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# Boundary fluxes for nonlocal diffusion ☆

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**Abstract**

We study a nonlocal diffusion operator in a bounded smooth domain prescribing the flux through the boundary. This problem may be seen as a generalization of the usual Neumann problem for the heat equation. First, we prove existence, uniqueness and a comparison principle. Next, we study the behavior of solutions for some prescribed boundary data including blowing up ones. Finally, we look at a nonlinear flux boundary condition.


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*Keywords:* Nonlocal diffusion; Boundary value problems

## 1. Introduction

The purpose of this article is to address the Neumann boundary value problem for a nonlocal diffusion equation.

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Let $J:\mathbb{R}^N \to \mathbb{R}$ be a nonnegative, symmetric $J(z) = J(-z)$ with $\int_{\mathbb{R}^N} J(z)\,dz = 1$. Assume also that $J$ is strictly positive in $B(0,d)$ and vanishes in $\mathbb{R}^N \setminus B(0,d)$. Equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes, see [1–3, 6,7,11,14]. As stated in [7] if $u(x,t)$ is thought of as a density at the point $x$ at time $t$ and $J(x-y)$ is thought of as the probability distribution of jumping from location $y$ to location $x$, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position $x$ from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location $x$ to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density $u$ satisfies Eq. (1.1).

Equation (1.1), so called nonlocal diffusion equation, shares many properties with the classical heat equation $u_t = \Delta u$ such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if $J$ is compactly supported, perturbations propagate with infinite speed.

Given a bounded, connected and smooth domain, $\Omega$, one of the most common boundary conditions that has been imposed to the heat equation in the literature is the *Neumann boundary condition*, $\partial u/\partial\eta(x,t) = g(x,t)$, $x \in \partial\Omega$.

Let us state our model equation. We study

$$u_t(x,t) = \int_{\Omega} J(x-y)\big(u(y,t) - u(x,t)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,t)\,dy, \tag{1.2}$$

for $x \in \Omega$. In this model we have that the first integral takes into account the diffusion inside $\Omega$. In fact, as we have explained the integral $\int J(x-y)(u(y,t) - u(x,t))\,dy$ takes into account the individuals arriving or leaving position $x$ from other places. Since we are integrating in $\Omega$, we are imposing that diffusion takes place only in $\Omega$. The last term takes into account the prescribed flux (given by the data $g(x,t)$) of individuals from outside (that is individuals that enter or leave the domain according to the sign of $g$). This is what is called Neumann boundary conditions.

Our first result for this problem is the existence and uniqueness of solutions and a comparison principle.

**Theorem 1.1.** *For every $u_0 \in L^1(\Omega)$ and $g \in L^\infty_{\mathrm{loc}}((0,\infty); L^1(\mathbb{R}^N \setminus \Omega))$ there exists a unique solution $u$ of* (1.2) *such that $u \in C([0,\infty); L^1(\Omega))$ and $u(x,0) = u_0(x)$.*

*Moreover the solutions satisfy the following comparison property*:

$$\text{if } u_0(x) \leqslant v_0(x) \quad \text{in } \Omega, \quad \text{then } u(x,t) \leqslant v(x,t) \quad \text{in } \Omega \times [0,\infty).$$

*In addition the total mass in $\Omega$ satisfies*

$$\int_{\Omega} u(y,t)\,dy = \int_{\Omega} u_0(y)\,dy + \int_0^t \int_{\Omega} \int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,s)\,dy\,dx\,ds. \tag{1.3}$$

Once existence and uniqueness of solutions is proved an important aspect in evolution equations is the asymptotic behavior as time evolves. In this context, we study the asymptotic behavior of solutions for certain fluxes on the boundary.

First, we deal with a flux independent of time, that is, $g(x,t) = h(x)$. As happens for the heat equation, in this problem, when $h$ verifies a compatibility condition, we prove that solutions converge exponentially fast as $t \to \infty$ to the unique stationary solution of the problem with the same total mass as $u_0$. If the compatibility condition is violated then solutions become unbounded as $t \to \infty$. We have the following result.

**Theorem 1.2.** *Let in addition $J \in L^2(\mathbb{R}^N)$. Let $h \in L^1(\mathbb{R}^N \setminus \Omega)$ such that*

$$0 = \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy\,dx. \tag{1.4}$$

*Then there exists a unique solution $\varphi$ of the problem*

$$0 = \int_{\Omega} J(x-y)\big(\varphi(y) - \varphi(x)\big)\,dy + \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy \tag{1.5}$$

*that verifies $\int_\Omega u_0 = \int_\Omega \varphi$ and the asymptotic behavior of solutions of* (1.2) *is described as follows*: *there exists $\beta = \beta(J,\Omega) > 0$ such that*

$$\big\|u(t) - \varphi\big\|_{L^2(\Omega)} \leqslant e^{-\beta t}\|u_0 - \varphi\|_{L^2(\Omega)}. \tag{1.6}$$

*If* (1.4) *does not hold then solutions of* (1.2) *are unbounded.*

Next, we prescribe the boundary flux in such a way that it blows up in finite time. We consider a flux of the form

$$g(x,t) = h(x)(T-t)^{-\alpha}, \tag{1.7}$$

with a nonnegative and nontrivial function $h$.

For this problem we analyze the possibility that the solution becomes unbounded at time $T$ a phenomenon that is known as blow-up in the literature. For blowing-up solutions we also analyze the rate of blow-up (that is the speed at which solutions go to infinity at time $T$) and the blow-up set (that is the spatial location of the singularities).

We find that blow-up takes place in strips of width $d$ (recall that $J$ is positive in $B(0,d)$ and zero outside) around the support of $h$ with blow-up rates that increase as the strips get closer to the support of $h$.

Before stating our theorem we need some notation. We set $\Omega_0 = \Omega$, $\mathcal{B}_0 = \operatorname{supp}(h)$ and define recursively for $i \geqslant 1$

$$\mathcal{B}_i = \Big\{x \in \Omega \setminus \bigcup_{j<i} \mathcal{B}_j \colon\ d(x, \mathcal{B}_{i-1}) < d\Big\}$$

and

$$\Omega_i = \Omega_{i-1} \setminus \mathcal{B}_i.$$

We also define the functions $w_i, \tilde{w}_i : \mathbb{R}^N \to \mathbb{R}$ by

$$w_1(x) = \frac{1}{(\alpha - 1)} \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy,$$

$$w_i(x) = \frac{1}{(\alpha - i)} \int_{\mathbb{R}^N \setminus \Omega_i} J(x-y)w_{i-1}(y)\,dy \quad \text{for } 1 < i < \alpha,$$

$$\tilde{w}_1(x) = \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy$$

and

$$\tilde{w}_i(x) = \int_{\mathbb{R}^N \setminus \Omega_i} J(x-y)w_{i-1}(y)\,dy \quad \text{for } 1 < i \leqslant [\alpha].$$

We can now state our result.

**Theorem 1.3.** *Let in addition $J \in L^\infty(\mathbb{R}^N)$. Assume $h \in L^\infty(\mathbb{R}^N \setminus \Omega)$, $h \geqslant 0$, $\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy\,dx \neq 0$. Then, the solution of* (1.2) *with $g(x,t) = h(x)(T-t)^{-\alpha}$ blows up at time $T$ if and only if $\alpha \geqslant 1$.*

*If $\alpha > 1$ is not an integer the blow-up set, $B(u)$, is given by*

$$B(u) = \bigcup_{1 \leqslant i \leqslant [\alpha]} \mathcal{B}_i,$$

*with the asymptotic behavior*

$$(T-t)^{\alpha - i} u(x,t) \to w_i(x) \quad \textit{uniformly in } \mathcal{B}_i \textit{ as } t \to T$$

*for each $i$ such that $1 \leqslant i \leqslant [\alpha]$.*

*If $\alpha$ is an integer the blow-up set, $B(u)$, is given by*

$$B(u) = \bigcup_{1 \leqslant i \leqslant \alpha} \mathcal{B}_i,$$

*with the asymptotic behavior,*

$$(T-t)^{\alpha - i} u(x,t) \to w_i(x) \quad \textit{uniformly in } \mathcal{B}_i \textit{ as } t \to T$$

*for each $i$ such that $1 \leqslant i < \alpha$ and*

$$\frac{u(x,t)}{-\ln(T-t)} \to \tilde{w}_\alpha(x) \quad \textit{uniformly in } \mathcal{B}_\alpha \textit{ as } t \to T.$$

Observe that blow-up in the whole domain (global blow-up) is possible. Indeed this happens for large values of $\alpha$ (depending on $\Omega$, $h$ and $d$).

One can compare this result with the corresponding one for the heat equation with boundary flux $\partial u/\partial\eta(x,t) = h(x)(T-t)^\alpha$. For the heat equation solutions blow up if and only if $\alpha > 1/2$ and in this case $\max_x u(x,t) \sim (T-t)^{-\alpha+1/2}$. Therefore the occurrence of blow-up and the blow-up rate for nonlocal diffusion are different from the corresponding ones for the heat equation.

Finally we consider a nonlinear boundary condition of the form

$$g(y,t) = \bar{u}^p(y,t) \tag{1.8}$$

where $\bar{u}$ is the extension of $u$ from the boundary to the exterior of the domain in the following form: let us assume that a neighborhood of width $d$ of $\partial\Omega$ in $\mathbb{R}^N \setminus \Omega$ can be described by coordinates $(z,s)$ where $z \in \partial\Omega$ and $s$ is the distance from the point to the boundary, then we set $\bar{u}(z,s) = u(z)$. For this nonlinear boundary condition with nonlocal diffusion we have the following result.

**Theorem 1.4.** *Let in addition $J \in C(\mathbb{R}^N)$. Then, positive solutions blow-up in finite time if and only if $p > 1$. As for the blow-up rate, there exist constants $C$, $c > 0$ such that*

$$c(T-t)^{-1/(p-1)} \leqslant \max_x u(x,t) \leqslant C(T-t)^{-1/(p-1)}. \tag{1.9}$$

*Moreover, the blow-up set is contained in a neighborhood of $\partial\Omega$ of width $Kd$, where $K = [p/(p-1)]$.*

There is a large amount of literature dealing with blow-up for parabolic equations and systems see for example the survey [9], the book [13] and references therein. When blow-up is due to nonlinear boundary conditions see for example [10,12], the surveys [4,8] and the references therein. It is known that solutions of the heat equation with a nonlinear boundary condition given by a power blow up in finite time if and only if $p > 1$, the blow-up rate is given by $\|u(x,t)\|_{L^\infty(\Omega)} \sim (T-t)^{-1/(2(p-1))}$ and the blow-up set is contained in $\partial\Omega$. Hence the blow-up rate and set are different but the blow-up set contracts to the boundary as the support of $J$ becomes smaller. Observe that for $J$ fixed the blow-up set can be the whole domain $\Omega$ if $p$ is sufficiently close to 1.

*Organization of the paper.* In Section 2 we prove existence, uniqueness and the comparison principle, in Section 3 we deal with the problem with $g(x,t) = h(x)$, in Section 4 we analyze the blow-up problem and finally in Section 5 we study the problem with a nonlinear boundary condition.

## 2. Existence, uniqueness and a comparison principle

In this section we prove Theorem 1.1 and give as remarks several consequences of the proof that will be used later in the paper.

As in [5], existence and uniqueness will be a consequence of Banach's fixed point theorem so we give first some preliminaries.

Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = C\big([0, t_0]; L^1(\Omega)\big)$$

with the norm

$$|||w||| = \max_{0 \leqslant t \leqslant t_0} \big\|w(\cdot, t)\big\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $T : X_{t_0} \to X_{t_0}$ defined by

$$T_{w_0,g}(w)(x,t) = w_0(x) + \int_0^t \int_\Omega J(x-y)\big(w(y,s) - w(x,s)\big)\,dy\,ds + \int_0^t \int_{\mathbb{R}^N \setminus \Omega} J(x-y) g(y,s)\,dy\,ds. \tag{2.1}$$

The following lemma is the main ingredient in the proof of existence.

**Lemma 2.1.** *Let $w_0, z_0 \in L^1(\Omega)$, $g, h \in L^\infty((0,t_0); L^1(\mathbb{R}^N \setminus \Omega))$ and $w, z \in X_{t_0}$, then there exists a constant $C$ depending only on $\Omega$ and $J$ such that*

$$\big|\big|\big|T_{w_0,g}(w) - T_{z_0,h}(z)\big|\big|\big| \leqslant \|w_0 - z_0\|_{L^1(\Omega)} + Ct_0\big\{|||w - z||| + \|g - h\|_{L^\infty((0,t_0); L^1(\mathbb{R}^N \setminus \Omega))}\big\}.$$

**Proof.** We have

$$\begin{aligned}
&\int_\Omega \big|T_{w_0,g}(w)(x,t) - T_{z_0,h}(z)(x,t)\big|\,dx \\
&\quad \leqslant \int_\Omega |w_0 - z_0|(x)\,dx \\
&\qquad + \int_\Omega \left| \int_0^t \int_\Omega J(x-y)\big[\big(w(y,s) - z(y,s)\big) - \big(w(x,s) - z(x,s)\big)\big]\,dy\,ds \right| dx \\
&\qquad + \int_\Omega \left| \int_0^t \int_{\mathbb{R}^N \setminus \Omega} J(x-y)\big(g(y,s) - h(y,s)\big)\,dy\,ds \right| dx.
\end{aligned}$$

Hence

$$\int_{\Omega} \left|T_{w_0,g}(w)(x,t) - T_{z_0,h}(z)(x,t)\right| dx$$
$$\leqslant \|w_0 - z_0\|_{L^1(\Omega)} + \int_0^t \int_{\Omega} \left|\big(w(y,s) - z(y,s)\big)\right| dy\, ds + \int_0^t \int_{\Omega} \left|\big(w(x,s) - z(x,s)\big)\right| dx\, ds$$
$$+ \int_0^t \int_{\mathbb{R}^N\setminus\Omega} \left|g(y,s) - h(y,s)\right| dy\, ds.$$

Therefore, we obtain,

$$\left|\!\left|\!\left| T_{w_0,g}(w) - T_{z_0,h}(z) \right|\!\right|\!\right| \leqslant \|w_0 - z_0\|_{L^1(\Omega)} + Ct_0\left\{ |\!|\!| w - z |\!|\!| + \|g - h\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N\setminus\Omega))}\right\},$$

as we wanted to prove. □

**Theorem 2.1.** *For every $u_0 \in L^1(\Omega)$ there exists a unique solution $u$ of* (1.2) *such that $u \in C([0,\infty); L^1(\Omega))$. Moreover, the total mass in $\Omega$ verifies,*

$$\int_{\Omega} u(y,t)\, dy = \int_{\Omega} u_0(y)\, dy + \int_0^t \int_{\Omega} \int_{\mathbb{R}^N\setminus\Omega} J(x-y) g(y,s)\, dy\, dx\, ds. \tag{2.2}$$

**Proof.** We check first that $T_{u_0,g}$ maps $X_{t_0}$ into $X_{t_0}$. From (2.1) we see that for $0 < t_1 < t_2 \leqslant t_0$,

$$\left\| T_{u_0,g}(w)(t_2) - T_{u_0,g}(w)(t_1) \right\|_{L^1(\Omega)}$$
$$\leqslant 2 \int_{t_1}^{t_2} \int_{\Omega} \left|w(y,s)\right| dy\, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^N\setminus\Omega} \left|g(y,s)\right| dy\, ds.$$

On the other hand, again from (2.1)

$$\left\| T_{u_0,g}(w)(t) - w_0 \right\|_{L^1(\Omega)} \leqslant Ct\left\{ |\!|\!| w |\!|\!| + \|g\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N\setminus\Omega))}\right\}.$$

These two estimates give that $T_{u_0,g}(w) \in C([0,t_0]; L^1(\Omega))$. Hence $T_{u_0,g}$ maps $X_{t_0}$ into $X_{t_0}$.

Choose $t_0$ such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$, $g \equiv h$ in Lemma 2.1 we get that $T_{u_0,g}$ is a strict contraction in $X_{t_0}$ and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$.

We finally prove that if $u$ is the solution, then the integral in $\Omega$ of $u$ satisfies (2.2). Since

$$u(x,t)-u_0(x)=\int_0^t\int_\Omega J(x-y)\big(u(y,s)-u(x,s)\big)\,dy\,ds$$

$$+\int_0^t\int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,s)\,dy\,ds.$$

We can integrate in $x$ and apply Fubini's theorem to obtain

$$\int_\Omega u(x,t)\,dx-\int_\Omega u_0(x)\,dx=\int_0^t\int_\Omega\int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,s)\,dy\,dx\,ds$$

and the theorem is proved. □

Now we give some consequences that we state as remarks for the sake of future references.

**Remark 2.1.** Solutions of (1.2) depend continuously on the initial condition and boundary data. Let $u$ be a solution of (1.2) with initial datum $u_0$ and $v$ a solution of (1.2) with $g$ replaced by $h$ and initial datum $v_0$. Then for every $t_0>0$ there exists a constant $C=C(t_0)$ such that

$$\max_{0\leqslant t\leqslant t_0}\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\Omega)}\leqslant C\big\|u(\cdot,0)-v(\cdot,0)\big\|_{L^1(\Omega)}+C\|g-h\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N\setminus\Omega))}.$$

**Remark 2.2.** The function $u$ is a solution of (1.2) if and only if

$$u(x,t)=e^{-A(x)t}u_0(x)+\int_0^t\int_\Omega e^{-A(x)(t-s)}J(x-y)u(y,s)\,dy\,ds$$

$$+\int_0^t\int_{\mathbb{R}^N\setminus\Omega} e^{-A(x)(t-s)}J(x-y)g(y,s)\,dy\,ds, \tag{2.3}$$

where $A(x)=\int_\Omega J(x-y)\,dy$.

Observe that $A(x)\geqslant\alpha>0$ $(x\in\overline{\Omega})$ for a certain constant $\alpha$.

**Remark 2.3.** From the previous remark we get that if $u\in L^\infty(\Omega\times(0,T))$, $u_0\in C^k(\overline{\Omega})$ with $0\leqslant k\leqslant\infty$, $g\in L^\infty(\mathbb{R}^N\setminus\Omega\times(0,T))$ and $J\in W^{k,1}(\mathbb{R}^N)$, then $u(\cdot,t)\in C^k(\overline{\Omega}\times[0,T])$.

On the other hand, if $J\in L^\infty(\mathbb{R}^N)$, $u_0\in L^\infty(\Omega)$ and $g\in L^1(\mathbb{R}^N\setminus\Omega\times(0,T))$, there holds that $u\in L^\infty(\Omega\times(0,T))$. (See Corollary 2.3 for an explicit bound in the case of continuous solutions.)

We now define what we understand by sub and supersolutions.

**Definition 2.1.** A function $u \in C([0,T); L^1(\Omega))$ is a supersolution of (1.2) if $u(x,0) \geqslant u_0(x)$ and

$$u_t(x,t) \geqslant \int_{\Omega} J(x-y)\big(u(y,t)-u(x,t)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,t)\,dy. \tag{2.4}$$

Subsolutions are defined analogously by reversing the inequalities.

**Lemma 2.2.** *Let $u_0 \in C(\overline{\Omega})$, $u_0 \geqslant 0$, and $u \in C(\overline{\Omega}\times[0,T])$ a supersolution to* (1.2) *with $g \geqslant 0$. Then, $u \geqslant 0$.*

**Proof.** Assume that $u(x,t)$ is negative somewhere. Let $v(x,t) = u(x,t) + \varepsilon t$ with $\varepsilon$ so small such that $v$ is still negative somewhere. Then, If we take $(x_0,t_0)$ a point where $v$ attains its negative minimum, there holds that $t_0 > 0$ and

$$v_t(x_0,t_0) = u_t(x_0,t_0) + \varepsilon > \int_{\Omega} J(x_0-y)\big(u(y,t_0)-u(x_0,t_0)\big)\,dy$$

$$= \int_{\Omega} J(x_0-y)\big(v(y,t_0)-v(x_0,t_0)\big)\,dy \geqslant 0$$

which is a contradiction. Thus, $u \geqslant 0$. □

**Corollary 2.1.** *Let $J \in L^\infty(\mathbb{R}^N)$. Let $u_0$ and $v_0$ in $L^1(\Omega)$ with $u_0 \geqslant v_0$ and $g,h \in L^\infty((0,T); L^1(\mathbb{R}^N\setminus\Omega))$ with $g \geqslant h$. Let $u$ be a solution of* (1.2) *with $u(x,0)=u_0$ and Neumann datum $g$ and $v$ be a solution of* (1.2) *with $v(x,0)=v_0$ and Neumann datum $h$. Then, $u \geqslant v$ a.e.*

**Proof.** Let $w = u - v$. Then, $w$ is a supersolution with initial datum $u_0 - v_0 \geqslant 0$ and boundary datum $g - h \geqslant 0$. Using the continuity of solutions with respect to the initial and Neumann data and the fact that $J \in L^\infty(\mathbb{R}^N)$, we may assume that $u, v \in C(\overline{\Omega}\times[0,T])$. By Lemma 2.2 we obtain that $w = u - v \geqslant 0$. So the corollary is proved. □

**Corollary 2.2.** *Let $u \in C(\overline{\Omega}\times[0,T])$* (*respectively $v$*) *be a supersolution* (*respectively subsolution*) *of* (1.2). *Then, $u \geqslant v$.*

**Proof.** It follows the lines of the proof of the previous corollary. □

**Corollary 2.3.** *Let $u$ be a continuous solution of* (1.2) *with $u(x,0)=u_0$ and Neumann datum $g \in L^\infty((\mathbb{R}^N\setminus\Omega)\times(0,T))$. Then,*

$$u(x,t) \leqslant \sup_{\Omega} u_0 + \int_0^t \sup_{\Omega} \int_{\mathbb{R}^N\setminus\Omega} J(x-y)g(y,s)\,dy\,ds. \tag{2.5}$$

**Proof.** Let

$$v(t) = \sup_{\Omega} u_0 + \int_0^t \sup_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} J(x-y) g(y,s)\, dy\, ds.$$

Then, $v$ is a continuous supersolution of (1.2). By the previous corollary we get the estimate (2.5). □

**Corollary 2.4.** *If* $J \in L^\infty(\mathbb{R}^N)$ *and* $u \in C([0,T]; L^1(\Omega))$ *is a solution of* (1.2) *with* $u(x,0) \in L^\infty(\Omega)$*,* $g \in L^\infty(0,T; L^1(\mathbb{R}^N \setminus \Omega))$ *then,* (2.5) *holds.*

**Proof.** Let $u_n$ be the solution of (1.2) with $u_n(x,0) = u_0^n(x)$ and Neumann datum $g_n \in L^\infty((\mathbb{R}^N \setminus \Omega) \times (0,T))$ such that $g_n \to g$ in $L^1((\mathbb{R}^N \setminus \Omega) \times (0,T))$ and $u_0^n \to u_0$ in $L^1(\Omega)$ with $\|u_0^n\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)}$.

The result follows from the application of Corollary 2.3 to the functions $u_n \in C(\overline{\Omega} \times [0,T])$ and taking limits as $n \to \infty$. □

## 3. Asymptotic behavior for $g(x,t) = h(x)$

In this section we study the asymptotic behavior, as $t \to \infty$, of the solutions of problem (1.2) in the case that the boundary data is time independent. So we will assume throughout this section that $g(x,t) = h(x)$ and that $J \in L^2(\mathbb{R}^N)$. We start by analyzing the corresponding stationary problem so we consider the equation

$$0 = \int_{\Omega} J(x-y)\big(\varphi(y) - \varphi(x)\big)\, dy + \int_{\mathbb{R}^N \setminus \Omega} J(x-y) h(y)\, dy. \tag{3.1}$$

Integrating in $\Omega$, it is clear that a necessary condition for the existence of a solution $\varphi$ is that

$$0 = \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} J(x-y) h(y)\, dy\, dx. \tag{3.2}$$

We will prove, by means of Fredholm's alternative, that condition (3.2) is sufficient for existence and that the solution is unique up to an additive constant.

To do this we write (3.1) in the form

$$\varphi(x) - K(\varphi)(x) = b(x), \tag{3.3}$$

where

$$b(x) = a(x) \int_{\mathbb{R}^N \setminus \Omega} J(x-y) h(y)\, dy,$$

$$K(\varphi)(x) = a(x) \int_{\Omega} J(x-y) \varphi(y)\, dy$$

and

$$a(x) = \left( \int_{\Omega} J(x-y)\,dy \right)^{-1}.$$

We consider the measure

$$d\mu = \frac{dx}{a(x)}$$

and its corresponding space $L^2_\mu$ of square integrable functions with respect to this measure.

We observe that, due to our assumptions on $J$, the operator $K$ maps $L^2_\mu$ into $L^2_\mu$ and as an operator $K : L^2_\mu \to L^2_\mu$ is compact and self adjoint.

We look now at the kernel of $I - K$ in $L^2_\mu$. We will show that this kernel consist only of constant functions. In fact, let $\varphi \in \ker(I - K)$. Then $\varphi$ satisfies

$$\varphi(x) = a(x) \int_{\Omega} J(x-y)\varphi(y)\,dy.$$

In particular, since $J \in L^2(\mathbb{R}^N)$, $\varphi$ is a continuous function. Set $A = \max_{x \in \overline{\Omega}} \varphi(x)$ and consider the set

$$\mathcal{A} = \bigl\{ x \in \overline{\Omega} \bigm| \varphi(x) = A \bigr\}.$$

The set $\mathcal{A}$ is clearly closed and nonempty. We claim that it is also open in $\overline{\Omega}$. Let $x_0 \in \mathcal{A}$. We have then

$$\varphi(x_0) = a(x_0) \int_{\Omega} J(x_0 - y)\varphi(y)\,dy.$$

Since $a(x_0) = (\int_\Omega J(x_0 - y)\,dy)^{-1}$ and $\varphi(y) \leqslant \varphi(x_0)$ this implies $\varphi(y) = \varphi(x_0)$ for all $y \in \Omega \cap B(x_0, d)$, and hence $\mathcal{A}$ is open as claimed. Consequently, as $\Omega$ is connected, $\mathcal{A} = \overline{\Omega}$ and $\varphi$ is constant.

According to Fredholm's alternative, problem (3.1) has a solution if and only if

$$\int_{\Omega} b(x) \frac{dx}{a(x)} = 0$$

or equivalently

$$\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy\,dx = 0.$$

We have proved

**Theorem 3.1.** *Problem* (3.1) *has a solution if and only if condition* (3.2) *holds. Moreover any two solutions differ by an additive constant.*

We will address now the problem of the asymptotic behavior of the solution of (1.2). The next proposition shows the existence of a Lyapunov functional for solutions of (1.2). Its proof is a direct computation and will be omitted.

**Proposition 3.1.** *Let $u(x,t)$ be the solution of* (1.2). *Let us define*

$$F(u)(t) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)^2\,dy\,dx - \int_\Omega\int_{\mathbb{R}^N\setminus\Omega} J(x-y)h(y)u(x,t)\,dy\,dx. \tag{3.4}$$

*Then*

$$\frac{\partial}{\partial t}F(u)(t) = -2\int_\Omega (u_t)^2(x,t)\,dx.$$

We are now in a position to state and prove a result on the asymptotic behavior of continuous solutions.

**Theorem 3.2.** *Let $u$ be a continuous solution of* (1.2) *with $g(x,t)=h(x)$ where $h$ satisfies the compatibility condition* (3.2). *Let $\varphi$ be the unique solution of* (3.1) *such that*

$$\int_\Omega \varphi(x)\,dx = \int_\Omega u_0(x)\,dx.$$

*Then*

$$u(x,t) \to \varphi(x) \quad \text{as } t\to\infty \tag{3.5}$$

*uniformly in $\overline{\Omega}$.*

*When* (3.2) *does not hold, solutions of* (1.2) *are unbounded.*

**Proof.** Set $w(x,t)=u(x,t)-\varphi(x)$. Then $w$ satisfies

$$w_t(x,t) = \int_\Omega J(x-y)\big(w(y,t)-w(x,t)\big)\,dy$$

and $\int_\Omega w(x,t)\,dx \equiv 0$.

By the estimate given in Corollary 2.3 we have that $\|w\|_{L^\infty(\Omega\times[0,\infty))}$ is bounded in $\overline{\Omega}\times[0,\infty)$ by $\|u_0-\varphi\|_{L^\infty(\Omega)}$.

Setting $A(x) = \int_\Omega J(x-y)\,dy$ and integrating, the above equation can be written as

$$w(x,t) = e^{-A(x)t} w(x,0) + \int_0^t e^{-A(x)(t-s)} \int_\Omega J(x-y) w(y,s)\,dy\,ds.$$

We note that $A(x)$ is a smooth function and that there exists $\alpha > 0$ such that $A(x) \geqslant \alpha$ for all $x \in \overline{\Omega}$. We observe that for $x_1, x_2 \in \overline{\Omega}$ one has

$$\left|e^{-A(x_1)t} - e^{-A(x_2)t}\right| \leqslant e^{-\alpha t} t \left|A(x_1) - A(x_2)\right|.$$

With this inequality in mind it is not difficult to obtain, via a triangle inequality argument, the estimate

$$\left|w(x_1,t) - w(x_2,t)\right| \leqslant D\left(\left|A(x_1) - A(x_2)\right| + \left|w(x_1,0) - w(x_2,0)\right|\right)$$

where the constant $D$ is independent of $t$. This implies that the functions $w(\cdot,t)$ are equicontinuous. Since they are also uniformly bounded, they are precompact in the uniform convergence topology.

Let $t_n$ be a sequence such that $t_n \to \infty$ as $n \to \infty$. Then the sequence $w(\cdot,t_n)$ has a subsequence, that we still denote by $w(\cdot,t_n)$, that converges uniformly as $n \to \infty$ to a continuous function $\psi$. A standard argument, using the Lyapunov functional of Proposition 3.1, proves that $\psi$ is a solution of the corresponding stationary problem and hence $\psi$ is constant. As $\int_\Omega w(x,t)\,dx \equiv 0$ this constant must be 0. Since this holds for every sequence $t_n$, with $t_n \to \infty$, we have proved that $w(\cdot,t) \to 0$ uniformly as $t \to \infty$ as we wanted to show.

When (3.2) does not hold the equation satisfied by the total mass, (2.2), implies that $u$ is unbounded. □

We end this section with a proof of the exponential rate of convergence to steady states of solutions in $L^2$. This proof does not use a Lyapunov argument. It is based on energy estimates.

First, we prove a lemma that can be viewed as a Poincaré type inequality for our operator.

**Lemma 3.1.** *There exists a constant $C > 0$ such that for every $u \in L^2(\Omega)$ it holds*

$$\int_\Omega \left(u(x) - \langle u\rangle\right)^2 dx \leqslant C \int_\Omega \int_\Omega J(x-y)\left(u(y) - u(x)\right)^2 dy\,dx,$$

*where $\langle u\rangle$ is the mean value of $u$ in $\Omega$, that is*

$$\langle u\rangle = \frac{1}{|\Omega|} \int_\Omega u(x)\,dx.$$

**Proof.** We can assume that $\langle u\rangle = 0$. Now let us take a partition of $\mathbb{R}^N$ in non-overlapping cubes, $T_i$, of diameter of length $h$. Using an approximation argument we can consider functions $u$ that are constant on each of the cubes $T_i$, $u|_{T_i} = a_i$. We will only consider cubes $T_i$ such that

$T_i \cap \Omega \neq \emptyset$. For this type of functions we have to prove that there exists a constant $C$ independent of the partition such that

$$\sum_i |T_i| a_i^2 \leqslant C \sum_i \sum_k \int_{T_i} \int_{T_k} J(x-y)\, dy\, dx\, (a_i - a_k)^2.$$

Recall that there exist $\sigma > 0$ and $r > 0$ such that $J(x-y) \geqslant \sigma$ for any $|x-y| < 2r$. If the centers of two cubes $T_i$, $T_k$, are at distance less than $r$ and $h < r$ we have

$$\int_{T_i} \int_{T_k} J(x-y)\, dy\, dx\, (a_i - a_k)^2 \geqslant \sigma |T_i||T_k|(a_i - a_k)^2.$$

Given $T_i$, $T_k$ two cubes intersecting $\Omega$ there exists a number $\ell$, depending only on $\Omega$ and $r$ but not on $h$, such that there exist a collection of at most $\ell$, not necessarily pairwise adjacent, cubes $T_{j_1}, \dots, T_{j_\ell}$ intersecting $\Omega$ with $T_{j_1} = T_i$, $T_{j_\ell} = T_k$ and such that the distance between the centers of $T_{j_m}$ and $T_{j_{m+1}}$ is less than $r$. Since all the involved cubes have the same measure, we have

$$\begin{aligned} |T_i||T_k|(a_i - a_k)^2 &\leqslant \ell^2 \left( \sum_{m=1}^{\ell-1} |T_{j_m}||T_{j_{m+1}}|(a_{j_m} - a_{j_{m+1}})^2 \right) \\ &\leqslant \frac{\ell^2}{\sigma} \sum_{m=1}^{\ell-1} \int_{T_{j_m}} \int_{T_{j_{m+1}}} J(x-y)\, dy\, dx\, (a_{j_m} - a_{j_{m+1}})^2. \end{aligned} \tag{3.6}$$

The intermediate cubes used in (3.6) corresponding to each pair $T_i$, $T_k$ can be chosen in such a way that no pair of cubes is used more than a fixed number of times (depending only on the diameter of $\Omega$ and $r$) when varying the pairs $T_i$, $T_k$. Therefore, there exists a constant $C$, depending only on $J$ and $\Omega$ but not on $h$, such that

$$\sum_i \sum_k |T_i||T_k|(a_i - a_k)^2 \leqslant C \sum_i \sum_k \int_{T_i} \int_{T_k} J(x-y)\, dy\, dx\, (a_i - a_k)^2.$$

On the other hand, as we are assuming that

$$\sum_i |T_i| a_i = 0,$$

we get

$$\sum_i \sum_k |T_i||T_k|(a_i - a_k)^2 \geqslant 2|\Omega| \sum_i |T_i|(a_i)^2$$

and the result follows. □

Now let us take the best $J$-Poincaré constant that is given by

$$\beta = \inf_{u \in L^2(\Omega)} \frac{\int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx}{\int_\Omega (u(x)-\langle u\rangle)^2\,dx}. \tag{3.7}$$

Note that by Lemma 3.1 $\beta$ is strictly positive and depends only on $J$ and $\Omega$.

Now let us prove the exponential convergence of $u(x,t)$ to the mean value of the initial datum when the boundary datum vanishes, i.e., $h=0$.

**Theorem 3.3.** *For every $u_0 \in L^2(\Omega)$ the solution $u(x,t)$ of* (1.2) *with $h=0$, satisfies*

$$\left\| u(\cdot,t) - \langle u_0\rangle \right\|^2_{L^2(\Omega)} \leqslant e^{-\beta t} \left\| u_0 - \langle u_0\rangle \right\|^2_{L^2(\Omega)}. \tag{3.8}$$

*Here $\beta$ is given by* (3.7).

**Proof.** Let

$$H(t) = \frac{1}{2}\int_\Omega \big(u(x,t)-\langle u_0\rangle\big)^2\,dx.$$

Differentiating with respect to $t$ and using (3.7), recall that $\langle u\rangle = \langle u_0\rangle$, we obtain

$$H'(t) = -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)^2\,dy\,dx \leqslant -\beta\frac{1}{2}\int_\Omega \big(u(x,t)-\langle u_0\rangle\big)^2\,dx.$$

Hence

$$H'(t) \leqslant -\beta H(t).$$

Therefore, integrating we obtain,

$$H(t) \leqslant e^{-\beta t} H(0).$$

As we wanted to prove. □

As a corollary we obtain exponential decay to the steady state for solutions of (1.2) with $h \neq 0$.

**Corollary 3.1.** *For every $u_0 \in L^2(\Omega)$ the solution of* (1.2), *$u(x,t)$, verifies*

$$\|u-\varphi\|^2_{L^2(\Omega)} \leqslant e^{-\beta t}\|u_0-\varphi\|^2_{L^2(\Omega)}. \tag{3.9}$$

*Here $\varphi$ is the unique stationary solution with the same mean value as the initial datum, and $\beta$ is given by* (3.7).

**Proof.** It follows from Theorem 3.3 by considering that $v = u - \varphi$ is a solution of (1.2) with $h=0$. □

## 4. Blow-up for $g(y,t) = h(y)(T-t)^{-\alpha}$

Now we analyze the asymptotic behavior of solutions of (1.2) when the flux at the boundary is given by

$$g(y,t) = h(y)(T-t)^{-\alpha}$$

with $h \geqslant 0$ and $\int_\Omega \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy\,dx > 0$. We will also assume that the initial data, and hence the solution, is nonnegative. Throughout this section $u(x,t)$ will denote the solution of (1.2) with boundary and initial data as described above. Also in this section we will assume, without loss of generality, that $T < 1$. This makes the quantity $-\ln(T-t)$ positive which helps to avoid overloading the notation.

Throughout this section we will assume that $J \in L^\infty(\mathbb{R}^N)$ and we will use the notation introduced in the introduction.

First, we prove that $\alpha = 1$ is the critical exponent to obtain blowing-up solutions.

**Lemma 4.1.** *The solution $u(x,t)$ blows up at time $T$ if and only if $\alpha \geqslant 1$.*

**Proof.** Set

$$M(t) = \int_\Omega u(x,t)\,dx,$$

then one has

$$M'(t) = \frac{1}{(T-t)^\alpha} \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} J(x-y)h(y)\,dy\,dx \geqslant \frac{c}{(T-t)^\alpha}.$$

Hence, if $\alpha \geqslant 1$ $M(t)$ is unbounded as $t \nearrow T$ and the same is true for the solution $u(x,t)$.

On the other hand, if $\alpha < 1$ we consider the solution of the ordinary differential equation

$$z'(t) = \frac{C}{(T-t)^\alpha} \quad \text{with } z(0) = z_0,$$

that is a supersolution of our problem if $C$ and $z_0$ are large enough. Since $z(t)$ remains bounded up to time $T$, a comparison argument shows that so does $u(x,t)$. □

**Lemma 4.2.** *There exists a constant $C$ such that for each integer $i$ such that $1 \leqslant i \leqslant \alpha$, the solution $u(x,t)$ verifies*

$$u(x,t) \leqslant \frac{C}{(T-t)^{\alpha-i}} \quad \text{in } \Omega_{i-1} \text{ if } i \neq \alpha$$

*and*

$$u(x,t) \leqslant -C\ln(T-t) \quad \text{in } \Omega_{i-1} \text{ if } i = \alpha.$$

**Proof.** If $\alpha > 1$ we have that

$$z(t) = \frac{C_1}{(T-t)^{\alpha-1}}$$

is a supersolution to our problem for $C_1$ large enough, therefore

$$u(x,t) \leqslant \frac{C_1}{(T-t)^{\alpha-1}} \quad \text{in } \Omega_0. \tag{4.1}$$

If $\alpha = 1$ the argument can be easily modified to get

$$u(x,t) \leqslant -C_1 \ln(T-t) \quad \text{in } \Omega_0.$$

Now for $x \in \Omega_1$ we have

$$u_t(x,t) = \int_{\Omega} J(x-y)\big(u(y,t) - u(x,t)\big)\,dy \tag{4.2}$$

which implies

$$u_t(x,t) \leqslant \int_{\Omega_1} J(x-y)\big(u(y,t) - u(x,t)\big)\,dy + \int_{\mathcal{B}_1} J(x-y)u(y,t)\,dy. \tag{4.3}$$

Assume that $\alpha > 2$. In this case, in view of (4.1), we can use the function

$$z(t) = \frac{C_2}{(T-t)^{\alpha-2}},$$

with $C_2$ large enough, as a supersolution in $\Omega_1$ to obtain that

$$u(x,t) \leqslant \frac{C_2}{(T-t)^{\alpha-2}} \quad \text{in } \Omega_1.$$

As before if $\alpha = 2$ we get

$$u(x,t) \leqslant -C_2 \ln(T-t) \quad \text{in } \Omega_1.$$

The previous argument can be repeated to obtain the conclusion of the lemma with the constant $C = \max_{1 \leqslant j \leqslant [\alpha]} C_j$. □

We can describe now precisely the blow-up set and profile of a blowing-up solution.

**Theorem 4.1.** *If $\alpha > 1$ is not an integer the blow-up set, $B(u)$, is given by*

$$B(u) = \bigcup_{1 \leqslant i \leqslant [\alpha]} \mathcal{B}_i,$$

*with the asymptotic behavior*

$$(T-t)^{\alpha-i}u(x,t)\to w_i(x) \quad \textit{uniformly in } \mathcal{B}_i \textit{ as } t\to T$$

*for each $i$ such that $1\leqslant i\leqslant[\alpha]$.*

*If $\alpha$ is an integer the blow-up set, $B(u)$, is given by*

$$B(u)=\bigcup_{1\leqslant i\leqslant\alpha}\mathcal{B}_i,$$

*with the asymptotic behavior*

$$(T-t)^{\alpha-i}u(x,t)\to w_i(x) \quad \textit{uniformly in } \mathcal{B}_i \textit{ as } t\to T$$

*for each $i$ such that $1\leqslant i<\alpha$ and*

$$\frac{u(x,t)}{-\ln(T-t)}\to\tilde{w}_\alpha(x) \quad \textit{uniformly in } \mathcal{B}_\alpha \textit{ as } t\to T.$$

**Proof.** We have

$$u_t(x,t)=\int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)\,dy+\int_{\mathbb{R}^N\setminus\Omega}J(x-y)\frac{h(y)}{(T-t)^\alpha}\,dy. \tag{4.4}$$

We prove first the theorem in the case when $\alpha=1$. Integrating (4.4) in $t$ and using that, by Lemma 4.2, $u(x,t)\leqslant -C\ln(T-t)$ we get

$$\left|-\frac{u(x,t)}{\ln(T-t)}-\tilde{w}_1(x)\right|\leqslant\frac{u(x,0)}{-\ln(T-t)}+C\frac{1}{-\ln(T-t)}\int_0^t\ln(T-r)\,dr+\frac{\ln T}{-\ln(T-t)}\tilde{w}_1(x).$$

This proves that

$$\lim_{t\to T}\frac{u(x,t)}{-\ln(T-t)}=\tilde{w}_1(x) \quad \text{uniformly in } \Omega_0.$$

Also if $x\in\Omega_1$ (4.4) reads

$$u_t(x,t)=\int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)\,dy.$$

Integrating in $t$ and using again Lemma 4.2 we have

$$u(x,t)\leqslant u(x,0)-C\int_0^t\ln(T-r)\,dr.$$

Hence $u$ is bounded in $\Omega_1$ and the theorem is proved if $\alpha=1$.

Assume now that $\alpha > 1$ and consider the change of variables

$$v_1(x,s) = (T-t)^{\alpha-1} u(x,t), \quad s = -\ln(T-t).$$

Since $u$ verifies (4.4), $v_1$ satisfies

$$(v_1)_s(x,s) = e^{-s} \int_{\Omega} J(x-y)\big(v_1(y,s) - v_1(x,s)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)h(y)\,dy - (\alpha-1)v_1(x,s). \tag{4.5}$$

Integrating in $s$ we obtain

$$\begin{aligned} &v_1(x,s) - w_1(x) \\ &\quad = e^{-(\alpha-1)s} v_1(x,0) + e^{-(\alpha-1)s} \int_0^s e^{(\alpha-2)r} \int_{\Omega} J(x-y)\big(v_1(y,r) - v_1(x,r)\big)\,dy\,dr \\ &\qquad - e^{-(\alpha-1)s} w_1(x). \end{aligned} \tag{4.6}$$

If $\alpha \neq 2$ since, by the previous lemma, $v_1$ is bounded we get

$$\big|v_1(x,s) - w_1(x)\big| \leqslant C\big(e^{-s} + e^{-(\alpha-1)s}\big) \tag{4.7}$$

for some constant $C$. This implies that

$$(T-t)^{\alpha-1} u(x,t) \to w_1(x) \tag{4.8}$$

uniformly in $\Omega_0$ as $t \to T$.

We note that if $\alpha < 2$, since $w_1(x)$ vanishes in $\Omega_1$, (4.7) implies

$$\big|v_1(x,s)\big| \leqslant Ce^{-(\alpha-1)s} \quad \text{for } x \in \Omega_1$$

and hence

$$u(x,t) \leqslant C \quad \text{for } x \in \Omega_1.$$

Consequently, if $1 < \alpha < 2$ the blow-up set of $u$ is $\Omega_0 \setminus \Omega_1 = \mathcal{B}_1$ and the asymptotic behavior at the blow-up time is given by (4.8).

We have to handle now the case $\alpha = 2$ which is slightly different. In this case instead of estimate (4.7) there holds

$$\big|v_1(x,s) - w_1(x)\big| \leqslant C\big(se^{-s} + e^{-(\alpha-1)s}\big). \tag{4.9}$$

This still implies that

$$(T-t)^{\alpha-1} u(x,t) \to w_1(x) \tag{4.10}$$

uniformly in $\Omega_0$ as $t \to T$ but does not ensure that $u$ is bounded in $\Omega_1$.

If $x \in \Omega_1$ one has

$$u_t(x,t) = \int_{\Omega_1} J(x-y)\big(u(y,t)-u(x,t)\big)\,dy + \int_{\mathcal{B}_1} J(x-y)u(y,t)\,dy - \int_{\mathcal{B}_1} J(x-y)u(x,t)\,dy.$$

Integrating in $t$ we obtain

$$u(x,t) - \int_0^t \int_{\mathcal{B}_1} J(x-y)u(y,r)\,dy\,dr = u(x,0) + I_1 + I_2,$$

where

$$I_1(x,t) = \int_0^t \int_{\Omega_1} J(x-y)\big(u(y,r)-u(x,r)\big)\,dy\,dr$$

and

$$I_2(x,t) = -\int_0^t \int_{\mathcal{B}_1} J(x-y)u(x,r)\,dy\,dr.$$

Now using the fact that for $z \in \Omega_1$ one has $u(z,t) \leqslant -C\ln(T-t)$, it can be checked that

$$\frac{I_1(x,t)}{\ln(T-t)} \to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t \to T$$

and also

$$\frac{I_2(x,t)}{\ln(T-t)} \to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t \to T.$$

Moreover since $(T-t)u(y,t) \to w_1(y)$ uniformly in $\mathcal{B}_1$ as $t \to T$ one has

$$-\frac{1}{\ln(T-t)} \int_0^t \int_{\mathcal{B}_1} J(x-y)u(y,r)\,dy\,dr \to \int_{\mathcal{B}_1} J(x-y)w_1(y)\,dy$$

uniformly in $\Omega_1$ as $t \to T$.

Putting together this information we deduce that

$$-\frac{u(x,t)}{\ln(T-t)} \to \int_{\mathcal{B}_1} J(x-y)w_1(y)\,dy$$

uniformly in $\Omega_1$ as $t \to T$.

Finally since $u(x,t) \leqslant -C\ln(T-t)$ in $\Omega_1$ we can argue as in the proof of the case $\alpha = 1$ to show that $u$ remains bounded in $\Omega_2$. So we have shown that if $\alpha = 2$, then the blow-up set, $B(u)$, of $u$ is given by

$$B(u) = \mathcal{B}_1 \cup \mathcal{B}_2$$

with the asymptotic behavior

$$\lim_{t\to T}(T-t)u(x,t) = w_1(x) \quad \text{uniformly in } \Omega_0$$

and

$$\lim_{t\to T}\frac{u(x,t)}{-\ln(T-t)} = \tilde{w}_2(x) \quad \text{uniformly in } \Omega_1.$$

If $\alpha > 2$ setting

$$v_2(x,s) = (T-t)^{\alpha-2}u(x,t), \quad s = -\ln(T-t),$$

we obtain for $x \in \Omega_1$ the equation

$$(v_2)_s(x,s) = e^{-s}\int_{\Omega} J(x-y)\big(v_2(y,s) - v_2(x,s)\big)\,dy - (\alpha-2)v_2(x,s). \tag{4.11}$$

This can be written as

$$\begin{aligned}(v_2)_s(x,s) = e^{-s}\int_{\Omega_1} & J(x-y)\big(v_2(y,s) - v_2(x,s)\big)\,dy \\ & + \int_{B_1} J(x-y)v_1(y,s)\,dy - v_1(x,s)\int_{B_1} J(x-y)\,dy \\ & - (\alpha-2)v_2(x,s).\end{aligned} \tag{4.12}$$

Again integrating in $s$, after observing that by (4.7) $|v_1(x,s)| \leqslant Ce^{-s}$ since $x \in \Omega_1$, we obtain that

$$\left| v_2(x,s) - e^{-(\alpha-2)s}\int_0^s e^{-(\alpha-2)r}\int_{\mathcal{B}_1} J(x-y)v_1(y,r)\,dy\,dr \right| \leqslant C\big(e^{-s} + e^{-(\alpha-2)s}\big) \tag{4.13}$$

for some constant $C$ provided that $\alpha \neq 3$.

Also by (4.7) one has that

$$\left| \int_{\mathcal{B}_1} J(x-y)v_1(y,s)\,dy - \int_{\mathcal{B}_1} J(x-y)w_1(y)\,dy \right| \leqslant Ce^{-s}$$

and consequently

$$\left| w_2(x) - e^{-(\alpha-2)s} \int_0^s e^{-(\alpha-2)r} \int_{\mathcal{B}_1} J(x-y) v_1(y,r)\,dy\,dr \right| \leqslant Ce^{-s}.$$

This, together with (4.13), implies that for all $x \in \Omega_1$

$$\left| v_2(x,s) - w_2(x) \right| \leqslant C\left(e^{-s} + e^{-(\alpha-2)s}\right)$$

for some constant $C$ and hence

$$(T-t)^{\alpha-2} u(x,t) \to w_2(x)$$

uniformly in $\Omega_1$.

The above procedure can be iterated to obtain for all integers $i$ such that $1 < i < \alpha$

$$\left| v_i(x,s) - w_i(x) \right| \leqslant C\left(e^{-s} + e^{-(\alpha-i)s}\right) \quad \text{for all } x \in \Omega_{i-1} \tag{4.14}$$

for some constant $C$ and hence

$$(T-t)^{\alpha-i} u(x,t) \to w_i(x) \tag{4.15}$$

uniformly in $\Omega_{i-1}$. Moreover, it follows from (4.14) that for $x \in \Omega_{[\alpha]}$, if $\alpha$ is not an integer,

$$\left| v_{[\alpha]}(x,s) \right| \leqslant Ce^{-(\alpha-[\alpha])s}$$

and hence

$$u(x,t) \leqslant C \quad \text{for all } x \in \Omega_{[\alpha]}.$$

In this fashion we have proved that, if $\alpha$ is not an integer, the blow-up set of $u$ is $\bigcup_{1 \leqslant i \leqslant [\alpha]} \mathcal{B}_i$ and the behavior of $u$ near time $T$ in $\mathcal{B}_i$ is given by (4.15). This proves the theorem in the case that $\alpha$ is not an integer.

In the case that $\alpha$ is an integer one can argue as in the proof of the case $\alpha = 2$ to obtain the result in that case. □

## 5. Blow-up with a nonlinear boundary condition

In this section we deal with the problem

$$\begin{aligned} u_t &= \int_{\Omega} J(x-y)\big(u(y,t) - u(x,t)\big)\,dy + \int_{\mathbb{R}^N \setminus \Omega} J(x-y) \bar{u}^p(y,t)\,dy, \\ u(x,0) &= u_0(x). \end{aligned} \tag{5.1}$$

Here we assume that $J \in C(\mathbb{R}^N)$, $u_0 \in C(\overline{\Omega})$, $u_0 \geqslant 0$ and $\bar{u}$ is the extension of $u$ to a neighborhood of $\overline{\Omega}$ defined as follows: take a small neighborhood of $\partial\Omega$ in $\mathbb{R}^N \setminus \Omega$ in such a way that there exist coordinates $(s,z) \in (0,s_0) \times \partial\Omega$ that describe that neighborhood in the form

$y = z + s\eta(z)$ where $z \in \partial\Omega$ and $\eta(z)$ is the exterior unit normal vector to $\partial\Omega$ at $z$. We set

$$\bar{u}(y,t) = u(z,t).$$

We also assume that $d < s_0$ therefore for any $x \in \overline{\Omega}$, $B_d(x) \cap (\mathbb{R}^N \setminus \Omega)$ is contained in the above mentioned neighborhood.

We address now the problem of local existence in time and uniqueness of solutions.

As in the previous sections we set

$$A(x) = \int_{\Omega} J(x-y)\,dy$$

and observe that there exists $\alpha > 0$ such that $A(x) \geqslant \alpha$ for all $x \in \overline{\Omega}$.

As earlier we obtain a solution of (5.1) as a fixed point of the operator $T$ defined by

$$Tu(x,t) = e^{-A(x)t}u_0(x) + \int_0^t e^{-A(x)(t-s)} \int_{\Omega} J(x-y)u(y,s)\,dy\,ds + \int_0^t e^{-A(x)(t-s)} \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}^p(y,s)\,dy\,ds.$$

We split the proof of existence into two cases. We deal first with the case $p \geqslant 1$ since in this case we have uniqueness of solutions. In this direction we have the following theorem.

**Theorem 5.1.**

(a) *Let* $p \geqslant 1$. *There exists* $t_0 > 0$ *such that problem* (5.1) *has a unique solution defined in* $[0, t_0)$.
(b) *Let* $p < 1$. *There exists* $t_0 > 0$ *such that problem* (5.1) *has at least one solution defined in* $[0, t_0)$.

**Proof.** Fix $M \geqslant \|u_0\|_\infty$, $t_0 > 0$ and set

$$X = \Big\{ u \in C\big(\overline{\Omega} \times [0,t_0)\big) \;\Big|\; u \geqslant 0,\ |||u||| \equiv \sup_{(x,t)\in\overline{\Omega}\times[0,t_0)} \big|u(x,t)\big| \leqslant 2M \Big\}.$$

If $t_0$ is chosen small enough, then $T$ maps $X$ into $X$. Indeed, we have for $t \leqslant t_0$ and $u \in X$

$$\begin{aligned}\big|Tu(x,t)\big| &\leqslant e^{-A(x)t}u_0(x) + \int_0^t e^{-A(x)(t-s)} \int_{\Omega} J(x-y)\big|u(y,s)\big|\,dy\,ds \\ &\quad + \int_0^t e^{-A(x)(t-s)} \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\big|\bar{u}^p(y,s)\big|\,dy\,ds \\ &\leqslant M + t_0\big(2M + (2M)^p\big) \leqslant 2M\end{aligned}$$

if $t_0$ is small.

Proof of (a): We will prove that for $p \geqslant 1$ we can choose $t_0$ in such a way that $T$ is a strict contraction. In fact, for $t \leqslant t_0$ and $u_1, u_2 \in X$

$$\begin{aligned}\left|Tu_1(x,t) - Tu_2(x,t)\right| &\leqslant \int_0^t e^{-A(x)(t-s)} \int_\Omega J(x-y)\left|u_1(y,s) - u_2(y,s)\right| dy\, ds \\ &\quad + \int_0^t e^{-A(x)(t-s)} \int_{\mathbb{R}^N \setminus \Omega} J(x-y)\left|\bar{u}_1^p(y,s) - \bar{u}_2^p(y,s)\right| dy\, ds \\ &\leqslant t_0\left(1 + p(2M)^{p-1}\right) |||u_1 - u_2|||\end{aligned}$$

and part (a) of the theorem follows via Banach's fixed point theorem.

Proof of (b): We have that $T$ maps $X$ into $X$ if $t_0$ is small enough. We claim that the operator $T : X \to X$ is compact. Indeed, for $t_1, t_2 \leqslant t_0$, $u \in X$ and $x_1, x_2 \in \overline{\Omega}$ we have

$$\begin{aligned}&\left|Tu(x_1,t_1) - Tu(x_2,t_2)\right| \\ &\quad \leqslant \left|e^{-A(x_1)t_1} u_0(x_1) - e^{-A(x_2)t_2} u_0(x_2)\right| \\ &\qquad + \left|\int_0^{t_1} e^{-A(x_1)(t_1-s)} \int_\Omega J(x_1-y) u(y,s)\, dy\, ds - \int_0^{t_2} e^{-A(x_2)(t_2-s)} \int_\Omega J(x_2-y) u(y,s)\, dy\, ds\right| \\ &\qquad + \left|\int_0^{t_1} e^{-A(x_1)(t_1-s)} \int_{\mathbb{R}^N \setminus \Omega} J(x_1-y) \bar{u}^p(y,s)\, dy\, ds\right. \\ &\qquad \left. - \int_0^{t_2} e^{-A(x_2)(t_2-s)} \int_{\mathbb{R}^N \setminus \Omega} J(x_2-y) \bar{u}^p(y,s)\, dy\, ds\right|.\end{aligned}$$

As in the proof of Theorem 3.2 we have that for $x_1, x_2 \in \overline{\Omega}$ one has

$$\left|e^{-A(x_1)t} - e^{-A(x_2)t}\right| \leqslant e^{-\alpha t} t \left|A(x_1) - A(x_2)\right|$$

with $\alpha > 0$. This inequality plus the fact that $J$ is integrable imply, via a triangle inequality argument, that the family $\{Tu \mid u \in X\}$ is equicontinuous and, since it is bounded, it is precompact in $(X, |||\cdot|||)$. Consequently, since $T$ is clearly continuous in $X$, it is a compact operator and the claim is proved. Part (b) of the theorem now follows from Schauder's fixed point theorem. □

**Remark 5.1.** We observe that the same argument of the proof of part (a) of Theorem 5.1 provides existence of a unique solution if the boundary nonlinearity takes the form $f(\bar{u})$ with $f$ locally Lipschitz.

Now we prove a comparison lemma for solutions of (5.1).

**Lemma 5.1.** *Let $u$ be a continuous subsolution and $v$ be a continuous supersolution of problem* (5.1) *defined in* $[0, t_0)$. *Assume* $u(x,0) < v(x,0)$ *for all* $x \in \overline{\Omega}$. *Then*

$$u(x,t) < v(x,t)$$

*for all* $(x,t) \in \overline{\Omega} \times [0, t_0)$.

**Proof.** Assume, for a contradiction, that the lemma is not true. Then, by continuity, there exist $x_1 \in \overline{\Omega}$ and $0 < t_1 < t_0$ such that $u(x_1, t_1) = v(x_1, t_1)$ and $u(x,t) \leqslant v(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0, t_1)$. We have now

$$\begin{aligned} 0 &= u(x_1,t_1) - v(x_1,t_1) \\ &= e^{-A(x_1)t_1}\big(u(x_1,0) - v(x_1,0)\big) \\ &\quad + \int_0^{t_1} e^{-A(x_1)(t_1-s)} \int_{\Omega} J(x_1-y)\big(u(y,s) - v(y,s)\big)\,dy\,ds \\ &\quad + \int_0^{t_1} e^{-A(x_1)(t_1-s)} \int_{\mathbb{R}^N\setminus\Omega} J(x_1-y)\big(\bar{u}^p(y,s) - \bar{v}^p(y,s)\big)\,dy\,ds < 0 \end{aligned}$$

a contradiction that proves the lemma. □

Now we use this comparison result to prove the lack of uniqueness for $p < 1$.

**Proposition 5.1.** *In the case $p < 1$ with $u_0 \equiv 0$ there exists a nontrivial solution of problem* (5.1). *Hence this problem does not have uniqueness.*

**Proof.** Let $b(t)$ be a positive solution of $b' = b^p$ with $b(0) = 0$ and $0 \leqslant a(x) \leqslant \gamma$ be a continuous function with $a(x) \equiv \gamma$ on $\partial\Omega$. Let $\gamma > 0$ be so small as to have

$$\gamma^p \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy > 2\gamma$$

for every $x \in \Omega$. Then,

$$v(x,t) = a(x)b(t)$$

is a subsolution to our problem for a certain interval of time, $(0, t_0)$.

Let $\varepsilon > 0$ be given and consider a locally Lipschitz function $f_\varepsilon$ such that $f_\varepsilon(s) = s^p$ for $s \geqslant \varepsilon/2$. It follows from Remark 5.1 that there exists a unique solution, $w_\varepsilon$, of (5.1) with the boundary nonlinearity replaced by $f_\varepsilon(\bar{w})$ and initial data $w_\varepsilon(x,0) \equiv \varepsilon$. By the comparison principle $w_\varepsilon \geqslant \varepsilon$ and hence it is a supersolution of (5.1).

By comparison, the sequence $w_\varepsilon$ is monotone increasing in $\varepsilon$. In particular, for every $\varepsilon$ $w_\varepsilon$ is defined on the interval $[0, t_1]$ where $w_1$ is defined. Therefore, by monotone convergence, we obtain that the limit

$$w = \lim_{\varepsilon \to 0} w_\varepsilon$$

is a solution with $w(x,0) = 0$.

Using again the comparison principle we obtain that $w_\varepsilon(x,t) > v(x,t)$ for $0 < t < \min\{t_0, t_1\}$. Hence, $w(x,t) > 0$ for every $0 < t < \min\{t_0, t_1\}$ and all $x \in \operatorname{supp}(a)$. $\square$

We address now the blow-up problem for solutions of (5.1). In this direction we have the following theorem.

**Theorem 5.2.**

(a) *Let $p > 1$, then every nontrivial solution of* (5.1) *blows up in finite time.*
(b) *Let $p \leqslant 1$, then every solution of* (5.1) *is globally defined in time, by this we mean that it exists for all $t \in [0, \infty)$.*

**Proof.** Proof of (a): Let $u$ be a solution of (5.1) and assume, for a contradiction, that it is globally defined in time.

Since $u \geqslant 0$ and $\int_\Omega J(x-y)\,dy \leqslant 1$ we have for $x \in \overline{\Omega}$,

$$u_t(x,t) \geqslant -u(x,t) + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}^p(y,t)\,dy.$$

Here we have used that the equation is satisfied for $x \in \partial\Omega$.

Integrating on $\partial\Omega$, denoting by $dS_x$ the surface area element of $\partial\Omega$, we get

$$\frac{d}{dt}\int_{\partial\Omega} u(x,t)\,dS_x \geqslant -\int_{\partial\Omega} u(x,t)\,dS_x + \int_{\partial\Omega}\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}^p(y,t)\,dy\,dS_x.$$

Since

$$\int_{\partial\Omega}\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy\,dS_x > 0$$

an application on Jensen's inequality implies that

$$\frac{d}{dt}\int_{\partial\Omega} u(x,t)\,dS_x \geqslant -\int_{\partial\Omega} u(x,t)\,dS_x + C\left(\int_{\partial\Omega}\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}(y,t)\,dy\,dS_x\right)^p$$

for some constant $C > 0$.

Now,

$$\int_{\partial\Omega}\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}(y,t)\,dy\,dS_x$$
$$\geqslant \frac{1}{2}\int_{\partial\Omega}\int_0^{\varepsilon}\int_{\partial\Omega} J\big(x-\sigma-s\eta(\sigma)\big)u(\sigma,t)\,dS_\sigma\,ds\,dS_x$$

$$= \frac{1}{2} \int_{\partial\Omega} \int_{\partial\Omega} \left[ \int_0^\varepsilon J\big(x - \sigma - s\eta(\sigma)\big)\, ds \right] u(\sigma, t)\, dS_\sigma\, dS_x$$

$$\geqslant \delta |\partial\Omega| \int_{\partial\Omega} u(\sigma, t)\, dS_\sigma,$$

if $0 < \varepsilon < d$ is small enough. (Recall that $J(z) \geqslant c > 0$ for $|z| < d/2$.)

Thus, if we call

$$m(t) = \int_{\partial\Omega} u(x, t)\, dS_x,$$

we have

$$m'(t) \geqslant -m(t) + \gamma m^p(t). \tag{5.2}$$

This implies that $m(t) \to \infty$ in finite time if for some $t_0$, $m(t_0)$ is large enough.

Since we are assuming that $u(x, t)$ is defined for every $t > 0$, it holds that $m(t)$ is defined (and finite) for all $t > 0$. Let us see that this leads to a contradiction. Let $v(x, t)$ be the solution of (1.2) with $g = 0$ and $v(x, 0) = u_0$. By Theorem 3.2 we get that

$$v(x, t) \to \frac{1}{|\Omega|} \int_\Omega u_0,$$

uniformly in $\Omega$. Since $u$ is a supersolution for the problem satisfied by $v$, there exists $t_1 > 0$ such that for $t \geqslant t_1$,

$$u(x, t) \geqslant \frac{1}{2|\Omega|} \int_\Omega u_0 = c_0 > 0.$$

Therefore,

$$M(t) = \int_\Omega u(x, t)\, dx$$

$$\geqslant M(t_1) + \int_{t_1}^t \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} J(x - y) \bar{u}^p(y, s)\, dy\, dx\, ds$$

$$\geqslant M(t_1) + (t - t_1) c c_0^p.$$

Arguing as before we get that

$$u(x, t) \geqslant \frac{1}{2|\Omega|} M(t_2)$$

for $t$ large enough. Now

$$m(t_0) = \int_{\partial\Omega} u(x,t_0)\,dx \geqslant \frac{|\partial\Omega|}{2|\Omega|} M(t_2) \geqslant C\big(M(t_1) + (t_2 - t_1)cc_0^p\big).$$

This implies that $m(t_0)$ is as large as we need if $t_0$ is large enough, hence $m(t)$ is not defined for all times and we conclude that $u$ blows up in finite time.

Proof of (b): Let $p < 1$ and let $u$ be a solution of (5.1). Set $v(t) = C(t+1)^{\frac{1}{1-p}}$. It is directly checked that

$$v'(t) = \frac{C^{\frac{1}{1-p}}}{1-p} v^p(t).$$

Picking $C$ such that

$$\frac{C^{\frac{1}{1-p}}}{1-p} \geqslant \max_{x\in\overline{\Omega}} \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy$$

we have that $v$ is a supersolution of (5.1). Moreover taking $C$ larger, if necessary, such that $u(x,0) < v(0)$ in $\overline{\Omega}$ we obtain by Lemma 5.1 that

$$u(x,t) \leqslant v(t)$$

as long as $u$ is defined. This implies the theorem in the case $p < 1$. The case $p = 1$ is proved in the same fashion but using $v(t) = Ce^t$ as a supersolution. □

Our next result is an estimate of the blow-up rate of blowing-up solutions of (5.1).

**Theorem 5.3.** *Let $u$ be a solution of* (5.1) *that blows up at time $T$. Then there exists a constant $C$ such that*

$$(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)} \leqslant \big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant C(T-t)^{-1/(p-1)}. \tag{5.3}$$

**Proof.** Let

$$v(t) = (p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)}.$$

One can easily check that $v$ is a supersolution of our problem. If for some $t_0 \in (0,T)$ one has

$$\big\|u(\cdot,t_0)\big\|_{L^\infty(\Omega)} < v(t_0),$$

then there exists $\widetilde{T} > T$ such that

$$u(x,t_0) < (p-1)^{-1/(p-1)}(\widetilde{T} - t_0)^{-1/(p-1)}.$$

Let $\tilde{v}(t) := (p-1)^{-1/(p-1)}(\widetilde{T}-t)^{-1/(p-1)}$ that is also a supersolution to our problem in the interval $[t_0, \widetilde{T})$.

Using a comparison argument, we obtain

$$u(x,t) < (p-1)^{-1/(p-1)}(\widetilde{T}-t)^{-1/(p-1)},$$

for all $t \in (t_0, T)$. This contradicts the fact that $\widetilde{T} > T$ and hence

$$(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)} \leqslant \|u(\cdot,t)\|_{L^\infty(\Omega)}.$$

The proof of the reverse inequality is more involved. By Eq. (5.2), if

$$m(t) = \int_{\partial\Omega} u(x,t)\,dS_x \to \infty \quad \text{as } t \nearrow T,$$

we have

$$m(t) \leqslant C(T-t)^{-1/(p-1)}. \tag{5.4}$$

Now, we claim that

$$(T-t)^{1/(p-1)} \int_0^t \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\bar{u}^p(y,s)\,dy\,ds \leqslant C, \tag{5.5}$$

for all $x \in \partial\Omega$. In fact, if this does not hold, there exists a sequence $(x_n, t_n)$ with $x_n \in \partial\Omega$, $t_n \nearrow T$, such that

$$(T-t_n)^{1/(p-1)} \int_0^{t_n} \int_{\mathbb{R}^N\setminus\Omega} J(x_n-y)\bar{u}^p(y,s)\,dy\,ds \to \infty.$$

By compactness we may assume that $x_n \to x_0 \in \partial\Omega$. Hence

$$(T-t_n)^{1/(p-1)} \int_0^{t_n} \int_{(\mathbb{R}^N\setminus\Omega)\cap B(x_0,2d)} \bar{u}^p(y,s)\,dy\,ds \to \infty.$$

Therefore there exists a point $x_1 \in \partial\Omega$ such that for a subsequence that we still call $t_n$,

$$(T-t_n)^{1/(p-1)} \int_0^{t_n} \int_{(\mathbb{R}^N\setminus\Omega)\cap B(x_1,d/4)} \bar{u}^p(y,s)\,dy\,ds \to \infty.$$

Since every function involved is nonnegative and $J(z) \geqslant c > 0$ for $|z| < d/2$ we get

$$(T - t_n)^{1/(p-1)} \int_0^{t_n} \int_{\mathbb{R}^N \setminus \Omega} J(\hat{x} - y)\bar{u}^p(y,s)\,dy\,ds \to \infty,$$

for every $\hat{x} \in \partial\Omega \cap \{|\hat{x} - x_1| < d/4\}$.

Using (2.3), we get

$$(T - t_n)^{1/(p-1)} u(\hat{x}, t_n) \geqslant c(T - t_n)^{1/(p-1)} \int_0^{t_n} \int_{\mathbb{R}^N \setminus \Omega} J(\hat{x} - y)\bar{u}^p(y,s)\,dy\,ds \to \infty.$$

Therefore,

$$(T - t_n)^{1/(p-1)} m(t_n) = (T - t_n)^{1/(p-1)} \int_{\partial\Omega} u(x, t_n)\,dS_x \to \infty,$$

which contradicts (5.4). The claim is proved.

Using again that $J(z) \geqslant c > 0$ for $z < d/2$ we get that (5.5) holds for every $x \in \overline{\Omega}$. In fact, first we see that for every $x \in \partial\Omega$

$$(T - t)^{1/(p-1)} \int_0^{t} \int_{\partial\Omega \cap B_{d/4}(x)} u^p(\sigma, s)\,dS_\sigma\,ds \leqslant C.$$

Then, since $\partial\Omega$ is compact we deduce that

$$(T - t)^{1/(p-1)} \int_0^{t} \int_{\partial\Omega} u^p(\sigma, s)\,dS_\sigma\,ds \leqslant C.$$

This immediately implies, by using that $J \in L^\infty$, that (5.5) holds for every $x \in \overline{\Omega}$.

Now, let for $t_0 < T$,

$$M = \max_{\overline{\Omega} \times [0, t_0]} (T - t)^{1/(p-1)} u(x, t) = (T - t_1)^{1/(p-1)} u(x_1, t_1).$$

This implies by using again (2.3) that

$$M \leqslant C + \int_0^{t_1} e^{-A(x_1)(t_1 - s)} \int_{\Omega} J(x_1 - y) M\,dy\,ds \leqslant C + \left(1 - e^{-A(x_1)t_1}\right) M.$$

So that, since $A(x) \geqslant \alpha > 0$,

$$M \leqslant C,$$

with $C$ independent of $t_0$. The result follows. □

**Corollary 5.1.** *Let u be a solution of* (5.1) *that blows up at time $T$. Then, the blow-up set, $B(u)$, verifies*

$$B(u) \subset \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leqslant Kd\} \tag{5.6}$$

*where* $K = [p/(p-1)]$.

**Proof.** The proof follows from the results in Section 4. □

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