Functions operating on multivariate distribution and survival functions—With applications to classical mean-values and to copulas

Paul Ressel
Katholische Universität Eichstät-Ingolstadt, Ostenstraße 28, 85072 Eichstätt, Germany

Abstract

Functions operating on multivariate distribution and survival functions are characterized, based on a theorem of Morillas, for which a new proof is presented. These results are applied to determine those classical mean values on $[0, 1]^n$ which are distribution functions of probability measures on $[0, 1]^n$. As it turns out, the arithmetic mean plays a universal role for the characterization of distribution as well as survival functions. Another consequence is a far reaching generalization of Kimberling's theorem, tightly connected to Archimedean copulas.

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0. Introduction

In [7], the notion of a multivariate distribution function (“d.f.”) was extended to include also non-grounded functions, and one such example is the arithmetic mean $M_t(x_1, \ldots, x_n) := \frac{1}{n} \sum_{i \leq n} x_i$ on $[0, 1]^n$. Since also the geometric mean $M_0(x) := \left( \prod_{i \leq n} x_i \right)^{1/n}$ is obviously a d.f. on $[0, 1]^n$, in this case grounded, it seemed to be an interesting question to consider the full scale of classical mean-values, given for $t \in \mathbb{R} \setminus \{0\}$ by

$$M_t(x) := \left( \frac{1}{n} \sum_{i = 1}^{n} x_i^t \right)^{1/t},$$

complemented by $M_0, M_{-\infty}(x) := \min_{i \leq n} x_i$ and $M_{\infty}(x) := \max_{i \leq n} x_i$, and to determine which of those are d.f.s on $[0, 1]^n$, see Theorem 6.

In order to give a complete answer we need a characterization of the functions $f$ on $[0,1]$ operating on multivariate d.f.s., i.e. for which $f \circ F$ is a d.f. whenever $F$ is. The key to the solution of this problem was offered by Morillas [5], with an elementary but very complicated proof. We will present a new one, shorter and easier to read, cf. Theorem 1. We shall also characterize those mean values which are max-infinitely divisible, i.e. every positive power of which is likewise a d.f. Similar questions for multivariate survival functions will also be answered.

It turns out that the crucial feature of the functions $f$ involved is their $n$-(absolute) monotonicity, a property scrutinized already in the 1940s. The new characterization expressed in Theorem 3 might be of independent interest.

E-mail address: paul.ressel@ku-eichstaett.de.

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The intimate connection between multivariate d.f.s and their copulas leads in a natural way to some applications to this special class of d.f.s; see Theorem 5 and Corollary 3. Kimberling’s results from 1974, often cited particularly in connection with Archimedean copulas, are given their probably most general form in Theorem 4.

1. Distribution functions which are not necessarily grounded

Let $A_1,\ldots,A_n \subseteq \mathbb{R} = [-\infty,\infty]$ be non-empty subsets, $A := \prod_{i=1}^n A_i$, and $\phi: A \rightarrow \mathbb{R}$ be some function. For $a, b \in A$ we put

$$D^b_\phi := \phi(b) - \phi(a_1, b_2, \ldots, b_n) - \cdots - \phi(b_1, \ldots, b_{n-1}, a_n)$$

$$= \phi(a_1, a_2, b_3, \ldots, b_n) + \cdots + \phi(b_1, \ldots, b_{n-2}, a_{n-1}, a_n)$$

$$- \phi(a_1, a_2, a_3, b_4, \ldots, b_n) - \cdots - \cdots + (-1)^n \phi(a).$$

**Definition.** $\phi$ is $n$-increasing iff $D^b_\phi \geq 0 \forall a < b \in A$; $\phi$ is called fully $n$-increasing iff $\phi$ with $k$ of the variables fixed is $(n-k)$-increasing in the remaining variables, for every choice of these variables, and for every $k = 0, 1, \ldots, n-1$.

Here $a < b$ means $a_i < b_i$ for all $i$, however the condition $D^b_\phi \geq 0 \forall a \leq b \in A$ would be equivalent, since $D^b_\phi = 0$ if $a_i = b_i$ for some $i$. Therefore, if $\phi$ depends on fewer than $n$ variables, we have always $D^b_\phi = 0$. An important special case will be in the following that $\phi$ is of the form

$$\phi(x) = \sum_{i=1}^n f_i(x_i)$$

for univariate functions $f_1,\ldots,f_n$. In this case, $\phi$ is fully $n$-increasing if and only if every $f_i$ is increasing in the usual sense. One such example is the arithmetic mean $\phi(x) = \frac{1}{n} \sum_{i=1}^n x_i = M_1(x)$ on $[0,1]^n$.

If $v_i := \inf A_i \in A_i$ for all $i \leq n$, then $\phi$ is grounded iff $\phi(a) = 0$ for all $a \in A$ with $a_i = v_i$ for some $i$. A grounded function is fully $n$-increasing iff it is (simply) $n$-increasing, cf. Theorem 2 in [7]. This result appears also in [5], Lemma 2.2. The just mentioned arithmetic mean is an example of a non-grounded $\phi$.

In [7] (Theorem 1), it was shown that a function $\phi: A \rightarrow \mathbb{R}$ is fully $n$-increasing if and only if $\phi$ is completely monotone on the semigroup $(A, \wedge)$, and this was used to prove the following.

**Correspondence Theorem.** Suppose $A_1,\ldots,A_n \subseteq \mathbb{R}$ is non-empty such that $\sup A_i \in A_i$ for $i = 1,\ldots,n$, and put $A := \prod_{i=1}^n A_i$. Then for a function $\phi: A \rightarrow \mathbb{R}_+$ the following two properties are equivalent.

(i) $\phi$ is fully $n$-increasing and right continuous
(ii) There is a (unique) finite measure $\mu$ on $\overline{A}$ (closure in $\mathbb{R}^n$) such that

$$\phi(a) = \mu(\overline{A} \cap [-\infty, a]), \quad a \in A.$$

(cf. [7], Theorem 7).

For $\phi = M_1$ on $A = [0,1]^n$, the corresponding measure is $\mu = \frac{1}{n} \sum_{i \leq n} \mathbb{1}_{[0,e_i]}$, where $e_1,\ldots,e_n$ are the usual unit vectors and $\mathbb{1}$ denotes the one-dimensional Lebesgue measure.

2. Fully differentiable multivariate distribution functions

The subsets $A_i$ considered in 1. will be intervals $A_i = [v_i, u_i] \subseteq \mathbb{R}$ in the following.

Let $v, u \in \mathbb{R}^n$, $v < u$, and define for $\alpha \subseteq \mathbb{N} := \{1,\ldots,n\}$

$$R_\alpha := \{a \in [v, u] | a_i > v_i \iff i \in \alpha\}.$$

Then $R_\alpha := [v, u]$, and the (disjoint) union

$$\bigcup_{\alpha \subseteq \mathbb{N}} R_\alpha = \{a \in [v, u] | a_i = v_i \text{ for some } i \in \mathbb{N}\}$$

will be called the lower-left boundary of $[v, u]$; note that $R_\emptyset = \{v\}$.

Let $F: [v, u] \rightarrow \mathbb{R}$ be any function; then for $\alpha \neq \emptyset$ the restriction of $F$ to $R_\alpha$ is a function of $x_\alpha := (x_i)_{i \in \alpha}$ where $x_\alpha \in \prod_{i \in \alpha} [v_i, u_i]$; $F(\cdot)|_{R_\alpha}$. We will call $F$ fully differentiable if $F$ is continuous on $[v, u]$ and $F^{(\alpha)} := F \big|_{R_\alpha}$ is $|\alpha|$ times continuously differentiable on $[v_\alpha, u_\alpha]$, for any $\emptyset \neq \alpha \subseteq \mathbb{N}$.

By $F_\alpha$, we will denote the partial derivative of $F$ with respect to $x_i$, $i \in \alpha$, i.e. for $\alpha = \{i_1,\ldots,i_k\}$

$$F_\alpha := \frac{\partial^k F}{\partial x_{i_1} \ldots \partial x_{i_k}}.$$
If \( F \) is fully differentiable then for \( v < a < b < u \)

\[
D^n_bF = \int_{[a,b]} F_\alpha d\mathbb{R}^n
\]

by Fubini's theorem, and a similar result holds of course for every restriction \( F^a \) to the part \( R_a \) of the lower-left boundary of \([v, u]\), for \( \emptyset \neq \alpha \subseteq n \). Hence we can state the following lemma.

**Lemma 1.** Let \( F; [v, u] \rightarrow [0, 1] \) be fully differentiable, \( F(u) = 1 \). Then \( F \) is a multivariate distribution function if and only if

\[
(F^a)_\alpha \geq 0 \quad \text{on} \quad [v, u], \quad u_\alpha \forall \emptyset \neq \alpha \subseteq n.
\]

Note that in this case \( \mu | [v, u] \) is absolutely continuous in the usual sense, \( \mu \) denoting the probability measure whose d.f. is \( F \), and \( \mu([v, u]) = 0 \) since \( F \) is continuous. Similarly \( \mu | R_a \) is absolutely continuous with respect to \( \mathbb{R}^n, \emptyset \neq \alpha \subseteq n \), and

\[
\mu | [v, u], \quad u_\alpha = (F^a)_\alpha \circ \mathbb{R}^n.
\]

Note that \( \mu([v]) = F(v) \) may well be positive, and that for grounded \( F \) the conditions of Lemma 1 reduce to \( F_\alpha \geq 0 \) on \([v, u]\).

**Example.** Consider \( F(x, y) := \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2 \), \((x, y) \in [0, 1]^2 = [v, u]\). Here

\[
R_{[1]} = [0, 1] \times \{0\}, \quad R_{[2]} = \{0\} \times [0, 1], \quad R_{[1, 2]} = [0, 1]^2
\]

\[
(F^{[1]})_x = \frac{x}{4} = (F^{[2]})_y.
\]

\[
F_{[1]}(x, y) = \frac{\sqrt{x} + \sqrt{y}}{4\sqrt{x}}, \quad F_{[1,2]}(x, y) = \frac{1}{8} \cdot \frac{1}{\sqrt{xy}}.
\]

We see that \( F \) is a fully differentiable d.f., say of the probability measure \( \mu \) on \([0, 1]^2 \), where \( \mu | [0, 1]^2 \) has the density \( F_{[1,2]} \), and \( \mu | [0, 1] \times \{0\} \) as well as \( \mu | \{0\} \times [0, 1] \) has the constant density \( \frac{1}{4} \), the total masses being

\[
\mu([0, 1]^2) = \frac{1}{2}, \quad \mu([0, 1] \times \{0\}) = \mu(\{0\} \times [0, 1]) = \frac{1}{4}, \quad \mu(\{0, 0\}) = 0.
\]

Of special interest will be in the following the question for which functions \( f; [0, 1] \rightarrow \mathbb{R} \) the composition \( f \circ F \) with an \( n \)-variable d.f. \( F \) is again a d.f. Suppose \( f \) is also \( n \) times continuously differentiable, then the chain rule gives

\[
(F \circ F)_\alpha = \sum_{k=1}^n (f^{\#}_k) \cdot \sum_{\pi \in P_k} \prod_{\alpha \in \pi} F_\alpha
\]

where \( P_k \) is the family of all partitions \( \pi \) of \( n \) consisting of \( k \) subsets. This formula is well-known, and easily shown by induction; since a corresponding formula holds obviously also for subsets of \( [n] \), we can combine it with Lemma 1 to obtain the following lemma.

**Lemma 2.** Let \( F; [v, u] \rightarrow [0, 1] \) be a fully differentiable distribution function, and let \( f; [0, 1] \rightarrow [0, 1] \) be \( n \) times differentiable with \( f^{\#}_k \geq 0 \) for \( k = 1, \ldots, n \) and \( f(1) = 1 \). Then \( f \circ F \) is again a distribution function.

Note that \( f(0) = 0 \) is not required; if \( v \) is the measure corresponding to \( f \circ F \), then \( v([v]) = f(F(v)) = f(0) \) in case \( F(v) = \mu([v]) = 0 \).

### 3. Functions operating on multivariate d.f.s and a new proof of Morillas' theorem

In Lemma 1, we saw a sufficient condition for a function \( f \) on \([0,1] \) to operate on multivariate d.f.s, applicable however only to sufficiently smooth functions. A natural question arises to find necessary and sufficient conditions for \( f \) such that \( f \circ F \) is again a d.f. for each multivariate d.f. \( F \) on \( \mathbb{R}^n \). Obviously, the crucial point will be the property “fully \( n \)-increasing”. This problem has in fact been solved by Morillas [5], in an admirable paper which surprisingly has not really been taken up as it seems. One reason might be that Morillas' proof is (in her own words) of elementary algebraic nature but very involved. We intend to give another proof, perhaps not elementary, but (hopefully) easier to “digest”.

Let \( c, d \in \mathbb{R}, c < d, n \in \mathbb{N}, \) and let \( f; [c, d] \rightarrow \mathbb{R} \) be any function. For \( t \in [c, d]\) and \( h > 0 \) such that \( t + nh \leq d \) we define

\[
\Delta_h f(t) := f(t + h) - f(t), \quad \Delta_h^2 := \Delta_h \circ \Delta_h, \quad \text{etc.}
\]
up to
\[ \Delta^n f(t) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(t + jh). \]
complemented by \( \Delta^n f(t) := f(t) \) (independent of \( h \)).

If \( \Delta^n f(t) \geq 0 \) for \( k = 0, 1, \ldots, n \), then we call \( f \) \( n \)-absolutely monotone. So, \( f \) is 2-absolutely monotone iff \( f \) is non-negative, increasing and convex; \( f \) then automatically continuous on \([c, d]\), and in fact also at the left boundary point \( c \), but not necessarily at \( d \), since plainly one might add whatever positive number to \( f(d) \) without violating the defining inequalities. However, the limit \( f(d−) = \lim_{t \to d−} f(t) \) exists, and \( f(d−) \leq f(d) \).

If \( f \) is \( n \)-absolutely monotone and \( n > 2 \), then a theorem of Boas and Widder [1] shows the existence, continuity and non-negativity (in the open interval \([c, d]\)) of \( f’, f’’, \ldots, f^{(n-2)} \) and the monotonicity as well as convexity of \( f^{(n-2)} \). In particular, \( f’, f’’, \ldots, f^{(n-2)} \) are all non-negative, increasing and convex on \([c, d]\), and therefore continuously extendible to \( c \), though not necessarily to \( d \). We shall however not make use of this result. In the other direction, if \( f \geq 0 \) is \( n \) times differentiable with non-negative derivatives \( f', f'', \ldots, f^{(n)} \), then clearly \( f \) is \( n \)-absolutely monotone.

Later on we will need a closely related notion: with \( \nabla_h := -\Delta_h \) we call \( g: [c, d] \to \mathbb{R} \) \( n \)-monotone if \( \nabla^n g(t) \geq 0 \) for \( k = 0, 1, \ldots, n \) and all \( t \in [c, d] \) such that \( t + kh \leq d \). This is equivalent with \( g(−s) \) being \( n \)-absolutely monotone on \( [−d, −c] \), so that the already mentioned theorem of Boas and Widder [1] implies that \( g [c, d] \) is \( n \) times differentiable and \( g^{(n-2)}  | \mathbb{R} \times [c, d] \) is non-negative, decreasing and convex. (This was actually used as a definition by Williamson [9], who astonishingly did not mention the result of Boas and Widder.) In particular, \( g \) is automatically continuous at \( d \), but not necessarily at \( c \).

A function \( f \) defined on any interval \( I \subseteq \mathbb{R} \) is called \( n \)-absolutely monotone if every restriction of \( f \) to a compact subinterval of \( I \) has this property.

A function which is \( n \)-monotone for every \( n \in \mathbb{N} \), is called completely monotone. This notion will appear below in connection with Kimberling’s theorem.

In the proof of Theorem 1, we will need the following lemma.

**Lemma 3.** Let \( F: \mathbb{R}^n \to [0, 1] \) be the d.f. of a probability measure \( \mu \) with finite support. Then there are \( C^\infty \)-d.f.’s \( F_m \) on \( \mathbb{R}^n \) converging pointwise to \( F \).

**Proof.** Let \( \{y_1, \ldots, y_k\} := \text{supp}(\mu) \); then \( \alpha_j := \mu(\{y_j\}) > 0 \) for all \( j \leq k \). Choose \( C^\infty \)-functions \( g_{j,m} \geq 0 \) on \( \mathbb{R}^n \) with
\[ \int g_{j,m} \, d\mathbb{L}^n = \alpha_j \quad \text{and} \quad \text{supp} (g_{j,m}) \subseteq \left[ y_j - \frac{1}{m} , y_j + \frac{1}{m} \right], \]
where \( \mathbb{L} \) \( \in \mathbb{R}^n \), and let \( F_m \) be the d.f. of
\[ \left( \sum_{j=1}^{k} g_{j,m} \right) \circ \mathbb{L}^n, \quad m \in \mathbb{N}. \]
Then for each \( x \in \mathbb{R}^n \) there exists \( m_x \in \mathbb{N} \) such that \( F_{m_x}(x) = F(x) \forall m \geq m_x. \)

In the following theorem, the equivalence of (i) and (ii) has been shown by Morillas [5, Theorem 2.3]. Condition (iii), added by us, will turn out in the following as quite important, since it is easily applicable. It can also be easily derived from Lemma 2.3 in [5].

**Theorem 1.** Let \( n \geq 2, c < d \) in \( \mathbb{R} \), and let \( f: [c, d] \to \mathbb{R} \) be an arbitrary function. Then the following three properties are equivalent.

(i) \( f \) is \( n \)-absolutely monotone.
(ii) If \( \varphi: \{0, 1\}^n \to [c, d] \) is fully \( n \)-increasing, then so is \( f \circ \varphi \).
(iii) For any \( k \in \mathbb{N}, t \in [c, d], h > 0 \) with \( t + kh \leq d \) we have
\[ D_k^n f \left( t + h \sum_{i=1}^{k} x_i \right) \geq 0 \]
where \( 1_k = (1, \ldots, 1) \) and \( 0_k = (0, \ldots, 0) \) are both of length \( k \).

**Proof.** (i) \( \iff \) (ii): suppose \( f \) to be \( n \)-absolutely monotone, then \( f \) is increasing and convex, implying \( f \) to be continuous on \([c, d]\), let \( f \) be the continuous extension of \( f \mid [c, d] \) to \([c, d] \), then \( f(d) \leq f(d) \) since \( f \) is increasing. We shall proceed by approximation with \( C^\infty \)-functions. Assuming without restriction that \([c, d] = [0, 1] \), let \( \varphi: \{0, 1\}^n \to [0, 1] \) be fully \( n \)-increasing, and, excluding \( \varphi \equiv 0 \), let \( \varphi(1, \ldots, 1) = 1 \). By Theorem 7 in [7], \( \varphi \) is the d.f. of a probability measure on \([0, 1]^n \), which, considered as a probability measure on \( \mathbb{R}^n \), has a d.f. \( F \) extending \( \varphi \) from \([0, 1]^n \) to \( \mathbb{R}^n \), and so there is by Lemma 3...
a sequence of $C^\infty$-d.f.'s $\{F_m\}$ converging pointwise to $F$. Let $f_1, f_2, \ldots$ be the sequence of Bernstein polynomials for $\overline{f}$, they converge (even uniformly) to $\overline{f}$ on $[0, 1]$. These polynomials are defined by

$$f_k(s) = \sum_{i=0}^{k} \binom{k}{i} \frac{1}{k^i} (1 - s)^{k-i}$$

(note that $\overline{f} \left( \frac{1}{k} \right) = f \left( \frac{1}{k} \right) \forall i < k$), and their derivatives can be expressed as

$$f_k^{(i)}(s) = k(k-1) \cdots (k-(j-1)) \sum_{j=i}^{k} \binom{k-j}{i} \frac{1}{k^j} (1 - s)^{k-j}$$

(cf. [3], page 12), showing $f_1, f_2, \ldots$ to be likewise $n$-absolutely monotone on $[0,1]$).

By Lemma 2, $f_k \circ M_n$ is a d.f. for each $k, m \in \mathbb{N}$, in particular fully $n$-increasing. Letting first $m$ tend to infinity and then $k$, we get the conclusion (for $f$, since $f(1) \geq \overline{f}(1)$).

(ii) $\implies$ (iii). $\varphi : [0, 1]^n \to [0, 1]$, defined by $\varphi(x) := t + h \sum_{i=1}^{k} x_i$ is obviously fully $n$-increasing.

(iii) $\implies$ (i). This follows immediately from

$$D_M f \left( t + h \sum_{i=1}^{k} x_i \right) = \Delta h f(t)$$

for $k$, $h$ as indicated in (iii). $\square$

As a consequence, we can state the following result, in which the arithmetic mean $M_1$ plays a surprising special rôle:

**Theorem 2.** Let $f : [0, 1] \to [0, 1]$ be continuous at $1$ and fulfill $f(0) = 0, f(1) = 1$, and let $n \geq 2$. Then the following three properties are equivalent:

(i) $f \circ M$ is a d.f. for every $n$-dimensional distribution function $F$

(ii) $f \circ M_1$ is a d.f. on $[0, 1]^n$

(iii) $f$ is $n$-absolutely monotone.

If $f$ is furthermore continuous on $[0, 1]$, $n$ times differentiable on $[0, 1]$, and $f^{(k)}(t) \geq 0 \forall \ t \in ]0, 1[ \text{ and } \forall \ k = 1, \ldots, n$, then $f$ has the properties (i)–(iii).

**Proof.** (ii) $\implies$ (iii): We shall show condition (iii) of Theorem 1. Let $k \in \mathbb{N}, \ t \in [0, 1], \ h > 0$ such that $t + kh \leq 1$. Let $0 < t' \leq t + k(n-1)\frac{h}{n}$ and choose $y \in [0, 1 - h], \ z \in [0, 1]$ such that

$$\frac{k}{n}y + \frac{n-k}{n}z = t',$$

which is possible because $t' \leq 1 - \frac{kh}{n}$. Then

$$H(x_1, \ldots, x_k) := (y + hx_1, \ldots, y + hx_k, z, \ldots, z)$$

maps $[0, 1]^k$ into $[0, 1]^n$, and is of course a homomorphism w.r. to "$\wedge$", so that $f \circ M_1 \circ H$ is (fully) $k$-increasing. Since

$$M_1(H(x)) = t' + \frac{h}{n} \cdot \sum_{i=1}^{k} x_i$$

we get

$$\left( \Delta h \right)^k f(t') = D_M f(\circ M_1 \circ H) \geq 0.$$
where the summation is over all \((m_1, \ldots, m_n) \in \mathbb{N}^n_0\) with \(m_1 + \cdots + m_n = k\), for which \(m_2 + 2m_3 + \cdots + (n - 1)m_n \leq (n - 1)k\).

This shows that \(\Delta^k_0 f(t)\) is a sum of terms of the form

\[
\left( \frac{\Delta}{h} \right)^k f \left( t + i \cdot \frac{h}{n} \right)
\]

where \(i \in \{0, 1, \ldots, k(n-1)\}\), and therefore non-negative.

(iii) \(\implies\) (i). As noted before, \(f\) is continuous on \([0, 1]\), and in 1 by assumption, ensuring right continuity of \(f \circ F\) for every d.f. \(F\) on \(\mathbb{R}^n\). Theorem 1 now gives the conclusion.

If \(f\) is \(n\)-times differentiable in \([0, 1]\) with non-negative derivatives, let \(k \leq n\), \(h > 0\), and \(t \in [0, 1]\) be given such that \(t + kh \leq 1\). Then a repeated application of the mean value theorem yields some \(\theta \in [0, 1]\) such that

\[
\Delta^k_0 f(t) = f^{(k)}(\theta) \cdot h^k \geq 0. \quad \square
\]

**Remark 1.** The condition \(f(0) = 0\) is not essential; it only ensures that \(f \circ F\) is again a “proper” d.f. (i.e., of a measure living on \(\mathbb{R}^n\)) if \(F\) is. Allowing \(f(0) > 0\) enlarges the frame to include also probability measures charging the lower left boundary of \(\mathbb{R}^n\).

**Remark 2.** The preceding proof shows that for a function \(f\) on a real interval to be \(n\)-absolutely monotone, it suffices to restrict the \(h\)-values involved to a given interval \([0, h_0]\), for some \(h_0 > 0\). This will be used in the proof of the following result which is interesting on its own.

**Theorem 3.** Let \(u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n\), and let \(f: [0, u_0] \to \mathbb{R}_+\) be given. Then \(f(\sum_{i=1}^n x_i)\) is fully \(n\)-increasing on \([0, u]\) if and only if \(f\) is \(n\)-absolutely monotone.

**Proof.** Let \(\sigma(x) := \sum_{i=1}^n x_i\) denote the sum function on \(\mathbb{R}^n\); since \(\sigma\) is fully \(n\)-increasing, so is therefore \(f \circ \sigma\) on \([0, u]\) by Theorem 1, if \(f\) is \(n\)-absolutely monotone.

Now suppose \(f \circ \sigma\) to be fully \(n\)-increasing on \([0, u]\), and put \(h_0 := \min_{1 \leq i \leq n} u_i\). Let \(t \in [0, u_0]\) and \(h \in [0, h_0]\) be given such that \(t + nh \leq u_0\). Choose \(b_i \in [h, u_i]\) with \(t + nh = \sum_{i=1}^n b_i\), and put \(a_i := b_i - h\), \(i = 1, \ldots, n\). Then

\[
0 \leq D^k_0 f(t) \leq \Delta^k_0 f(t).
\]

For \(k \in \{1, \ldots, n-1\}\), \(t \in [0, u_0], h \in [0, h_0]\) such that \(t + kh \leq u_0\) choose \(b_i \in [h, u_i]\) for \(1 \leq i \leq k\), and \(b_i \in [0, u_i]\) for \(k < i \leq n\), with \(\sum_{i=1}^n b_i = t + kh\). Put \(a_i := b_i - h\) for \(i = 1, \ldots, k\), \(a' := (a_1, \ldots, a_k)\), \(b' := (b_1, \ldots, b_k)\). Then we have likewise

\[
0 \leq D^k_0 \left( f \left( \sum_{i=1}^n x_i + b_{k+1} + \cdots + b_n \right) \right) = \Delta^k_0 f(t).
\]

Hence \(f\) is indeed \(n\)-absolutely monotone. \(\square\)

**Corollary 1.** Let \(l_1, \ldots, l_n \subseteq \mathbb{R}_+\) be non-degenerate intervals containing \(0\), put \(l_0 := l_1 + \cdots + l_n\), and let \(f: l_0 \to \mathbb{R}\) be any given function. Then \(f(\sum_{i=1}^n x_i)\) is fully \(n\)-increasing on \(\prod_{i=1}^n l_i\) iff \(f\) is \(n\)-absolutely monotone.

**Corollary 2.** For \(n \geq 2\) there is no d.f. on \(\mathbb{R}_+^n\) of the form \(f(\sum_{i=1}^n x_i)\), for some \(f: \mathbb{R}_+ \to \mathbb{R}\).

For, \(f\) would then be increasing, convex and bounded, hence constant.

4. Kimberling-type results and applications to copulas

In [2] Kimberling considered multivariate d.f.s of the form

\[
F(x_1, \ldots, x_n) = f \left( \sum_{i=1}^n f^{-1}(F_i(x_i)) \right)
\]

where \(F_1, \ldots, F_n\) are the marginals of \(F\). He showed that if \(f: \mathbb{R}_+ \to [0, 1]\) is continuous, completely monotone, \(f(0) = 1\) and \(\lim_{s \to \infty} f(s) = 0\), then \(F\) as above is always a d.f. Conversely, if \(F_1, F_2, \ldots\) is a sequence of continuous one-dimensional d.f.s, and \(f: \mathbb{R}_+ \to [0, 1]\) is strictly decreasing, \(f(0) = 1\), such that \(F\) as above is a d.f. for all \(n\), then \(f\) is necessarily completely monotone.

In [6] we showed already more general versions of these results, looking at d.f.s of the form

\[
F(x_1, \ldots, x_n) = f \left( \sum_{i=1}^n g_i(F_i(x_i)) \right)
\]
for the so-called log-operating functions $g$, though still having complete monotonicity of $f$ in view. ($g : [0, 1] \rightarrow [0, \infty]$ is log-operating iff $\exp(-g \circ F)$ is a d.f. for every one-dimensional d.f. $F$.) Based on the preceding results we can now prove far reaching generalizations: replacing $g_i \circ F_i$ by nearly arbitrary decreasing functions $h_i$, and requiring only $n$-monotonicity of $f$.

**Theorem 4.** Let $h_i : \mathbb{R}_+ \rightarrow [0, \infty]$ be right continuous and decreasing, $h_i(0) > 0$, $\lim_{t \to \infty} h_i(t) = 0$, $i = 1, \ldots, n (n \geq 2)$. Put $u_0 := \sum_{i \leq n} h_i(0)$, let $f : [0, u_0] \rightarrow \mathbb{R}_+$ fulfill $f(0) = \lim_{s \to 0} f(s) = 1$, and consider

$$F(x) := f \left( \sum_{i = 1}^{n} h_i(x_i) \right), \quad x \in \mathbb{R}_+^n.$$ 

(i) If $f$ is $n$-monotone on $[0, u_0] \cap \mathbb{R}_+$, then $F$ is a d.f. on $\mathbb{R}_+^n$.

(ii) If each $h_i$ is continuous, strictly decreasing on $[h_i(0), 0]$, and if $F$ is a d.f., then $f$ is $n$-monotone on $[0, u_0] \cap \mathbb{R}_+$.

**Proof.** (i) Let us first assume $u_0 < \infty$. Then $s \mapsto f(u_0 - s)$ is $n$-absolutely monotone on $[0, u_0]$, $x \mapsto u_0 - \sum_{i \leq n} h_i(x_i)$ is fully $n$-increasing, and so is their composition $F$ by Theorem 1. The case $u_0 = \infty$ is dealt with by considering first $c \wedge h_i$ for any $c > 0$, and then letting $c$ tend to $\infty$.

(ii) $h_i^{-1}$ is well-defined on $[0, h_i(0)]$. We complement it by $h_i^{-1}(0) := \sup\{h_i > 0\}$. Let now $0 < c_i < h_i(0)$ be given, $c := \sum_{i = 1}^{n} c_i, c_n(s) := f(c - s)$ for $s \in [0, c]$, $\tilde{h}_i(s) := c_i - h_i(s)$ for $s \in [h_i^{-1}(c_i), h_i^{-1}(0)]$, an interval mapped by $\tilde{h}_i$ onto $[0, c_i]$. Then $\tilde{H} := (\tilde{h}_1, \ldots, \tilde{h}_n)$ is a $\wedge$-isomorphism, and, with $\sigma$ denoting again the sum function,

$$g_n \circ \sigma \circ \tilde{H}(x_1, \ldots, x_n) = f\left( \sum_{i = 1}^{n} h_i(x_i) \right)$$

is fully $n$-increasing by assumption, hence so is $g_n \circ \sigma \circ (\sigma^{-1})$. (on $\prod_{i = \infty}^{0}$, $[0, c_i]$) and so $g_n$ is $n$-absolutely monotone by Theorem 3, i.e. $f$ is $n$-monotone on $[0, c]$. Since $c$ was any number in $[0, u_0], f$ is indeed $n$-monotone on $[0, u_0]$, and on $[0, u_0]$ for $u_0 < \infty$, since in that case we may choose $c_i = h_i(0)$ for all $i$. □

This result can be applied to Archimedean copulas, i.e. to copulas of the form

$$A_f(x) = f\left( \sum_{i = 1}^{n} f^{-1}(x_i) \right), \quad x \in [0, 1]^n$$

where $f : [0, \infty] \rightarrow [0, 1]$ is a decreasing bijection, called the generator of the copula (sometimes $f^{-1}$ is called such).

**Corollary 3.** $A_f$ is a copula iff $f$ is $n$-monotone.

This was proved in [5] (Theorem 3.5) and again in [4] (Theorem 2).

Let now $f : [0, 1] \rightarrow [0, 1]$ be any increasing surjection, then $f(0) = 0, f(1) = 1$, is continuous, and $s_0 := \sup\{f = 0\} < 1$. We define its pseudo-inverse by $f^{-1}(0) := 1$ and

$$f^{-1}(s) := \inf\{f > s\}, \quad 0 \leq s < 1;$$

$f^{-1}$ is right continuous and strictly increasing from $f^{-1}(0) = s_0$ to $1$, and $f \circ f^{-1} = \text{id}$. For any $n$-variate copula $C$ we have

$$f \circ C(s_1, \ldots, s_n) = f \circ C(s_1 \lor s_0, \ldots, s_n \lor s_0)$$

(\ast) for all $(s_1, \ldots, s_n) \in [0, 1]^n$.

**Theorem 5.** Let $f : [0, 1] \rightarrow [0, 1]$ be an $n$-absolutely monotone surjection, and let $C$ be an $n$-variate copula. Then

(i) $C_f(s_1, \ldots, s_n) := f \circ C(f^{-1}(s_1), \ldots, f^{-1}(s_n))$ is again a copula

(ii) if $C$ is a copula for the d.f. $F$, then $C_f$ is a copula for $f \circ F$.

Part (i) was shown already by Morillas in [5], Theorem 3.6.

**Proof.** (i) The composition of $C$ with the $\wedge$-homomorphism $(s_1, \ldots, s_n) \mapsto (f^{-1}(s_1), \ldots, f^{-1}(s_n))$ is fully $n$-increasing, and right continuous, hence a d.f. This holds in fact for any increasing surjection $f$. Our assumption that $f$ be $n$-absolutely monotone allows to apply Theorem 2, and shows $C_f$ to be a d.f. as well, and then in fact a copula.

(ii) The restriction $f \upharpoonright [s_0, 1]$ is strictly increasing, and $f^{-1}(f(s)) = s_0 \lor s$ for all $s \in [0, 1]$. Denoting by $F_1, \ldots, F_n$ the marginals of $F$, the d.f. $f \circ F$ has the marginals $f \circ F_i$, and

$$C_f(f \circ F_1(x_1), \ldots, f \circ F_n(x_n)) = f \circ C(f^{-1}(f(F_1(x_1))), \ldots) = f \circ C(s_0 \lor F_1(x_1), \ldots, s_0 \lor F_n(x_n)) = f \circ C(F_1(x_1), \ldots, F_n(x_n))$$

by (\ast)

$$= f \circ C(x_1, \ldots, x_n)$$

which had to be shown. □
Clearly, the condition on \( f \) to be \( n \)-absolutely monotone is not necessary in general for \( C_f \) to be a copula, as the example of the independence copula \( C(x) = \prod_{i \leq n} x_i \) and \( f(s) := \sqrt{s} \) shows, where \( C_f = C \).

**Remark 3.** It seems appropriate to point out that also non-grounded d.f.s are connected to their marginals by some copula. More precisely, let \( A_i \subseteq \mathbb{R} \) be non-empty such that \( u_i := \sup A_i \in A_i \) for \( i = 1, \ldots, n \), put \( A := \prod_{i \leq n} A_i \), and let \( \varphi : A \rightarrow \mathbb{R}_+ \) be fully \( n \)-increasing. Define the marginals \( \varphi_i \) of \( \varphi \) by

\[
\varphi_i(a_i) := \varphi(u_1, \ldots, u_{i-1}, a_i, u_{i+1}, \ldots, u_n), \quad a_i \in A_i,
\]

and note that for \( a_1 < b_1 \) in \( A_1 \) we have, making use of the fact that \( \varphi \) is \( 2 \)-increasing if the other variables are fixed, for \( x_i \in A_i, i = 2, \ldots, n \)

\[
\begin{align*}
\varphi(b_1, x_2, \ldots, x_n) - \varphi(a_1, x_2, \ldots, x_n) &\leq \varphi(b_1, u_2, x_3, \ldots, x_n) - \varphi(a_1, u_2, x_3, \ldots, x_n) \\
&\leq \cdots \leq \varphi(b_1, u_2, u_3, u_4, \ldots, x_n) - \varphi(a_1, u_2, u_3, u_4, \ldots, x_n) \\
&\leq \varphi_1(b_1) - \varphi_1(a_1),
\end{align*}
\]

and similarly for \( i = 2, \ldots, n \). This implies

\[
|\varphi(b) - \varphi(a)| \leq \sum_{i=1}^n |\varphi_i(b_i) - \varphi_i(a_i)|
\]

for arbitrary \( a, b \in A \), and ensures the existence of a function \( C \) on \( \prod_{i \leq n} \varphi_i(A_i) \) such that

\[
\varphi(x) = C(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \quad \forall x \in A.
\]

Assuming now \( \varphi(u_i) = 1 \) we also have \( \varphi_i(u_i) = 1 \) for \( i = 1, \ldots, n \), and extending \( C \) to \( \prod_{i=1}^n (\{0\} \cup \varphi_i(A_i)) \) by \( C(s) := 0 \) if \( s_i = 0 \) for some \( i \), \( C \) is immediately seen to be a subcopula, which may be further extended to a copula by Sklar’s theorem.

### 5. Mean values as distribution functions

The three classical means (arithmetic, geometric and harmonic) are embedded into a full scale \( (M_t)_{t \in \mathbb{R}} \) of mean values, defined for \( x_1, \ldots, x_n > 0 \) by

\[
M_t(x) := \left( \frac{1}{n} \sum_{i=1}^n x_i^t \right)^{1/t}
\]

and

\[
M_0(x) := \prod_{i=1}^n x_i
\]

where \( \lim_{t \rightarrow 0} M_t(x) = M_0(x) \), complemented by \( \lim_{t \rightarrow -\infty} M_t(x) = \max_{i \leq n} x_i =: M_\infty(x) \), \( \lim_{t \rightarrow +\infty} M_t(x) = \min_{i \leq n} x_i =: M_{-\infty}(x) \), and \( t \mapsto M_t(x) \) is strictly increasing from \( \min x_i \) to \( \max x_i \) for every non-constant sequence \( (x_i) \). Clearly, \( M_t(x) \) is also perfectly well-defined if some or all \( x_i \) are 0, and the restrictions of \( M_t \) to \( [0, 1]^n \) appear as possible candidates for distribution functions.

Some of these mean values have the additional property of being \textit{infinitely divisible}, meaning that every \( k \)-th root is likewise a d.f., \( k \in \mathbb{N} \). The underlying probability distribution (and also the d.f.) is then called \textit{max-infinitely divisible}. In [7] it was shown that a d.f. \( F \) is infinitely divisible iff \( \log F \big| \{F > 0\} \) is fully \( n \)-increasing (we remind this notion does not include non-negativity). If \( \varphi \) is fully \( n \)-increasing, the function \( -\varphi \) is called \textit{fully \( n \)-decreasing}, a property equivalent with being \textit{completely alternating} with respect to the semigroup operation “\(^{\wedge} \)”, and also with being \textit{negative definite}; see [7]. This has the important consequence, that a non-negative fully \( n \)-decreasing function can be composed with Bernstein functions on \( \mathbb{R}_+ \), leading to new fully \( n \)-decreasing functions.

**Theorem 6.** Let \( n \geq 2 \) be given. Then

\begin{itemize}
  \item[(i)] \( M_1 \mid [0, 1]^n \) is a distribution function exactly for \( t \in \{-\infty, \frac{1}{n-1} \} \cup \{ \frac{1}{n-2}, \ldots, \frac{1}{2}, 1 \} \)
  \item[(ii)] \( M_t \) is max-infinitely divisible iff \( t \in [-\infty, 0] \).
\end{itemize}

**Proof.**

\begin{itemize}
  \item[(i)] \( M_1(x) = \frac{1}{n} \sum_{i=1}^n x_i \) is fully \( n \)-increasing, and since products of d.f.’s are again d.f.’s, we see that for arbitrary \( k \in \mathbb{N} \)
  \[ M_{1/k}(x) = \left( \frac{1}{n} \sum_{i=1}^n \sqrt[k]{x_i} \right)^k \]
is likewise a d.f. Let now $0 < t < \frac{1}{n-1}$, put $f_t(u) := u^{1/t}$, and note that the first $n$ derivatives of $f_t$ are positive in $[0, 1]$, with

$$f_t^{(k)}(u) = \prod_{j=0}^{k-1} \left( \frac{1}{t} - j \right) u^{\frac{j}{t} - k}, \quad k = 1, \ldots, n,$$

whence $M_t(x) = f_t \left( \frac{1}{n} \sum_{i=1}^{n} x_i^t \right)$ is a d.f. by Theorem 2. If $t \in [\frac{1}{n-1}, 1] \setminus \left\{ \frac{1}{n-1}, \frac{1}{n-2}, \ldots, \frac{1}{2} \right\}$, then $t^{-1} \in [1, n-1 \setminus N]$ and at least one of the first $n$ derivatives of $f_t$ is negative throughout $[0, 1]$, so that $f_t$ is certainly not $n$-absolutely monotone. By Theorem 1 (iii), there is some $k \in \mathbb{N}$, $r \in [0, 1], h > 0$ with $r + kh \leq 1$ such that

$$D_{\frac{k}{n}}^h \left( f_t \circ \phi \right) < 0$$

where $\phi(x_1, \ldots, x_k) := r + h \sum_{i=1}^{k} x_i$ for $x_i \in [0, 1]$, $i \leq k$. Let us add now the dimension to the symbol $M_t$, i.e.

$$M_t^{[n]}(x_1, \ldots, x_n) := \left( \frac{1}{n} \sum_{i=1}^{n} x_i^t \right)^{1/t}, \quad x_i \in [0, 1].$$

Then, if $M_t^{[n]}$ was a d.f., so would be $M_t^{[k]}$ because of

$$M_t^{[k]}(x_1, \ldots, x_k) = M_t^{[n]}(x_1, \ldots, x_k, 0, \ldots, 0) \cdot \left( \frac{n}{k} \right)^{1/t}.$$

The function $H: [0, 1]^k \rightarrow [0, 1]^k$, $(H(x))_i := (r + khx_i)^{1/t}$ ($i = 1, \ldots, k$) is a homomorphism with respect to "\wedge", hence also $M_t^{[k]} \circ H$ is then fully $k$-increasing, but

$$M_t^{[k]}(H(x)) = \left( r + h \sum_{i=1}^{k} x_i \right)^{1/t} = f_t(\phi(x))$$

for $x \in [0, 1]^k$ leads now to a contradiction.

We conclude that $M_t$ is not a d.f.

The geometric mean $M_0$ is the tensor product of one-dimensional d.f.'s, and so obviously an $n$-dimensional d.f., which furthermore is clearly max-ininitely divisible. For $t < 0$ put $s := -t$, then (for $x_i > 0$)

$$- \log M_t(x) = \frac{1}{s} \log \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{x_i} \right)^s \right],$$

and $\psi(x) := \frac{1}{n} \sum_{i=1}^{n} x_i^{-s}$ is fully $n$-decreasing, a property preserved if applying a Bernstein function such as the log-function, hence $M_t = \exp[(- \log M_1)]$ is max-ininitely divisible.

It remains to show that for no $t > 0$ the d.f. $M_t$ is max-ininitely divisible (among the possible values of $t$): if it was then

$$\sqrt[n]{M_t(x)} = \left( \frac{1}{n} \right)^{1/(kt)} \cdot \left( \sum_{i=1}^{n} x_i^t \right)^{1/(kt)}$$

would be fully $n$-increasing for each $k \in \mathbb{N}$. Setting $x_i = 0$ for $i > 2$ in case $n > 2$, we consider $\psi(x_1, x_2) := M_t(x_1, x_2, 0, \ldots, 0)$, and choose $k$ such that $kt > 1$. Then

$$D_{\left(0, 0\right)}^{\left(1, 1\right)} \left( \sqrt[n]{\psi} \right) = \sqrt[n]{\psi(1, 1)} + \sqrt[n]{\psi(0, 0)} - \sqrt[n]{\psi(0, 1)} - \sqrt[n]{\psi(1, 0)} = \left( \frac{1}{n} \right)^{1/(kt)} \cdot \left( 2^{1/(kt)} - 2 \right) < 0,$$

a contradiction, thus finishing our proof. \(\square\)

With the exception of $M_{-\infty}$, all other $M_t$'s are, if d.f.s, even fully differentiable, and for $t \leq 0$ absolutely continuous in the usual sense. Straightforward calculation leads (for $t \neq 0$) to

$$(M_t)_x = \frac{1}{n^{1/t}}(1 - t)(1 - 2t) \cdots (1 - (k - 1)t) [M_t(x)]^{1/k} \prod_{i=1}^{k} x_i^{t-1}, \quad k \in \mathbb{N}$$

and $\mu_t$ denoting by $\mu_t$ the probability measure on $[0, 1]^n$ with d.f. $M_t$ – we have for $-\infty < t \leq 0$

$$\mu_t = (M_t)_x \circ Z^n, \quad Z^n,$$
where \((M_0)_n(x) = n^{-n} \cdot \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n} - 1}\). For \(t \in \left[0, \frac{1}{n-1}\right] \cup \left\{ \frac{1}{n-2}, \ldots, \frac{1}{2}, 1 \right\}\) the d.f. \(M_t\) is no longer grounded, and correspondingly \(\mu\) no longer absolutely continuous in the usual sense. Keeping the notation introduced in 2, we have
\[
R_\alpha = \left\{ a \in [0, 1]^n \mid a_i > 0 \iff i \in \alpha \right\} \quad \text{for } \alpha \subseteq \mathbb{N}
\]
and
\[
\mu_\alpha \mid R_\alpha = (M_\alpha^n)_\alpha \odot \mathcal{A}^{|\alpha|}, \quad \text{for } \emptyset \neq \alpha \subseteq \mathbb{N},
\]
where for \(\alpha = \emptyset\)
\[
(M_\alpha^n)_\emptyset (x_1, \ldots, x_k) = (M_\emptyset)_k(x_1, \ldots, x_k, 0, \ldots, 0),
\]
and of course \(\mu_\emptyset (R_\emptyset) = \mu_\emptyset (\{0\}) = M_\emptyset (0) = 0\). Note that \(\mu_\emptyset (R_\emptyset) = 0\) for \(t \in \left\{ \frac{1}{n-1}, \frac{1}{n-2}, \ldots, 1 \right\}\), even \(\mu_\emptyset (\{0, 1\}^n) = 0\), since the upper right boundary of \([0, 1]^n\) carries no \(\mu_\emptyset\)-mass, for whatever \(t\).

It may be of interest to know the copulas \(C_t\) associated with those mean values \(M_t\) which are d.f.s. Of course \(C_0\) is the independence copula, and \(M_{-\infty}(x) = \min_{i \leq n} x_i = C_{-\infty}(x)\) is already a copula. For other values of \(t\) an easy calculation yields
\[
C_t(s_1, \ldots, s_n) = \left[ 1 + \sum_{i=1}^{n} (s_i^t - 1) \right]^{1/t}
\]
at least for \(s_i \geq 1 - \frac{1}{t}, i = 1, \ldots, n\). Here, if \(t < 0\), \(-\log C_t\) is fully \(n\)-decreasing (cf. the proof of Theorem 6), ensuring \(C_t\) to be a copula which is max-infinitely divisible as a d.f. For \(0 < t \leq \frac{1}{n-1}\) we use the fact that \(r \mapsto r^{1/t}\) is then \(n\)-absolutely monotone on \(\mathbb{R}\), together with Theorem 1, to see that \(C_t\) is fully \(n\)-increasing, and hence a copula. These copulas are known as Mardia–Takahashi–Clayton copulas. The special case where \(t = \frac{1}{n-1}\) was used already in \([8]\), Appendix 2. For the remaining finitely many values \(t \in \left\{ \frac{1}{n-2}, \frac{1}{n-3}, \ldots, 1 \right\}\) \(C_t\) is only a subcopula on \(\{0\} \cup \left[ 1 - \frac{1}{t}, 1 \right]\) \(n\), which can be extended to a copula by Sklar’s theorem.

### 6. Functions operating on multivariate survival functions

The notion \(n\)-increasing has a perfect counterpart for the max-operation: if we change in Definition 1 the restriction \(a < b\) to \(a > b\), then the function \(\varphi\) considered there is called (fully) \(n\)-\(\max\) increasing. Here “\(n\)-increasing” refers to the direction of the neutral element which in this case is the smallest element. In \([7]\), Theorem 10 the following analogue to the corresponding theorem for distribution functions was proved.

Let \(A_i \subseteq \mathbb{R}\) be non-empty, \(1 \leq i \leq n\), such that \(v_i := \inf A_i \in A_i\) for all \(i\). Then \(\varphi : A \longrightarrow \mathbb{R}_+\) is of the form
\[
\varphi(a) = \mu(\bar{A} \cap [a, \infty)), \quad a \in A
\]
for some probability measure \(\mu\) on \(\bar{A}\) (closure in \(\mathbb{R}^n\)) if and only if \(\varphi\) is fully \(n\)-\(\max\) increasing, left continuous, and \(\varphi(v) = 1\).

\(\varphi\) is called the survival function ("s.f." of \(\mu\).

**Remark 4.** Note that traditionally \(a \mapsto \mu(\bar{A} \cap [a, \infty))\) is called the s.f. of \(\mu\), but this function does “cover” neither the lower left nor the upper right boundary of \(A\); using it the one-to-one relation between \(\mu\) and its s.f. would be lost. In the one-dimensional case one might of course still prefer to use \(\mu([a, \infty))\), in order to have d.f. and s.f. summing up to 1, since especially on \(\mathbb{R}_+\) one mostly considers distributions not charging the origin, and certainly not the point \(\infty\).

It will not be surprising that now the already introduced \(n\)-monotone functions play an important rôle. We have the following analogue of Theorem 1.

**Theorem 7.** Let \(n \geq 2\), and let \(g : [0, 1] \longrightarrow \mathbb{R}\) be any function. Then the following three properties are equivalent.

(i) \(g\) is \(n\)-monotone.
(ii) If \(\varphi : [0, 1]^n \longrightarrow [0, 1]\) is fully \(n\)-\(\max\) increasing, then so is \(g \circ (1 - \varphi)\).
(iii) For any \(k \in \mathbb{N}\), \(t \in [0, 1], h > 0\) with \(t + kh \leq 1\), we have
\[
D^{\varphi}_{\frac{1}{2n}} g \left( t + h \sum_{i=1}^{k} x_i \right) \geq 0.
\]

**Proof.** (i) \(\Longrightarrow\) (ii). For \(x \in [0, 1]^n\) define
\[
H(x) := (1 - x_1, \ldots, 1 - x_n),
\]
then \(H\) is a semigroup isomorphism from \(([0, 1]^n, \land)\) to \(([0, 1]^n, \lor)\), \(\varphi \circ H\) is fully \(n\)-increasing, and \(s \mapsto g(1 - s)\) is \(n\)-absolutely monotone; hence by Theorem 1 \(g \circ (1 - \varphi \circ H)\) is fully \(n\)-increasing, which in turn means that \(g \circ (1 - \varphi)\) is fully \(n\)-\(\max\) increasing.
\[ (\text{ii}) \implies (\text{iii}). \varphi(x_1, \ldots, x_n) := 1 - t - h \sum_{i=1}^{k} x_i \text{ maps } [0, 1]^n \text{ to } [0, 1], \text{ and is fully n-max increasing, so is therefore } g \circ (1 - \varphi), \text{ whence} \]

\[ D_{\mathbb{L}}^0 (g \circ (1 - \varphi)) = D_{\mathbb{L}}^0 \left( g \left( t + h \cdot \sum_{i=1}^{k} x_i \right) \right) \geq 0. \]

(iii) \implies (i). This follows immediately from

\[ D_{\mathbb{L}}^0 g \left( t + h \sum_{i=1}^{k} x_i \right) = \nabla_h g(t). \quad \square \]

If for \( \varphi: [0, 1]^n \rightarrow \mathbb{R} \) we put \( \overline{\varphi}(x) := \varphi(1 - x) \), where \( 1 = (1, \ldots, 1) \), then obviously

\[ D_{\mathbb{L}}^0 \overline{\varphi} = D_{\mathbb{L}}^{1-b} \overline{\varphi} \quad \text{for } a, b \in [0, 1]^n, \]

whence \( \overline{\varphi} \) is (fully) n-max increasing iff \( \varphi \) is (fully) n-increasing. This will be used to prove the following analogue of Theorem 2, where again the arithmetic mean plays a special rôle.

**Theorem 8.** Let \( g: [0, 1] \rightarrow [0, 1] \) be continuous at 0 and fulfill \( g(0) = 1, g(1) = 0 \), and let \( n \geq 2 \). Then the following three properties are equivalent.

(i) \( g \circ (1 - G) \) is a s.f. for every n-dimensional s.f. \( G \).

(ii) \( g \circ M_1 \) is a s.f. on \([0, 1]^n\).

(iii) \( g \) is n-monotone.

If \( g \) is continuous on \([0, 1], n \text{ times differentiable on } ]0, 1[, \) and \((-1)^k g^{(k)}(t) \leq 0 \ \forall t \in ]0, 1[ \) and \( k = 1, \ldots, n \), then \( g \) has the properties (i)–(iii).

**Proof.** (i) \implies (ii). If \( M_1(x) = \frac{1}{n} \sum_{i=1}^{n} (1 - x_i) \) is fully n-max increasing, and therefore a s.f.

(ii) \implies (iii). For \( f(s) := g(1 - s) \), we have \( \overline{f} \circ M_1 = g \circ M_1 \), hence \( f \circ M_1 \) is a d.f. By Theorem 1, \( f \) is n-absolutely monotone, i.e. \( g \) is n-monotone.

(iii) \implies (i). Since \( g \) is continuous on all of \([0, 1], g \circ (1 - G) \) is left continuous and fully n-max-increasing by Theorem 7.

The sufficiency of the conditions on \( g \) at the end follows like in Theorem 2 by applying the mean value theorem, since

\[ \nabla_h f(t) = (-1)^k h^k f^{(k)}(\theta) \]

for some \( \theta \). \quad \square

We saw in Corollary 2 that no d.f. on \( \mathbb{R}^+ \) can have the form \( f(\sum x_i) \), for \( n \geq 2 \). For s.f.’s this is different.

**Theorem 9.** Let \( g: \mathbb{R}^+ \rightarrow [0, 1] \) be such that \( g(0) = 1 = \lim_{t \rightarrow 0} g(s) \), and let \( h_i: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be increasing and left continuous with \( h_i(0) = 0, i = 1, \ldots, n \). Then we have the following.

(i) \( G(x) := g \left( \sum_{i=1}^{n} x_i \right) \) is a s.f. on \( \mathbb{R}^+ \), (of some probability measure on \( \mathbb{R}^+ \)) iff \( g \) is n-monotone.

(ii) If \( g \) is n-monotone then \( g \left( \sum_{i=1}^{n} h_i(x_i) \right) \) is a s.f.

(iii) If \( h_i \) are furthermore bijections, \( i = 1, \ldots, n \), and \( g \left( \sum_{i=1}^{n} h_i(x_i) \right) \) is a s.f., then \( g \) is n-monotone.

Part (i) of this theorem was shown in [4] (Proposition 2) by completely different methods.

**Proof.** (i) Suppose first \( G \) to be a s.f. Fix \( 0 < c < \infty \) and consider \( g_c(s) := g(cs) \), for \( s \in [0, 1] \). Then \( g_c(M_1(x)) = g \left( \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \right) = G \left( \frac{c}{n} \cdot x \right) \) is a s.f. on \([0, 1]^n\), implying \( g_c \mid [0, 1] \) to be n-monotone by Theorem 8; that is, \( g \mid [0, c] \) is n-monotone for each \( c > 0 \), hence on all of \( \mathbb{R}^+ \).

If \( g \) is n-monotone then by the same reasoning \( x \mapsto G \left( \frac{x}{n} \right) \) is a s.f. on \([0, 1]^n\), for every \( c > 0 \), hence \( G \) itself is a s.f.

(ii) We observe that

\[ H(x) := (h_1(x_1), \ldots, h_n(x_n)) \]

is a semigroup homomorphism of \((\mathbb{R}^+)^n, \circ)\), the result then follows from (i).

(iii) This follows since now \( H \) is even an isomorphism. \quad \square
7. Classical norms and multivariate survival functions

Let \( G \) be the survival function (s.f.) of a probability measure \( \mu \) on \( \mathbb{R}^n \). If \( \sqrt{G} \) is also a s.f. for every \( k \in \mathbb{N} \), \( G \) is called infinitely divisible, and \( \mu \) min-infinitely divisible.

Two well known examples are

\[
G_1(x) := \exp\left(-\sum_{i=1}^{n} x_i\right), \quad x \in \mathbb{R}^n
\]

and

\[
G_\infty(x) := \exp\left(-\max_{i \leq n} x_i\right), \quad x \in \mathbb{R}^n
\]

\( G_1 \) being the s.f. of standard exponential iid random variables \( X_1, \ldots, X_n \), and \( G_\infty \) the s.f. of \( (X_1, \ldots, X_1) \), i.e. of one standard exponential random variable “living” on the diagonal \( \{(u, \ldots, u) \in \mathbb{R}^n \mid u \geq 0\} \). A natural question seems to be, for which values of \( t \in \mathbb{R} \setminus \{0\} \) the functions

\[
N_t(x) := \left(\sum_{i=1}^{n} x_i^t\right)^{1/t}, \quad x \in \mathbb{R}^n
\]

for which \( t > 0 \) are just the classical (semi-) norms, have the property that \( \exp(-N_t) \) is a s.f., and we may complement this family by

\[
N_\infty(x) := \max_{i \leq n} x_i, \quad N_{-\infty}(x) := \min_{i \leq n} x_i.
\]

Due to homogeneity, if \( \exp(-N_t) \) is a s.f., it is automatically infinitely divisible. Clearly, \( M_t = n^{-1/t} \cdot N_t \) could just as well be considered, but here \( N_t \) seems more natural, with the exception of \( t = 0 \). As it turns out, precisely the \( t \)-values corresponding to norms lead to survival functions.

**Theorem 10.** Let \( n \geq 2 \) be given. Then \( \exp(-N_t) \) is a survival function on \( \mathbb{R}^n \) if and only if \( t \in [1, \infty) \), \( \exp(-M_0) \) is not a survival function.

**Proof.** By [7], Theorems 4 and 10, we have to show that \( N_t \) is fully \( n \)-max decreasing if \( t \geq 1 \). For any \( t > 0 \) the function \( x \mapsto \sum_{i=1}^{n} x_i^t \) is fully \( n \)-max decreasing, cf. Remark 1 in [7] and its obvious analogue for the maximum operation. Since fully \( n \)-max decreasing is the same as being completely alternating (or negative definite on the semigroup \((\mathbb{R}^n_+, \lor)\), and since \( s \mapsto s^{1/t} \) is a Bernstein function on \( \mathbb{R}_+ \) for \( t \geq 1 \), their composition \( N_t \) is in this case likewise fully \( n \)-max decreasing.

For \( t \in ]-\infty, 1[ \setminus \{0\} \) we use derivatives. Since

\[
D_a^b \psi = (-1)^{|a|} D_a^b \psi, \quad a, b \in \mathbb{R}^n
\]

for any function \( \psi \) of \( n \) real variables, in case of \( C^n \)-functions we can state that \( \psi \) is fully \( n \)-max decreasing if and only if

\[
(-1)^{|a|} \psi_a \leq 0 \quad \text{for all } \emptyset \neq a \subseteq n.
\]

Therefore, in order to show that \( N_t \) does not have this property, it clearly suffices to consider the case \( n = 2 \). Now

\[
\frac{\partial^2 N_t}{\partial x_1 \partial x_2} = \left(\frac{1}{t} - 1\right) t(x_1, x_2)^{t-1}(x_1^t + x_2^t)^{\frac{1}{t} - 2}
\]

which is positive for \( 0 < t < 1 \) and also for \( t < 0 \).

For the geometric mean \( M_0 \) we have

\[
\frac{\partial^k M_0}{\partial x_1 \cdots \partial x_k} = \frac{1}{n^k} \cdot \prod_{i=1}^{k} x_i^{\frac{1}{n} - 1} > 0
\]

for \( k = 1, \ldots, n \). So \( M_0 \) cannot be fully \( n \)-max decreasing, and neither is \( M_{-\infty} \) because of (again with \( n = 2 \))

\[
D_{(1,1)}^{(0,0)}(M_{-\infty}) = 1 > 0. \quad \square
\]

Combining the last two theorems we obtain the following.

**Corollary 4.** Let \( h_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be increasing and left continuous, \( h_i(0) = 0 \) and \( \lim_{s \rightarrow \infty} h_i(s) = \infty \), for \( i = 1, \ldots, n \). Then for any \( t \in [1, \infty] \)

\[
\exp[-N_t(h_1(x_1), \ldots, h_n(x_n))]
\]

is an infinitely divisible s.f. on \( \mathbb{R}_+^n \).
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References