F}(\bar{x}_k)| \leqslant \psi|_{W_F}(u_k) + C_F,$$

where we have set $C_F = 0$ if $F \notin B_{\bar{x}_k}$.

(c) Let $\varphi_2 = \varphi \,|\, J$ for some $J \subset \mathbb{N}$. Assume further that φ_2 has the following property: *For every $k = 2, ..., n$ and every family $f = \{f_1, ..., f_l\} \subset \cup K^s(\varphi)$ of type I or II w.r.t. $\bar{x}_k$, either $f_i|_J(\bar{x}_k) = 0$ for all $i = 1, ..., l$ or $f_i|_J(\bar{x}_k) = f_i(\bar{x}_k)$ for all $i = 1, ..., l$.*

For $k = 2, ..., n$, we let

$$L_k = \{e_f \colon f \text{ belongs to some family of type I or II w.r.t. } \bar{x}_k \text{ and } \operatorname{supp} f \cap J \neq \varnothing\}$$

and we set $\psi_2 = \psi \,|\, \bigcup_{k=2}^n L_k$. Then it follows from (b) that

$$|\varphi_2(\bar{x}_k)| \leqslant \psi_2(u_k) + \frac{1}{m_{j_k}}, \qquad k = 2, ..., n.$$

3. THE SPACE X

We pass now to the construction of a space X not containing any unconditional basic sequence. It is based on the modification $X_{M(1),u}$. Let $K=\bigcup_n \bigcup_j K_j^n$ be the norming set of the space $X_{M(1),u}$. Consider the countable set

$$G=\{(x_1^*, x_2^*, ..., x_k^*); k\in\mathbb{N}, x_i^*\in K, i=1, ..., k \text{ and } x_1^*<x_2^*<\cdots<x_k^*\}.$$

There exists a one to one function $\Phi\colon G\to\{2j\}_{j=1}^\infty$ such that for every $(x_1^*, ..., x_k^*)\in G$, if j_1 is minimal such that $x_1^*\in\mathscr{A}_{j_1}$ and $j_l=\Phi(x_1^*, ..., x_{l-1}^*)$, $l=2, 3, ..., k$, then

$$\Phi(x_1^*, ..., x_k^*)>\max\{j_1, ..., j_k\}.$$

Definition of the Space. For $n=0, 1, 2, ...,$ we define by induction sets $\{L_j^n\}_{j=1}^\infty$ such that L_j^n is a subset of K_j^n.

For $j=1, 2, ...,$ we set $L_j^0=\{\pm e_n\colon n\in\mathbb{N}\}$. Suppose that the $\{L_j^n\}_{j=1}^\infty$ have been defined. We set $L^n=\bigcup_{j=1}^\infty L_j^n$ and

$$\begin{aligned} L_1^{n+1}=\pm L_1^n\cup\{\tfrac{1}{2}(x_1^*+\cdots+x_d^*)\colon & d\in\mathbb{N}, x_i^*\in L^n,\\ & d\leqslant \min\operatorname{supp} x_1^*<\cdots<\min\operatorname{supp} x_d^*,\\ & \operatorname{supp} x_i^*\cap\operatorname{supp} x_l^*=\varnothing \text{ for } i\neq l\}, \end{aligned}$$

and for $j\geqslant 1$,

$$L_{2j}^{n+1}=\pm L_{2j}^n\cup\left\{\frac{1}{m_{2j}}(x_1^*+\cdots+x_d^*)\colon d\in\mathbb{N}, x_i^*\in L^n,\right.$$

$$\left.(\operatorname{supp} x_1^*, ..., \operatorname{supp} x_d^*) \text{ is } \mathscr{M}_{2j}\text{-admissible}\right\},$$

$$L_{2j+1}'^{n+1}=\pm L_{2j+1}^n\cup\left\{\frac{1}{2_{2j+1}}(x_1^*+\cdots+x_d^*)\colon d\in\mathbb{N},\right.$$

$$x_1^*\in L_{2k}^n \text{ for some } k>2j+1,$$

$$x_i^*\in L^n_{\Phi(x_1^*, ..., x_{i-1}^*)} \text{ for } 1<i\leqslant d$$

$$\left.\text{and } (\operatorname{supp} x_1^*, ..., \operatorname{supp} x_d^*) \text{ is } \mathscr{M}_{2j+1}\text{-admissible}\right\},$$

$$L_{2j+1}^{n+1}=\{E_s x^*\colon x^*\in L_{2j+1}'^{n+1}, s\in\mathbb{N}, E_s=\{s, s+1, ...\}\}.$$

This completes the definition of L_j^n, $n=0, 1, 2, ..., j=1, 2, ...$ It is obvious that each L_j^n is a subset of the corresponding set K_j^n.

We set $\mathscr{B}_j = \bigcup_{n=1}^{\infty} (L_j^n \setminus L^0)$ and we consider the norm on c_{00} defined by the set $L = L^0 \cup (\bigcup_{j=1}^{\infty} \mathscr{B}_j)$. The space X is the completion of c_{00} under this norm. It is easy to see that $\{e_n\}_{n=1}^{\infty}$ is a bimonotone basis for X.

Remark. The norming set L is closed under projections onto *intervals*, and has the property that for every j and every $\mathscr{M}_{2j}$-admissible family $f_1, f_2, ..., f_d$ contained in L, $(1/m_{2j})(f_1 + \cdots + f_d)$ belongs to L. It follows that for every $j = 1, 2, ...$ and every $\mathscr{M}_{2j}$-admissible family $x_1 < x_2 < \cdots < x_n$ in c_{00},

$$\left\| \sum_{k=1}^{n} x_k \right\| \geqslant \frac{1}{m_{2j}} \sum_{k=1}^{n} \|x_k\|.$$

For the same reason, for $\mathscr{S}$-admissible families $x_1 < x_2 < \cdots < x_n$, we have

$$\left\| \sum_{k=1}^{n} x_k \right\| \geqslant \tfrac{1}{2} \sum_{k=1}^{n} \|x_k\|.$$

We note however that such a relation is *not* true for $\mathscr{S}$-*allowable* families (x_i). Of course, if it were true, it would immediately imply that the basis $\{e_n\}$ is unconditional.

For $\varepsilon > 0$, $j = 2, ...,$ (ε, j)-special convex combinations are defined in X exactly as in $X_{M(1), u}$ (Definition 2.5). Rapidly increasing sequences and (ε, j)-R.I. special convex combinations in X are defined by Definitions 2.8 and 2.16, respectively, with $\tilde{K} = L$.

By the previous remark we get the following.

3.1. LEMMA. *For $j = 2, 3, ...$ and every normalized block sequence $\{x_k\}_{k=1}^{\infty}$ in X, there exists a finite normalized block sequence $\{y_s\}_{s=1}^{n}$ of $\{x_k\}$ such that $\sum_{s=1}^{n} a_s y_s$ is a seminormalized $(1/m_j^4, j)$-s.c.c.*

By Corollary 2.17, we have:

3.2. PROPOSITION. *Let $\sum_{k=1}^{n} b_k x_k$ be a $(1/m_j^2, j)$-R.I.s.c.c. in X. Then, for $i \in \mathbb{N}$, $\varphi \in \mathscr{B}_i$, we have the following:*

$$\left| \varphi\left(\sum_{k=1}^{n} b_k x_k \right) \right| \leqslant \frac{33}{m_i m_j}, \qquad \text{if} \quad i < j \tag{a}$$

$$\left| \varphi\left(\sum_{k=1}^{n} b_k x_k \right) \right| \leqslant \frac{16}{m_i} + 2b_1, \qquad \text{if} \quad i \geqslant j. \tag{b}$$

In particular, $\|\sum_{k=1}^{n} b_k x_k\| \leqslant 17/m_j$.

3.3. PROPOSITION. *Let* $j > 100$. *Suppose that* $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{y_k^*\}_{k=1}^n$, *and* $\{\theta_k\}_{k=1}^n$ *are such that*

(i) *There exists a rapidly increasing sequence (w.r.t.* X*)*

$$\{x_{(k,i)}: k=1, ..., n, i=1, ..., n_k\}$$

with $x_{(k,i)} < x_{(k,i+1)} < x_{(k+1,l)}$ *for all* $k<n$, $i<n_k$, $l \leqslant n_{k+1}$, *such that:*

(a) *Each* $x_{(k,i)}$ *is a seminormalized* $(1/m_{j_{(k,i)}}^4, j_{(k,i)})$*-s.c.c. where, for each* k, $2j_k+2 < j_{(k,i)}$, $i=1, ..., n_k$.

(b) *Each* y_k *is a* $(1/m_{2j_k}^4, 2j_k)$*-R.I.s.c.c. of* $\{x_{(k,i)}\}_{i=1}^{n_k}$ *of the form* $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$.

(c) *There exists a decreasing sequence* $\{b_k\}_{k=1}^n$ *such that* $\sum_{k=1}^n b_k y_k$ *is a* $(1/m_{2j+1}^4, 2j+1)$*-s.c.c.*

(ii) $y_k^* \in L_{2j_k}$, $y_k^*(y_k) \geqslant 1/4m_{2j_k}$ *and*

$$\text{supp } y_k^* \subset [\text{min supp } y_k, \text{max supp } y_k].$$

(iii) $1/17 \leqslant \theta_k \leqslant 4$ *and* $y_k^*(m_{2j_k}\theta_k y_k) = 1$.

(iv) $j_1 > 2j+1$ *and* $2j_k = \Phi(y_1^*, ..., y_{k-1}^*)$, $k=2, ..., n$.

Let $\varepsilon_k = (-1)^{k+1}$, $k=1, ..., n$. *Then*

$$\left\| \sum_{k=1}^n \varepsilon_k b_k \theta_k m_{2j_k} y_k \right\| \leqslant \frac{300}{m_{2j+1}^2}.$$

Before presenting the proof of Proposition 3.3 let us show how from it the main result of this section follows.

3.4. COROLLARY. *The space* X *is Hereditarily Indecomposable.*

Proof. It is clear by the choice of the sequences $\{y_k\}_{k=1}^n$, $\{y_k^*\}_{k=1}^n$ in Proposition 3.3 that the functional $\psi = 1/m_{2j+1} \sum_{k=1}^n y_k^*$ belongs to L and that $\psi(\sum_{k=1}^n b_k m_{2j_k}\theta_k y_k) = 1/m_{2j+1}$. It follows that

$$\left\| \sum_{k=1}^n b_k m_{2j_k} \theta_k y_k \right\| \geqslant \frac{1}{m_{2j+1}}.$$

To conclude that X is Hereditarily Indecomposable, it remains to show that, for every $j > 100$ and every block subspaces U and V of X, one can choose $\{y_k\}$ and $\{y_k^*\}$ satisfying the assumptions of Proposition 3.3 and such that $y_k \in U$ if k is odd, $y_k \in V$ if k is even. The proof of this is the same as that of Proposition 3.12 of [3], so we omit it. ∎

Proof of Proposition 3.3. Our aim is to show that for every $\varphi \in \bigcup_{i=1}^{\infty} \mathscr{B}_i$,

$$\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right) \leqslant \frac{300}{m_{2j+1}^2}.$$

The proof is given in several steps. We give a brief description.

For $k=1, ..., n$, set $z_k = \theta_k m_{2j_k} y_k$. Then by the assumptions about y_k and Proposition 3.2 we have $1 = y_k^*(z_k) \leqslant \|z_k\| \leqslant 17\theta_k \leqslant 68$.

We consider separately three cases for φ:

1st *Case*. $w(\varphi) = 1/m_{2j+1}$. Then φ has the form $\varphi = (1/m_{2j+1})(Ey_{k_1}^* + y_{k_1+1}^* + \cdots + y_{k_2}^* + z_{k_2+1}^* + \cdots + z_d^*)$ where E is an interval and $z_{k_2+1}^* \neq y_{k_2+1}^*$. For the action of φ on the part $\sum_{k=k_1+1}^{k_2-1} \varepsilon_k b_k z_k$ we have an obvious conditional (i.e., depending on the signs) estimate using the monotonicity of the sequence $(b_k)_{k=1}^n$:

$$\left|\varphi\left(\sum_{k=k_1+1}^{k_2-1} \varepsilon_k b_k z_k\right)\right| \leqslant \frac{1}{m_{2j+1}} b_{k_1+1}.$$

For the remaining part we get an unconditional estimate using Proposition 3.2. In particular, if $k > k_2+1$ then, since Φ is one to one, we have $j_{k_2+1} \neq j_k$ and, for $s = k_2+2, ..., d$, if t_s is such that $z_s^* \in \mathscr{B}_{2t_s}$ then $t_s \neq j_k$. In Lemma 3.5 we show that in this case $|\varphi(z_k)| \leqslant 1/m_{2j+2}^2$.

Using now the trivial estimates $|\varphi(z_k)| \leqslant 68$ for $k = k_1, k_2, k_2+1$ and $\varphi(z_k) = 0$ for $k < k_1$, as well as the fact that $\max b_k < 1/m_{2j+1}^4$, we obtain the desired result.

2nd *Case*. $w(\varphi) \leqslant 1/m_{2j+2}$. Then we get an unconditional estimate for $\varphi(\sum_{k=1}^n \varepsilon_k b_k z_k)$ directly, applying Proposition 3.2 (Lemma 3.7).

3rd *Case*. $w(\varphi) > 1/m_{2j+1}$. For $k = 1, ..., n$ we have $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$ where the sequence $\{x_{(k,i)} : k = 1, ..., n, i = 1, ..., n_k\}$ is a R.I.S. w.r.t. L. We fix an analysis $\{K^s(\varphi)\}$ of φ. It follows by Proposition 2.9 that there exist a functional $\psi \in \mathrm{co}(K')$ and blocks of the basis $u_{(k,i)}$, $k = 1, ..., n$, $i = 1, ..., n_k$ with $\|u_{(k,i)}\|_{\ell_1} \leqslant 16$ for all (k, i), such that, setting $v_k = \theta_k m_{2j_k} \sum_{i=1}^{n_k} b_{(k,i)} u_{(k,i)}$, $k = 1, ..., n$, we have

$$\left|\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k z_k\right)\right| \leqslant \psi\left(\sum_{k=1}^{n} b_k v_k\right) + \frac{1}{m_{2j+2}}.$$

However, since the estimate that we get in this way is unconditional, it is insufficient. So, we partition φ into two disjointly supported functionals φ_1 and φ_2, defined as follows.

For every $f \in \cup K^s(\varphi)$ of the form $f = 1/m_{2j+1}(Ey_{k_1}^* + y_{k_1+1}^* + \cdots + y_{k_2}^* + z_{k_2+1}^* + \cdots + z_d^*)$ in $\bigcup K^s(\varphi)$ where E is an interval, we set

$If = y^*_{k^f+2} + \cdots + y^*_{k_2}$ for an appropriate $k^f \geqslant k_1$. For the other functionals $f \in \cup K^s(\varphi)$, we set $If = 0$. We define $\varphi_1 = \varphi|_{\cup \operatorname{supp} If}$ and $\varphi_2 = \varphi - \varphi_1$. We let ψ_1 be the projection of ψ corresponding to φ_1 and $\psi_2 = \psi - \psi_1$.

In Lemma 3.9 we show that the pair φ_2, ψ_2 satisfies the assumption of Remark 2.19(c). It follows from this remark that

$$\left|\varphi_2\left(\sum_{k=1}^n \varepsilon_k b_k z_k\right)\right| \leqslant \psi_2\left(\sum_{k=1}^n b_k v_k\right) + \frac{1}{m_{2j+2}}.$$

Further, in Lemma 3.11(a) we show that $\psi_2(\sum_{k=1}^n b_k v_k) \leqslant 257/m^2_{2j+1}$.

Finally, in Lemma 3.11(b) we obtain a conditional estimate for $\varphi_1(\sum_{k=1}^n \varepsilon_k b_k z_k)$, namely,

$$\left|\varphi_1\left(\sum_{k=1}^n \varepsilon_k b_k z_k\right)\right| \leqslant \frac{4}{m^2_{2j+1}}.$$

3.5. Lemma. *Let j, $\{j_k\}_{k=1}^n$, and $\{y_k\}_{k=1}^n$ be as in Proposition* 3.3. *Suppose that $2j+1 < t_1 < \cdots < t_d$ and let $\{z^*_s\}_{s=1}^d$ be such that $z^*_1 < \cdots < z^*_d$, $z^*_s \in \mathscr{B}_{2t_s}$ and $(1/m_{2j+1})(z^*_1 + \cdots + z^*_d) \in \mathscr{B}_{2j+1}$. Assume that for some $k = 1, 2, ..., n$, $j_k \notin \{t_1, ..., t_d\}$. Then,*

$$\left|\left(\sum_{s=1}^d z^*_s\right)(m_{2j_k} y_k)\right| \leqslant \frac{1}{m^2_{2j+1}}.$$

Proof. Each y_k is a $(1/m^4_{2j_k}, 2j_k)$-R.I.s.c.c. of the form $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$. Let $s_1 \leqslant d$ be such that $s_1 = \max\{s \in \{1, ..., d\}: t_s < j_k\}$.

If $s \leqslant s_1$, by Proposition 3.2(a) we get $|z^*_s(y_k)| \leqslant 33/m_{2t_s} m_{2j_k}$ and so, using that $2j+1 < t_1 < \cdots < t_d$ and that the sequence $\{m_j\}$ is increasing sufficiently fast, we get

$$\left|\left(\sum_{s=1}^{s_1} z^*_s\right)(y_k)\right| \leqslant \frac{33}{m_{2j_k}} \sum_{s=1}^{s_1} \frac{1}{m_{2t_s}} \leqslant \frac{1}{2m^2_{2j+2} m_{2j_k}}. \qquad (*)$$

For every $s \geqslant s_1 + 1$ set

$$D_s = \left\{i: \operatorname{supp} x_{(k,i)} \cap \operatorname{supp} z^*_s = \operatorname{supp} x_{(k,i)} \cap \operatorname{supp} \sum_{t=s_1+1}^d z^*_t\right\}.$$

The sets D_s are disjoint. Put $I = \{s \geqslant s_1 + 1: D_s \neq \varnothing\}$ and

$$T = \left\{r: 1 \leqslant r \leqslant n_k, \operatorname{supp} x_{(k,r)} \cap \operatorname{supp} \sum_{t=s_1+1}^d z^*_t \neq \varnothing\right\} \Big\backslash \bigcup_{s \in I} D_s.$$

Then,

$$\left|\left(\sum_{s=s_1+1}^{d} z_s^*\right)(y_k)\right| \leqslant \sum_{s\in I}\left|z_s^*\left(\sum_{r\in D} b_{(k,r)}x_{(k,r)}\right)\right| + \left|\sum_{s=s_1+1}^{d} z_s^*\left(\sum_{r\in T} b_{(k,r)}x_{(k,r)}\right)\right|. \tag{1}$$

It follows from Proposition 3.2 (b) that for every $s\in I$,

$$\left|z_s^*\left(\sum_{r\in D_s} b_{(k,r)}x_{(k,r)}\right)\right| \leqslant \frac{16}{m_{2t_s}} + 2b_{(k,r_s)}, \tag{2}$$

where $r_s=\min D_s$. Since by the definition of D_s we have that $\{\max\operatorname{supp} x_{(k,r_s)}\}_{s\in I}\in\mathscr{M}_{2j+1}$, then

$$\sum_{s\in I} b_{(k,r_s)} \leqslant \frac{1}{m_{2j_k}^4}. \tag{3}$$

Since $(1/m_{2j+1})(z_1^*+\cdots+z_d^*)\in\mathscr{B}_{2j+1}$, as in Lemma 2.7(a) we have

$$\left|\left(\sum_{s=s_1+1}^{d} z_s^*\right)\left(\sum_{r\in T} b_{(k,r)}x_{(k,r)}\right)\right| \leqslant \frac{m_{2j+1}}{m_{2j_k}^4} < \frac{1}{m_{2j_k}^3}. \tag{4}$$

By (1), (2), (3), (4), using that $j_k<t_{s_1+1}$, $m_{i+1}\geqslant m_i^i$, and that $2j+2<2j_1$, we have that

$$\left|\left(\sum_{s=s_1+1}^{d} z_s^*\right)(y_k)\right| \leqslant 16\sum_{s=s_1+1}^{d}\frac{1}{m_{2t_s}} + \frac{2}{m_{2j_k}^4} + \frac{1}{m_{2j_k}^3} \leqslant \frac{1}{2m_{2j+2}^2 m_{2j_k}}. \tag{$**$}$$

Therefore, by $(*)$ and $(**)$, we get

$$\left|\left(\sum_{s=1}^{d} z_s^*\right)(m_{2j_k}y_k)\right| \leqslant \frac{1}{m_{2j+2}^2}.$$

3.6. Lemma. *Let* j, $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{y_k^*\}_{k=1}^n$, $\{\theta_k\}_{k=1}^n$, *and* $\{\varepsilon_k\}_{k=1}^n$ *be as in Proposition* 3.3. *For every* $\varphi\in\mathscr{B}_{2j+1}$ *we have*

$$\left|\varphi\left(\sum_{k=1}^{n}\varepsilon_k b_k\theta_k m_{2j_k}y_k\right)\right| \leqslant \frac{1}{m_{2j+1}^2}.$$

Proof. Let $\varphi=(1/m_{2j+1})(Ey_{k_1}^*+y_{k_1+1}^*+\cdots+y_{k_2}^*+z_{k_2+1}^*+\cdots+z_d^*)$, where $E=E_s$ for some s and $z_{k_2+1}^*\neq y_{k_2+1}^*$.

For $k=1, 2, ..., n$ we set $z_k=\theta_k m_{2j_k} y_k$, hence $y_k^*(z_k)=1$. Since $\{b_k\}$ is decreasing,

$$\left|\varphi\left(\sum_{k=k_1+1}^{k_2-1} \varepsilon_k b_k z_k\right)\right| = \frac{1}{m_{2j+1}} \left|\sum_{k=k_1+1}^{k_2-1} \varepsilon_k b_k y_k^*(z_k)\right|$$

$$= \frac{1}{m_{2j+1}} \left|\sum_{k=k_1+1}^{k_2-1} \varepsilon_k b_k\right| \leqslant \frac{1}{m_{2j+1}} b_{k_1+1}, \tag{a}$$

and

$$|\varphi(z_{k_1})| = \frac{1}{m_{2j+1}} |Ey_{k_1}^*(z_{k_1})| \leqslant \frac{1}{m_{2j+1}} \|z_{k_1}\| < \frac{68}{m_{2j+1}}. \tag{b}$$

For z_{k_2} we have

$$|\varphi(z_{k_2})| \leqslant \frac{1}{m_{2j+1}} |y_{k_2}^*(z_{k_2})| + \frac{1}{m_{2j+1}} \left|\left(\sum_{k=k_2+1}^{d} z_k^*\right)(z_{k_2})\right|.$$

If $k \geqslant k_2+1$, then $z_k^* \in B_{2t_k}$ where $2t_k = \Phi(y_1^*, ..., y_{k_1}^*, ..., z_{k-1}^*)$. Since Φ is one to one, $2t_k \neq \Phi(y_1^*, ..., y_{k_2-1}^*) = 2j_{k_2}$. Thus, by Lemma 3.5,

$$\frac{1}{m_{2j+1}} \left|\sum_{k=k_2+1}^{d} z_k^*(z_{k_2})\right| \leqslant \frac{1}{m_{2j+1}} \frac{\theta_k}{m_{2j+2}^2} < \frac{1}{m_{2j+1}},$$

and so,

$$|\varphi(z_{k_2})| \leqslant \frac{2}{m_{2j+1}}. \tag{c}$$

In a similar way, for z_{k_2+1} we have

$$|\varphi(z_{k_2+1})| \leqslant \frac{1}{m_{2j+1}} |z_{k_2+1}^*(z_{k_2+1})| + \frac{1}{m_{2j+1}} \left|\left(\sum_{k>k_2+1} z_k^*\right)(z_{k_2+1})\right| < \frac{69}{m_{2j+1}}. \tag{d}$$

If $k<k_1$, then $\varphi(z_k)=0$. By Lemma 3.5, for $k>k_2+1$ we have

$$|\varphi(z_k)| = \frac{1}{m_{2j+1}} \left|\sum_{p=k_2+1}^{d} z_p^*(z_k)\right| \leqslant \frac{1}{m_{2j+1}} \frac{\theta_k}{m_{2j+2}^2} < \frac{1}{m_{2j+2}^2}. \tag{e}$$

Putting (a)–(e) together and using that, since $\sum b_k y_k$ is a $(1/m_{2j+1}^4, 2j+1)$-s.c.c., $b_k < 1/m_{2j+1}^4$, we get the result.

3.7. LEMMA. *Under the assumptions of Proposition* 3.3, *let* $\varphi \in \mathcal{B}_r$ *for* $r \geqslant 2j+2$. *Then,*

$$\left|\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right)\right| \leqslant \frac{1}{m_{2j+1}^3}.$$

Proof. If $j_k > r$ then, by Proposition 3.2(a), $|\varphi(\theta_k m_{2j_k} y_k)| \leqslant 4(33/m_r) \leqslant 132/m_{2j+2}$.

If $j_k \leqslant r$ the, by Proposition 3.2(b), $|\varphi(\theta_k m_{2j_k} y_k)| \leqslant 64 m_{2j_k}/m_r + 8/m_{2j_k}^3$. So, for $j_k = r$, we have $|\varphi(\theta_k m_{2j_k} y_k)| \leqslant 65$ and, for $j_k < r$, using the lacunarity of the sequence $\{m_j\}_{j=1}^{\infty}$, we have $|\varphi(\theta_k m_{2j_k} y_k)| \leqslant 1/m_{2j_k}^2 \leqslant 1/m_{2j_1}^2$.

Since $\max b_k \leqslant 1/m_{2j+1}^4$, we get

$$\left|\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right)\right| \leqslant \frac{132}{m_{2j+2}} + \frac{1}{m_{2j+1}^2} + \frac{65}{m_{2j+1}^4} \leqslant \frac{1}{m_{2j+1}^2}. \quad \blacksquare$$

3.8. PROPOSITION. *Let* j, $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{y_k^*\}_{k=1}^n$, $\{\theta_k\}_{k=1}^n$, $\{\varepsilon_k\}_{k=1}^n$ *be as in Proposition* 3.3. *For every* $\varphi \in \mathcal{B}_r$, $r < 2j+1$, *we have*

$$\left|\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right)\right| \leqslant \frac{262}{m_{2j+1}^2}.$$

The proof is based on Proposition 2.9. We first need to introduce new notation and establish several lemmas. We have $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$ and the sequence $\{x_{(k,i)}, k = 1, ..., n, i = 1, ..., n_k\}$ is a R.I.S. w.r.t. L. By Proposition 2.9 there exist a functional $\psi \in K'$ and blocks of the basis $u_{(k,i)}$, $k = 1, ..., n$, $i = 1, ..., n_k$ with $\psi \in \mathcal{A}'_r$, $\operatorname{supp} u_{(k,i)} \subset \operatorname{supp} x_{(k,i)}$, $\|u_{(k,i)}\|_{\ell_1} \leqslant 16$ and such that

$$\begin{aligned}
&\left|\varphi\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k}\left(\sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}\right)\right)\right| \\
&\quad \leqslant \theta_1 m_{2j_1} b_1 b_{(1,1)} + \psi\left(\sum_{k=1}^{n} b_k \theta_k m_{2j_k}\left(\sum_{i=1}^{k_n} b_{(k,i)} u_{(k,i)}\right)\right) + \frac{1}{m_{2j+2}^2} \\
&\quad \leqslant \psi\left(\sum_{k=1}^{n} b_k \theta_k m_{2j_k}\left(\sum_{i=1}^{k_n} b_{(k,i)} u_{(k,i)}\right)\right) + \frac{1}{m_{2j+2}}.
\end{aligned}$$

Recall that the construction of ψ and $u_{(k,i)}$ is done via some analysis $\{K^s(\varphi)\}$ of φ and some restriction on the support of $x_{(k,i)}$ which we denote by $\bar{x}_{(k,i)}$. Let $\{K^s(\varphi)\}$ be the analysis of φ which we use to construct ψ.

Let $f \in \cup K^s(\varphi)$ be of the form $f = (1/m_{2j+1})(Ey^*_{k_1} + y^*_{k_1+1} + \cdots + y^*_{k_2} + z^*_{k_2+1} + \cdots + z^*_d)$, where E is an interval of integers $\{p, p+1, ...\}$. For each $t = 1, ..., n$, $y_t = \sum_{i=1}^{n_t} b_{(t,i)} x_{(t,i)}$. Put

$$k^f = \min\{t \in \{k_1, ..., k_2 - 2\}: \text{supp } Ey^*_t \cap \text{supp } \bar{x}_{(t,i)} \neq \varnothing \text{ for some } i \in \{1, 2, ..., n_t\}\}.$$

Set

$$If = \frac{1}{m_{2j+1}} (y^*_{k^f+2} + \cdots + y^*_{k_2}),$$

while for the other functionals in $\bigcup K^s(\varphi)$ set $If = 0$.

We set

$$\varphi_1 = \varphi|_{\cup \text{supp } If} \qquad \text{and} \qquad \varphi_2 = \varphi - \varphi_1.$$

Recall that, for $f \in \cup K^s(\varphi)$ which is a member of a family of type I or type II w.r.t. $\bar{x}_{(k,i)}$, we have defined $e_f = \min\{m: m \in \text{supp } f \cap \text{supp } \bar{x}_{(k,i)}\}$. Let

$$P = \cup\{F \subset \cup K^s(\varphi): F \text{ is a family of type I or type II w.r.t. some } \bar{x}_{(k,i)}\}.$$

The functional ψ is supported in the set $\{e_f: f \in P\}$. We set

$$\psi_1 = \psi|_{\{e_f:\, f \in P \text{ and } f \text{ is in the analysis of } \varphi_1\}} \qquad \text{and} \qquad \psi_2 = \psi - \psi_1.$$

As in the previous section without loss of generality we assume that supp $\varphi \cap$ supp $\bar{x}_{(1,1)} \neq \varnothing$.

3.9. Lemma. (a) *For every* $f, g \in \cup K^s(\varphi)$ *with* $f \neq g$ *and* $If \neq 0$, $Ig \neq 0$, *we have* supp $If \cap$ supp $Ig = \varnothing$.

(b) *Let* $F = \{f_1, ..., f_l\} \subset \cup K^s(\varphi)$ *be a family of type I or type II w.r.t.* $\bar{x}_{(k,i)}$. *Suppose that for some* $p \in \{1, ..., l\}$, supp $f_p \subseteq$ supp φ_1. *Then,* supp $f_r \subseteq$ supp φ_1 *for every* $r \in \{1, ..., l\}$.

(c) *Let* $F = \{f_1, ..., f_l\} \subset \cup K^s(\varphi)$ *be a family of type I or type II w.r.t.* $\bar{x}_{(k,i)}$. *Suppose that for some* $p = 1, ..., l$, supp $f_p \not\subseteq$ supp φ_1. *Then* $f_p|_{\text{supp } \varphi_2}(\bar{x}_{(k,i)}) = f_p(\bar{x}_{(k,i)})$.

(d) *Let* $F = \{f_1, ..., f_l\} \subset \cup K^s(\varphi)$ *be a family of type I or type II w.r.t.* $\bar{x}_{(k,i)}$. *If* supp $f_p \not\subseteq$ supp φ_1 *for some* $p = 1, ..., l$, *then, for all* $r = 1, ..., l$, $f_r|_{\text{supp } \varphi_2}(\bar{x}_{(k,i)}) = f_r(\bar{x}_{(k,i)})$.

Proof. (a) Let $f = (1/m_{2j+1}(Ey^*_{k_1} + \cdots + y^*_{k_2} + z^*_{k_2+1} + \cdots + z^*_{k_3})$ and $g = (1/m_{2j+1})(Ey^*_{t_1} + \cdots + y^*_{t_2} + z^*_{t_2+1} + \cdots + z^*_{t_3})$. If supp $f \cap$ supp $g \neq \varnothing$, then either supp $f \subset$ supp g or supp $g \subset$ supp f. Suppose that the first is true. Since supp $y_l^* \subseteq [\min \operatorname{supp} y_l, \max \operatorname{supp} y_l]$, it is impossible to have supp $f \subseteq$ supp y_l^* for any $t_1 \leqslant l \leqslant t_2$. It follows that supp $f \subseteq$ supp z_t^* for some $t_2 + 1 \leqslant t \leqslant t_3$. This implies that supp $If \cap$ supp $Ig = \emptyset$.

(b) Let $F = \{f_1, ..., f_l\}$ be a family of type I or type II w.r.t. $\bar{x}_{(k,i)}$ and suppose that supp $f_p \subset$ supp φ_1 for some p. If $\# F = 1$ there is nothing to prove. So assume that $\# F \geqslant 2$. Let g_F be the functional in $\bigcup K^s(\varphi)$ which contains F in its decomposition. Since $f_p \in \cup K^s(\varphi_1)$, we have that f_p belongs to the analysis of If for some $If = (1/m_{2j+1})(y_{k^f+2}^* + \cdots + y_{k_2}^*)$. It follows that $k^f + 2 \leqslant k \leqslant k_2$ and f_p belongs to the analysis of y_k^*. We have to show that supp $g_F \subset$ supp y_k^* or equivalently that g_F does not coincide with f. If $w(g_F) = \frac{1}{2}$ then we get supp $g_F \subseteq$ supp y_k^*, since $w(f) < \frac{1}{2}$. If $w(g_F) < \frac{1}{2}$ then, since $\# F \geqslant 2$, F is of type I and again we get supp $g_F \subseteq$ supp y_k^*, since $\bigcup_{f \in F}$ supp f intersects only supp $\bar{x}_{(k,i)}$.

(c) Suppose that supp $f_p \cap$ supp $Ig \neq \emptyset$ for some $g = (1/m_{2j+1})(Ey_{k_1}^* + \cdots + y_{k_2}^* + z_{k_2+1}^* + \cdots + z_{k_s}^*) \in \cup K^s(\varphi)$. Then either supp $f_p \subset$ supp g strictly or supp $g \subseteq$ supp f_p. In the first case we get that supp $f_p \subseteq$ supp y_l^* for some $k^g + 2 \leqslant l \leqslant k_2$ and so supp $f_p \subseteq$ supp φ_1, a contradiction. In the case supp $g \subseteq$ supp f_p, since supp $g \cap$ supp $\bar{x}_{(k^g,q)} \neq \emptyset$ for some q, we get by the definition of families of type I and type II w.r.t. $\bar{x}_{(k,i)}$ that $k \leqslant k^g$. So $Ig = (1/m_{2j+1})(y_{k^g+2}^* + \cdots + y_{k_2}^*)$ does not intersect $\bar{x}_{(k,i)}$. It follows that $(f_p - f_p|_{\operatorname{supp} Ig})(\bar{x}_{(k,i)}) = f_p(\bar{x}_{(k,i)})$. Since supp $\varphi_1 = \bigcup_g$ supp Ig, we conclude that $(f_p|_{\operatorname{supp} \varphi_2})(\bar{x}_{(k,i)}) = f_p(\bar{x}_{(k,i)})$.

(d) It follows from (b) and (c). ∎

3.10. LEMMA. *For φ_2 we have*

$$\left|\varphi_2\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k}\left(\sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}\right)\right)\right| \leqslant \psi_2\left(\sum_{k=1}^{n} b_k \theta_k m_{2j_k}\left(\sum_{i=1}^{n_k} b_{(k,i)} u_{(k,i)}\right)\right) + \frac{1}{m_{2j+2}}.$$

Proof. By Lemma 3.9(d) we have that φ_2 satisfies the assumptions of Remark 2.19(c). The proof follows from this remark. ∎

3.11. LEMMA.

$$\left|\varphi_2\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right)\right| \leqslant \frac{257}{m_{2j+1}^2}, \tag{a}$$

$$\left|\varphi_1\left(\sum_{k=1}^{n} \varepsilon_k b_k \theta_k m_{2j_k} y_k\right)\right| \leqslant \frac{4}{m_{2j+1}^2}. \tag{b}$$

Proof. (a) By Lemma 3.10 it suffices to estimate

$$\psi_2\left(\sum_{k=1}^{n} b_k\theta_k m_{2j_k}\left(\sum_{i=1}^{n_k} b_{(k,i)}u_{(k,i)}\right)\right).$$

Recall that $u_{(k,i)}$ is of the form $u_{(k,i)}=\sum_{m\in A_{(k,i)}} a_m e_m$, where $a_m>0$ and $\sum_{m\in A_{(k,i)}} a_m\leqslant 16$. Let $\{K^s(\psi_2)\}$ be the corresponding analysis of ψ_2. For $k=1,2,\dots,n$ set

$$D_1^k=\Big\{m\in\bigcup_{i=1}^{n_k} A_{(k,i)}\text{: for all } f\in\bigcup_s K^s(\psi_2)\text{ such that }$$

$$m\in\operatorname{supp} f,\ w(f)>\frac{1}{m_{2j_k}}\Big\},$$

$$D_2^k=\Big\{m\in\bigcup_{i=1}^{n_k} A_{(k,i)}\text{: there exists } f\in\bigcup_s K^s(\psi_2)\text{ such that }$$

$$m\in\operatorname{supp} f\text{ and } w(f)<\frac{1}{m_{2j_k}}\Big\},$$

$$D_3^k=\Big\{m\in\bigcup_{i=1}^{n_k} A_{(k,i)}\text{: } m\notin D_2^k,\text{ there exists } f\in\bigcup_s K^s(\psi_2)$$

$$\text{with } m\in\operatorname{supp} f,\ w(f)=\frac{1}{m_{2j_k}}$$

$$\text{and there exists } g\in\bigcup_s K^s(\psi_2)\text{ with}$$

$$\operatorname{supp} f\subset\operatorname{supp} g\text{ strictly and } w(g)\leqslant\frac{1}{m_{2j+2}}\Big\},$$

$$D_4^k=\Big\{m\in\bigcup_{i=1}^{n_k} A_{(k,i)}\text{: } m\notin D_2^k,\text{ there exists } f\in\bigcup_s K^s(\psi_2)$$

$$\text{with } m\in\operatorname{supp} f,\ w(f)=\frac{1}{m_{2j_k}}$$

$$\text{and for every } g\in\bigcup_s K^s(\psi_2)\text{ with}$$

$$\operatorname{supp} f\subset\operatorname{supp} g,\ w(g)\geqslant\frac{1}{m_{2j+1}}\Big\}.$$

Then, $\bigcup_{p=1}^{4} D_p^k = \bigcup_{i=1}^{n_k} \operatorname{supp} u_{(k,i)} \cap \operatorname{supp} \psi_2$. For every k,

$$\psi_2|_{D_2^k}\left(b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant b_k\theta_k m_{2j_k}\frac{16}{m_{2j_k+1}} < \frac{1}{m_{2j_k}},$$

thus

$$\psi_2|_{\cup_k D_2^k}\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant \sum_k \frac{1}{m_{2j_k}} < \frac{1}{m_{2j+2}}. \tag{1}$$

Also,

$$\psi_2|_{\cup_k D_3^k}\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant \sum_k b_k\theta_k\frac{16}{m_{2j+2}} \leqslant \frac{64}{m_{2j+2}}. \tag{2}$$

For $k = 1, 2, ..., n$, $|\psi_2|_{D_1^k}|_{2j_k-1}^* \leqslant 1$ (see Notation after Lemma 2.2). So, by Lemma 2.4(b),

$$\psi_2|_{D_1^k}\left(b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant b_k\theta_k m_{2j_k}\frac{32}{m_{2j_k}^2} \leqslant b_k\frac{128}{m_{2j_k}}.$$

Hence,

$$\psi_2|_{\cup_k D_1^k}\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant \sum_k b_k\frac{128}{m_{2j_k}} < \frac{1}{m_{2j+2}}. \tag{3}$$

For every $k = 1, ..., n$, $i = 1, ..., n_k$ and every $m \in \operatorname{supp} u_{(k,i)} \cap D_4^k$, there exists a unique functional $f^{(k,i,m)} \in \bigcup_s K^s(\psi_2)$ with $m \in \operatorname{supp} f$, $w(f) = 1/m_{2j_k}$ and such that, for all $g \in \bigcup_s K^s(\psi_2)$ with $\operatorname{supp} f \subset \operatorname{supp} g$ strictly, $w(g) \geqslant 1/m_{2j+1}$. By definition, for $k \neq p$ and $i = 1, ..., n_k$, $m \in \operatorname{supp} u_{(k,i)}$, we have $\operatorname{supp} f^{(k,i,m)} \cap D_4^p = \emptyset$. Also, if $f^{(k,i,m)} \neq f^{(k,r,n)}$, then $\operatorname{supp} f^{(k,i,m)} \cap \operatorname{supp} f^{(k,r,n)} = \emptyset$.

For each $k = 1, ..., n$, let $\{f^{k,t}\}_{t=1}^{r_k} \subset \cup K^s(\varphi)$ be a selection of mutually disjoint such functionals with $D_4^k = \bigcup_{t=1}^{r_k} \operatorname{supp} f^{k,t}$. For each such functional $f^{k,t}$, we set $H_t^k = \operatorname{supp} f^{k,t}$ and

$$a_{f^{k,t}} = \sum_{i=1}^{n_k} b_{(k,i)} \sum_{m \in H_t^k} a_m.$$

Then,

$$f^{k,t}\left(b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) \leqslant b_k\theta_k a_{f^{k,t}}. \tag{$*$}$$

CLAIM. *Let* $D_4=\bigcup_{k=1}^{n} D_4^k$. *Then* $\psi_2|_{D_4}(\sum_k b_k\theta_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)}))\leqslant 256/m_{2j+1}^2$.

Proof of the Claim. We shall define a functional $g\in K'$ with $|g|_{2j}^*\leqslant 1$ and blocks u_k of the basis so that $\|u_k\|_{\ell_1}\leqslant 16$, $\operatorname{supp} u_k\subseteq\bigcup_i \operatorname{supp} u_{(k,i)}$ and

$$\psi_2|_{D_4}\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)\leqslant g\left(2\sum_k b_k\theta_k u_k\right),$$

hence by Lemma 2.4(b) we shall have the result.

For $f=(1/m_q)\sum_{p=1}^{d} f_p\in\bigcup_s K^s(\psi_2|_{D_4})$ we set

$$J=\{1\leqslant p\leqslant d\colon f_p=f^{k,t}\text{ for some } k=1,\ldots,n,\ t=1,\ldots,r_k\},$$

$$T=\{1\leqslant p\leqslant d\colon \text{there exists } f^{k,t} \text{ with } \operatorname{supp} f^{k,t}\subset\operatorname{supp} f_p \text{ strictly}\}.$$

For every $f\in\bigcup_s K^s(\psi_2|_{D_4})$ such that $J\cup T=\varnothing$ we set $g_f=0$, while if $J\cup T\neq\varnothing$ we shall define a functional g_f with the following properties: Let $D_f=\bigcup_{p\in J\cup T}\operatorname{supp} f_p$ and $u_k=\sum a_{f^{k,t}}e_{f^{k,t}}$, where $e_{f^{k,t}}=e_{\min H_t^k}$. Then,

(a) $\operatorname{supp} g_f\subseteq\operatorname{supp} f$.

(b) $g_f\in K'$ and $w(g_f)\geqslant w(f)$,

(c) $f|_{D_f}(\sum_k b_k\theta_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)}))\leqslant g_f(2\sum_k b_k\theta_k u_k)$.

Let $s>0$ and suppose that the g_f have been defined for all $f\in\bigcup_{t=0}^{s-1}K^t(\psi_2|_{D_4})$ and let $f=(1/m_q)(f_1+\cdots+f_d)\in K^s(\psi_2|_{D_4})\backslash K^{s-1}(\psi_2|_{D_4})$ where the family $(f_p)_{p=1}^d$ is $\mathscr{M}_q'$-admissible if $q>1$, or $\mathscr{S}'$-allowable if $q=1$. We consider three cases:

Case (*i*). $1/m_q=1/m_{2j_{k_0}}$ for some k_0, $1\leqslant k_0\leqslant n$. Then $f=f^{k_0,t}$ for some t and we set $g_f=e^*_{f^{k_0,t}}$. By $(*)$ we get

$$\begin{aligned} f\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right) &= b_{k_0}\theta_{k_0}m_{2j_{k_0}}f\left(\sum_i b_{(k_0,i)}u_{(k_0,i)}\right)\\ &\leqslant b_{k_0}\theta_{k_0}a_{f^{k_0,t}}\\ &= b_{k_0}\theta_{k_0}a_{f^{k_0,t}}e^*_{f^{k_0,t}}(e_{f^{k_0,t}})\\ &= g_f(b_{k_0}\theta_{k_0}u_{k_0}). \end{aligned}$$

Case (*ii*). $1/m_q>1/m_{2j+1}$. Then if $J\cup T\neq\varnothing$, set

$$g_f=\frac{1}{m_q}\left(\sum_{p\in J}e^*_{f_p}+\sum_{p\in T}g_{f_p}\right).$$

For $p\in J$, $f_p=f^{k_p,t}$ for some (k_p,t) and by $(*)$,

$$f_p\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)\leqslant b_{k_p}\theta_{k_p}a_{f_p}e^*_{f_p}(e_{f_p}).$$

For $p\in T$ we obtain by the inductive hypothesis

$$f_p\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)\leqslant 2g_{f_p}\left(\sum_k b_k\theta_k u_k\right).$$

Therefore,

$$\begin{aligned} f&\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)\\ &=\frac{1}{m_q}\sum_{p\in J\cup T} f_p\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)\\ &\leqslant g_f\left(2\sum_k b_k\theta_k u_k\right).\end{aligned}$$

Since $\operatorname{supp} g_{f_p}\subseteq \operatorname{supp} f_p$, $e_{f_p}\in\operatorname{supp} f_p$ and $J\cap T=\emptyset$, we have that the family $\{e^*_{f_p}: p\in J\}\cup\{g_{f_p}: p\in T\}$ is $\mathscr{M}'_q$-admissible if $q>1$, or $\mathscr{S}'$-allowable if $q=1$, therefore $g_f\in\mathscr{A}'_q$.

Case (iii). $1/m_q=1/m_{2j+1}$. Suppose that $f_p\in T$. Then, by the definition of $f^{k,t}$ and T, $w(f_p)\geqslant 1/m_{2j+1}$. On the other hand, recall (Remark 2.19(a)) that ψ is defined through φ, so that every functional in $\bigcup K^s(\psi)$ has the same weight as the corresponding functional in $\bigcup K^s(\varphi)$. So, in this case, by the definition of L'_{2j+1}, we get that $w(f_p)<1/m_{2j+1}$ for every p. It follows that $T=\emptyset$.

Recalling also the definition of *If* and ψ_2, we get that in this case $\#J\leqslant 3$. Let $J=\{p_1,p_2,p_3\}$ and $f_{p_\lambda}=f^{k_\lambda,t_\lambda}$, $\lambda=1,2,3$. Set $g_f=\frac{1}{2}(e^*_{f_{p_1}}+e^*_{f_{p_2}}+e^*_{f_{p_3}})$. By $(*)$, $f_{p_\lambda}(\sum_k b_k\theta_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)}))\leqslant b_{k_\lambda}\theta_{k_\lambda}a_{f_{p_\lambda}}$, $\lambda=1,2,3$. Thus,

$$\begin{aligned} f|_{D_f}\left(\sum_k b_k\theta_k m_{2j_k}\left(\sum_i b_{(k,i)}u_{(k,i)}\right)\right)&\leqslant\sum_{\lambda=1}^{2} b_{k_\lambda}\theta_{k_\lambda}a_{f_{p_\lambda}}\\ &=\sum_{\lambda=1}^{3} b_{k_\lambda}\theta_{k_\lambda}a_{f_{p_\lambda}}(e^*_{f_{p_\lambda}})\\ &=g_f\left(2\sum_k b_k\theta_k u_k\right).\end{aligned}$$

This completes the proof of the Claim. By the Claim and relations (1), (2), (3), statement (a) follows.

(b) We have from Lemma 3.9(a) that for $f, f' \in \bigcup_s K^s(\varphi)$, $f \neq f'$,

$$\operatorname{supp} If \cap \operatorname{supp} If' = \emptyset. \tag{$**$}$$

For f with $If \neq 0$, let $If = (1/m_{2j+1})(y_p^* + \cdots + y_{p+q}^*)$. Since $\{b_k\}$ is decreasing,

$$\left| If\left(\sum_k \varepsilon_k b_k \theta_k m_{2j_k} y_k \right) \right| \leqslant \frac{b_p}{m_{2j+1}}. \tag{$***$}$$

Set

$$I_1 = \left\{ If\colon \text{there exists } h \in \bigcup_s K^s(\varphi) \text{ with } \operatorname{supp} If \subset \operatorname{supp} h \text{ strictly and } w(h) \leqslant \frac{1}{m_{2j+1}} \right\},$$

$$I_2 = \left\{ If\colon \text{for every } h \in \bigcup_s K^s(\varphi) \text{ with } \operatorname{supp} If \subset \operatorname{supp} h \text{ strictly, } w(h) \geqslant \frac{1}{m_{2j}} \right\}.$$

Set also

$$A_1 = \bigcup_{If \in I_1} \operatorname{supp} If \qquad \text{and} \qquad A_2 = \bigcup_{If \in I_2} \operatorname{supp} If.$$

Then, by ($**$) and ($***$),

$$\left| \varphi_1|_{A_1} \left(\sum_k \varepsilon_k b_k \theta_k m_{2j_k} y_k \right) \right| \leqslant \frac{1}{m_{2j+1}^2}.$$

For $If \in I_2$, we set

$$k(f) = \min\{l\colon y_l^* \text{ is in the decomposition of } If\},$$

$$T = \{k = 1, ..., n\colon k = k(f) \text{ for some } If \in I_2\}$$

and, for $k = k(f) \in T$, $l_k = \min(\operatorname{supp} y_k \cap \operatorname{supp} If)$.

Using ($**$) and ($***$) we construct in a similar way as in part (a) a functional $g \in K'$, $|g|_{2j}^* \leqslant 1$ such that

$$\left| \varphi_1|_{A_2} \left(\sum_k \varepsilon_k b_k \theta_k m_{2j_k} y_k \right) \right| \leqslant g\left(\sum_{k \in T} b_k e_{l_k} \right).$$

Then by Lemma 2.4(b) we have the result. This completes the proof of the lemma. Proposition 3.8 follows. ∎

Proposition 3.3 follows from Lemmas 3.6, 3.7, and Proposition 3.8.

3.12. *Remark*. The space X is reflexive.

The proof of this is similar to the proof of Theorem 1.27. We need to prove that: (a) The basis $(e_n)_n$ is boundedly complete. (b) The basis $(e_n)_n$ is shrinking. The proof of (a) is exactly the same as that of Theorem 1.27(a). For (b) we also follow the proof of Theorem 1.27(b). We just need to notice that the norming set L of X satisfies the properties of the set K which are used in that proof.

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