MATHEMATICS

SELF-COMPLEMENTARY AND SELF-CONVERSE ORIENTED GRAPHS

BY

M. R. SRIDHARAN

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Abstract

De Bruijn [1] applied his generalization of Pólya's fundamental theorem to provide an outline of a general method for enumerating self-complementary structures. This was used by Read [8] to carry out in detail the enumeration of self-complementary graphs and digraphs. Suitable modifications of the same scheme gave Harary and Palmer [4], [5] the basic clue for their enumeration of self-converse digraphs. In this paper we extend these results to obtain the formula for the number of self-complementary oriented graphs on \( n \) points and the generating function for self-converse oriented graphs in terms of the number of lines.

1. Self-complementary oriented graphs

The generating functions for self-complementary graphs and digraphs are respectively \( Z_{G_n}(0, 2, 0, 2, \ldots) \) and \( Z_{D_n}(0, 2, 0, 2, \ldots) \) where \( G_n \) is the configuration group \( S_n^{(2)} \) for graphs and \( D_n \) is the configuration group \( S_n^{[2]} \) for digraphs. When we write analogously \( Z_{Q_n}(0, 2, 0, 2, \ldots) \) for the generating function for self-complementary oriented graphs, \( Q_n \) being the configuration group as defined in [3], we come across a difficulty which is deeper than a mere notational one. For example in [3] Harary observes in one place (p. 221) that \( Q_n \) is a permutation group of degree \( p \) \((p - 1)/2 \) whereas in computing the contribution to oriented graphs certain cycles from some permutations are deleted and the resulting formula \( Z(Q_n) \) is not a proper cycle index in the usually accepted sense of the term. Since we see no obvious way of overcoming this difficulty, instead of deriving the generating function for self-complementary oriented graphs by applying de Bruijn's theorem [1] with \( Q_n \) acting on \( X^{(2)} \) and \( S_2 \) on \{0, 1\}, we start with Read's result for self-complementary digraphs and pick out the oriented graphs from these by eliminating all those permutations of \( D_n \) which give rise to non-oriented graphs.

The cycle index of \( D_n \) was first obtained by Pólya and was described by Harary [2] in his famous expository article dealing with graphical enumeration and is given as

\[
Z_{D_n}(f_1, f_2, \ldots, f_n, \ldots) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{r=0}^{[\frac{n-1}{2}]} \int_{2r+1}^{i_{2r+1}} \int_{2r+1}^{i_{2r+1}} f_{2r}^{\sigma_{2r+1}+1} \int_{2r}^{i_{2r+1}} f_{2r}^{\sigma_{2r}+1} \int_{2r}^{i_{2r}} f_{2r}^{\sigma_{2r}} \cdots \prod_{1 \leq \ell < r < n} \int_{\ell+1}^{i_{\ell+1}} f_{\ell+1}^{\sigma_{\ell+1}+1}
\]

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where \((q, r)\) is the g.c.d and \(\langle q, r \rangle\) is the l.c.m. of \(q\) and \(r\) and \(\alpha\) has cycle structure \((1^2 2^2 3^3 \ldots)\). The self-complementary digraphs on \(n\) points are obtained from this by putting \(j_1 = 0\) or 2 according as \(\nu\) is odd or even. Read [8] has shown that this leads to the conclusion that only permutations of \(S_n\) with cycle structure \((1^2 2^2 4^2 6^6 8^8 \ldots)\) with \(j_1 = 0\) or 1 give non-zero contributions to the expression \(Z_{D_n}(0, 2, 0, 2, \ldots)\).

We now make the modifications in this result so that we get self-complementary oriented graphs. We observe that if \(\alpha \in S_n\) has a cycle whose length is a multiple of four, the resulting self-complementary digraph is not oriented. For example, if \((1234)\) is a cycle of \(\alpha\), the corresponding cycles of the permutation \(g_\alpha\) of \(D_n\) induced by \(\alpha\) are

\[
\begin{align*}
&((1,2)(3,4)(4,1)), ((1,3)(2,4)(3,1)(4,2)), \\
&((2,1)(3,2)(4,3)(1,4))
\end{align*}
\]

Since a self-complementary digraph should satisfy the condition \(f(d) = \beta(g_\alpha d)\) where \(d \in X^{[2]}\) and \(\beta\) is the cycle of length 2 on the symbols 0, 1, the presence of the middle cycle requires that either \(f(1,3) = f(3,1) = 1\) or \(f(2,4) = f(4,2) = 1\). In either case we get a non-oriented digraph. Thus, only permutations of \(S_n\) with cycle structure of the form

\[
(1^1 2^2 6^6 10^2 \ldots)
\]

give nonzero contribution to the generating function.

The main observation in computing the contribution from a typical permutation of the above form is that the contribution gets halved from the corresponding contribution for self-complementary digraphs. The detailed contributions are given below with illustrations:

(i) A cycle of length \(\nu = 4\nu + 2\) in \(\alpha \in S_n\) gives rise to \(\nu - 1\) cycles of length \(\nu\) in \(g_\alpha\) but the contribution from such a cycle to the expression, in which the substitution \(j_\alpha = 0\) or 2 according as \(\nu\) is odd or even has to be made, comes out to be only \(\frac{\nu-2}{2} + 1\) cycles of length \(\nu\).

Example. \(\alpha = (123456)\) gives rise to five cycles of \(g_\alpha\).

\[
\begin{align*}
&((1,2)(2,3)(3,4)(4,5)(5,6)(6,1)) \quad ((2,1)(3,2)(4,3)(5,4)(6,5)(1,6)) \\
&((1,3)(2,4)(3,5)(4,6)(5,1)(6,2)) \quad ((3,1)(4,2)(5,3)(6,4)(1,5)(2,6)) \\
&((1,4)(2,5)(3,6)(4,1)(5,2)(6,3))
\end{align*}
\]

The pairs of cycles in the first two rows may be called converse pairs. If digraphs were permitted, each such pair will correspond to 4 possible combinations obtained from

\[
f(1,2) = \frac{1}{1} \quad \text{and} \quad f(2,1) = \frac{1}{1}
\]

But to obtain an oriented graph the contributions \(f(1,2) = f(2,1) = 1\) and \(f(1,2) = f(2,1) = 0\) have to be disallowed. The last cycle does not give rise
to any such difficulty. Thus the contribution from this cycle to the cycle index for self-complementary oriented graphs is \( f_6 \) where \( j_6 = \frac{6-2}{2} + 1 \). The total contribution from individual cycles of length \( v \) will therefore be

\[
\int_{v}^{(v/2-2)2+1} f_v
\]

(ii) A pair of distinct cycles of length \( v \) gives rise to \( 2v^2 \) point pairs \((i, j)\) which arrange themselves into \( 2v \) cycles of length \( v \) each. These being pairs of converse cycles, the contribution is \( f_v^* \). The total contribution from such pairs is therefore \( f_{v(v-1)2^2} \).

Example. \((12)(34)\) gives rise to the following converse pairs.

\[
((1,3)(2,4)) \quad (3,1)(4,2)) \quad ((3,4)(1,2)) \quad ((4,1)(3,2))
\]

(iii) Any two cycles of unequal lengths \( q = 4a + 2 \) and \( r = 4b + 2 \) give rise to \( (q, r) \) pairs of converse cycles of length \( \langle q, r \rangle \) each and the total contribution from such cycle pairs is \( f_{\langle q, r \rangle}^{(q, r)} \).

(iv) The \( j_1 \) (= 0 or 1) trivial cycles do not give any individual contribution and their contribution in pairs with other cycles can be obtained by the formula in (iii).

Thus we have the following lemma:

Lemma 1. The generating function for self-complementary oriented graphs is

\[
\bar{O}_n = \frac{1}{n!} \sum_{\alpha \in S_n} \prod_{\epsilon \in N} \int_{\epsilon}^{(n-2)/2+1} f_\epsilon + \frac{n}{2} (j_{\epsilon} - 1) f_\epsilon \prod_{\epsilon' \in N'} \int_{\epsilon'}^{(n-2)/2} f_\epsilon' j_{\epsilon'}
\]

where \( N = \{2, 6, 10, 14, \ldots \} \) and \( N' = \{1, 2, 6, 10, 14, \ldots \} \).

Since every cycle in the formula of lemma 1 is of even length the result of putting \( f_\epsilon = 0 \) or 2 according as \( \nu \) is odd or even leads to the following theorem:

Theorem 1. The number of self-complementary oriented graphs is

\[
Y_n = \frac{1}{n!} \sum_{\alpha \in S_n} 2^{\bar{O}_n(\alpha)}
\]

where

\[
\bar{O}_n(\alpha) = \sum_{\epsilon \in N} \frac{\nu}{2} j_\epsilon^2 + \sum_{\epsilon' \in N'} (q, r) j_q j_r.
\]

The numbers of self-complementary oriented graphs with up to 10 points are given in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_n )</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>12</td>
<td>88</td>
<td>176</td>
<td>2752</td>
<td>8784</td>
</tr>
</tbody>
</table>
Since self-complementary oriented graphs must necessarily have \( n(n-1)/2 \) lines and these must be tournaments, that is, complete oriented graphs, it is not difficult to pick them out from digraphs. For example, though there are 1670 digraphs on 5 points with 10 lines [2, p. 453] only 12 of them are oriented graphs [3, p. 224]. These can be easily drawn systematically from the complete oriented graphs on 4 points with 6 lines (Graphs 45–48 in Harary et al. [6]) by adding an additional point and 4 oriented lines. The eight self-complementary oriented graphs with 5 points and 10 lines are given below.

![Fig. 1. Self-complementary oriented graphs with five points.](image)

Since every self-complementary oriented graph is a complete oriented graph and the converse and complement of a complete oriented graph are the same, the self-dual oriented graphs are the same as the self-complementary oriented graphs. The enumeration obtained above therefore partially settles problem 6 in Harary's latest list of unsolved problems [7, p. 30].

2. **Self-Converse Oriented Graphs**

The generating function for self-converse digraphs has been obtained in [5] as

\[
F(S_n^*, S_n, 1+x, 1+x^2, \ldots) = \frac{1}{n!} \sum_{\alpha \in S_n^*} I(\alpha, 1+x, 1+x^2, \ldots)
\]

where

\[
I(\alpha) = \prod_{k=1}^{n} a_{(2,k)} a_{(2,k)}^{(k^2)/2} \prod_{1 \leq r < s \leq n} a_{(2,(r,s),(r,s))}^{(r+s)} I_{r+s}
\]

\[
\times \prod_{k \text{ odd}} a_{2k}^{(k^2)} \prod_{k \text{ even}} a_{k}^{(k-2)} a_{(2k)}^{(1-(\eta(k)))(2k)} a_{k}^{(1-\eta(k))} I_k
\]

where \( \eta(k) = 1 \) if \( k/2 \) is odd and 0 otherwise.
To select the oriented graphs from these, one has to delete those cycles of \( g_a \in S_n^*S_2 \) in which both the pairs of points \((i, j)\) and \((j, i)\) occur. This is so because in enumerating self-converse digraphs the restriction imposed on \( f \) is \( f(d) = f(g_a d) \) so that \((i, j)\) and \((j, i)\) will take the same weight 1 if they appear in the same cycle. Observing further that the action of \( g_a \in S_n^*S_2 \) on \( X^{[2]} \) induced by \( \alpha \in S_n \) is given by \( g_a (i, j) = (\alpha j, \alpha i) \) we arrive at the following conclusions.

(i) Odd cycles of \( \alpha \) give rise to inadmissible cycles of \( g_a \).

Example. \((12345)\) induces the following cycles in \( g_a \).

\[
((1,2)(3,2)(3,4)(5,4)(5,1)(2,1)(2,3)(4,3)(4,5)(1,5)) \nonumber \\
((1,3)(4,2)(3,5)(1,4)(5,2)(3,1)(2,4)(5,3)(4,1)(2,5)).
\]

These cycles are inadmissible because they contain the pairs \((i, j)\) and \((j, i)\) in the same cycle.

(ii) Any two odd cycles of \( \alpha \) give rise to inadmissible cycles of \( g_a \).

Example. \((1)\) \((234)\) gives rise to the cycle

\[
((1,2)(3,1)(1,4)(2,1)(1,3)(4,1)).
\]

In general \((i, j)\) and \((j, i)\) occur with a separation of \((p, q)\) elements, where \( p \) and \( q \) are the lengths of the two odd cycles. It can be verified that this holds even when \( p = q \).

(iii) An even cycle of \( \alpha \), whose length is a multiple of four, gives rise to one inadmissible cycle in \( g_a \) while the other cycles form converse pairs.

Example. \((1234)\) \(\rightarrow\) \((1,2)(3,2)(3,4)(1,4)) \((2,1)(2,3)(4,3)(4,1)) \nonumber \\
\((1,3)(4,2)(3,1)(2,4)).
\]

(iv) Even cycles of \( \alpha \) whose lengths are not multiples of 4 give rise to admissible cycles of \( g_a \) which pair off into converse cycles.

Example. \((123456)\) \(\rightarrow\) \((1,2)(3,2)(3,4)(5,4)(5,6)(1,6)) \nonumber \\
\((2,1)(2,3)(4,3)(4,5)(6,5)(6,1)) \\
\((1,3)(4,2)(3,5)(6,4)(5,1)(2,6)) \\
\((3,1)(2,4)(5,3)(4,6)(1,5)(6,2)) \\
\((1,4)(5,2)(3,6)) \((4,1)(2,5)(6,3)).
\]

(v) Cycle pairs of \( \alpha \) whose lengths are unequal and are both even or one even and one odd give rise to admissible cycles of \( g_a \) which pair off into converse cycles.

If \( I' (\alpha) \) denotes the expression corresponding to \( I (\alpha) \) for self-converse oriented graphs, the above considerations lead to the following calculations.
(1) The contribution to $I'(\alpha)$ from individual even cycles is

$$\prod_{k \in M} a_k^{(k-2)/2} \prod_{k \in N} a_k^{(k-2)/2} a_k a_{k/2}$$

where

$$M = \{4, 8, 12, \ldots\}$$

and

$$N = \{2, 6, 10, \ldots\}$$

(2) The contribution to $I'(\alpha)$ from pairs of even cycles of same length is

$$\prod_{k \in M \cup N} a_k^{(k-1)/2}.$$ 

Combining (1) and (2) we have the following lemma:

Lemma 2. The contribution to $I'(\alpha)$ from all even cycles is

$$\prod_{k \text{ even}} a_k^{i_k(k-2)/2} a_k^{\eta(k)/2}$$

where

$$\eta(k) = \begin{cases} 1 & \text{if } k \in N \\ 0 & \text{if } k \in M \end{cases}$$

(3) Pairs of cycles of $\alpha$ of unequal lengths give the contribution

$$\prod_{1 \leq r < s \leq n} a_{(r,s)}^{i_r i_s}$$

where $r$ and $s$ are not both odd.

Hence we have

Lemma 3.

(3) $I'(\alpha) = \prod_{k \text{ even}} a_k^{i_k(k-2)/2} a_k^{\eta(k)/2} \prod_{1 \leq r < s \leq n} a_{(r,s)}^{i_r i_s}$

Finally we have the following theorem:

Theorem 2. The generating function for self-converse oriented graphs is

$$0'_n(x) = \frac{1}{n} \sum_{\alpha \in S_n} I'(\alpha, 1+2x, 1+2x^2, 1+2x^3, \ldots)$$

The generating functions for these graphs with up to 6 points have been computed and are given below:

$$n \quad 0'_n(x)$$

2  $1 + x$

3  $1 + x + x^2 + 2x^3$

4  $1 + x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 2x^6$

5  $1 + x + 2x^2 + 5x^3 + 9x^4 + 14x^5 + 17x^6 + 18x^7 + 19x^8 + 8x^9 + 8x^{10}$

6  $1 + x + 2x^2 + 6x^3 + 13x^4 + 27x^5 + 45x^6 + 72x^7 + 104x^8 + 123x^9$

$+ 136x^{10} + 112x^{11} + 104x^{12} + 58x^{13} + 32x^{14} + 12x^{15}$.
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Department of Mathematics,
Indian Institute of Technology,
Madras, India.

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