On Locally Supersoluble Groups

B. A. F. Wehrfritz

Department of Mathematics, Queen Mary College,
Mile End Road, London, England, E1 4NS

Communicated by J. E. Roseblade

Received November 21, 1975

A group $G$ is supersoluble if and only if $G$ is finitely generated and has a series

$$\langle 1 \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

(*)

of normal subgroups with abelian factors such that for each $i = 1, 2, \ldots, n$ every subgroup of $G_i$ containing $G_{i-1}$ is normal in $G$ (cf. [4]). More generally, call a group $G$ parasoluble if it just has a series (*) with the above properties. The least $n$ for which a supersoluble (more generally a parasoluble) group $G$ has a series (*) of this type we call the paraheight of $G$. Paraheight corresponds to parasolubility as derived length does to solubility and central height to nilpotency.

An obvious question is whether a group that is locally supersoluble-of-paraheight-$n$ necessarily is parasoluble of paraheight $n$. This question is not as trivial as the corresponding questions for solubility and nilpotency. In [2] Hill shows that such a group is parasoluble with paraheight at most $(3.2^{n-1} - 1 - n)n + 1$. Brazier and Stewart [1] reduce this bound to $2n + 1$.

The main purpose of this note is to give examples to show that the best bound is never $n$ for any $n \geq 3$. We also take the opportunity to give a short proof leading to a slight improvement to the bound $2n + 1$ above. Let $\mathcal{P}_n$ denote the class of all parasoluble groups of paraheight at most $n$ and set $\rho(n)$ equal to the least integer $m$ for which $L\mathcal{P}_n \subseteq \mathcal{P}_m$. We prove that

$$\min\{2n - 3, n\} < \rho(n) \leq 2n - 2 \quad \text{for every} \quad n \geq 2.$$  

Let $H \subseteq K$ be normal subgroups of a group $G$. Suppose that there exists a series

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = K$$

of normal subgroups of $G$ such that for $i = 1, 2, \ldots, n$ every subgroup of $H_i$ containing $H_{i-1}$ is normal in $G$ and $H_i/H_{i-1}$ is abelian. The least $n$ for which
there is such a series we call the $G$-paraheight of $K/H$. We shall use the following simple fact. If a factor $A = H_i/H_{i-1}$ contains elements of infinite order then for each $g \in G$ either $a^g = a$ for all $a \in A$ or $a^g = a^{-1}$ for all $a \in A$.

**The Lower Bound**

Choose distinct odd primes $p$ and $q$ such that $q \equiv 1 \mod 4$. Then there exists an integer $r$ with $r^2 \equiv -1 \mod q$ and $1 < r < q - 1$. Let $A$ be the abelian group generated by elements $a_i$ for $i = 1, 2, \ldots$ subject to the relations $|a_i| = p^i$ for each $i$. Put $B = A \times \langle b \rangle$ where $|b| = \infty$ and $C = B \times \langle c \rangle \times \langle z \rangle$ where $|c| = |z| = q$.

Define automorphisms $d, g_i, i = 1, 2, \ldots$ of $C$ by

\[
\begin{align*}
    a_k^d &= a_k & a_k^{g_i} &= a_k^{-1} \\
    b^d &= b & b^{g_i} &= a_i b^{-1} \\
    c^d &= cz & c^{g_i} &= c^r \\
    z^d &= z & z^{g_i} &= z^{-1}
\end{align*}
\]

for each $i$ and $k$. Clearly $E = \langle g_i; 1 \leq i < \infty \rangle$ is abelian. Since $r^4 \equiv 1 \mod q$ we have

\[
c^{g_{i-1} d} g_i = c^{r^{g_i} d} g_1 = c^{2^r} = c^{d^r}.
\]

Thus $d^{g_i} = d^r$ for each $i$ and in particular $E$ normalizes $\langle d \rangle$. Clearly $|d| = q$.

Let $G$ denote the split extension of $C$ by the subgroup $\langle d \rangle E$ of Aut $C$. Note that $C\langle d \rangle = B \times \langle c, d \rangle$ where the two direct factors here are normal in $G$. Trivially $a_i^2 = [g_i, a_i]$, $b^a = a_i^{-1}[g_i, b]$, $c^{r^{-1}} = [c, g_1]$ and $d^{r^{-1}} = [d, g_1]$. It follows easily in view of the choice of $p$, $q$ and $r$ that $G' = A\langle b^2, c, d \rangle$.

Note that $\langle c, d \rangle \cong \text{Tr}_1(3, q)$ and the latter group has exponent $q$ (for $q > 2$) and a unique nontrivial cyclic normal subgroup, namely its centre, while the centre of $\langle c, d \rangle$ is $\langle z \rangle$. Suppose that $ab^e$ generates a cyclic normal subgroup of $G$ where $a \in A$ and $e \in \mathbb{Z}$. Choose an integer $i > |e|$. Then for some integer $t$ we have

\[
(ab^e)^i = (ab^e)^{g_i} = a^{-1} a_i^e b^{-e}.
\]

If $e \neq 0$ we have $t = -1$, whence $a_i^e = 1$, which contradicts the choice of $i$. Thus $e = 0$. We have now shown that if $x \in G'$ and $\langle x \rangle$ is normal in $G$ then $x \in A\langle x \rangle$. By the definition of $G$ every subgroup of $A\langle x \rangle$ is normal in $G$.

Now modulo $A\langle x \rangle$ the element $b^2$ has infinite order and is inverted by $g_1$,
while module $A\langle x \rangle$ the element $c$ is not inverted by $g_1$ since $r \equiv -1 \mod q$. 
Hence $G'/A\langle x \rangle$ has $G$-paraheight 2 and so $G$ has paraheight exactly 4.

For $n = 1, 2, \ldots$ set 
\[ G_n = \langle a_i, b, c, d, g_i : 1 \leq i \leq n \rangle \subseteq G. \]
Clearly $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots$ and $G = \bigcup_n G_n$. We prove that $G \in L \mathfrak{P}_3$ by showing that $G_n$ has paraheight at most 3. For each $i \leq n$ we have 
\[ (b^p)^{a_i} = a_i^p b - p^n = b - p^n. \]
Therefore $H_n = \langle a_i, b^p, z : 1 \leq i \leq n \rangle$ has $G_n$-paraheight 1. Now $\langle b, H_n \rangle/H_n$ is a cyclic normal $p$-subgroup of $G_n/H_n$ and $\langle c, d, H_n \rangle/H_n$ is a normal $q$-subgroup of $G_n/H_n$. Since $p \neq q$ it follows that $\langle b, c, d, H_n \rangle/H_n$ has $G_n$-paraheight 1. Clearly $G_n' \subseteq \langle b, c, d, H_n \rangle$, so $G_n$ has paraheight at most 3. (In fact $G_n$ has paraheight exactly 3 for every $n$ since $G_n'$ is not abelian.)

For $m \geq 1$ let $S$ denote the split extension of the free abelian group of rank $m$ on the standard basis by the cyclic group generated by the $m \times m$ matrix $(\alpha_{ij})$ acting in the obvious way, where $\alpha_{ij}$ is 0 if $i < j$ and 1 otherwise. It is easily seen that $S$ is supersoluble of paraheight $m$ and that the only parasoluble series (*) of $S$ of length $m$ is the upper central series of $S$. With $G$ as in the above example it is elementary to check that $G \times S$ has paraheight $m + 3$ while each finitely generated subgroup of it has paraheight at most $m + 2$. We have now proved the following result.

**Theorem 1.** $n < \rho(n)$ for every $n \geq 3$.

**The Upper Bound**

Let $\mathfrak{P}_n$ denote the class of all groups $G$ with a series 
\[ \langle 1 \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G \]
of normal subgroups of $G$ of length $n$ such that for $i = 1, 2, \ldots, n$ the group $G_i/G_{i-1}$ is abelian, every subgroup of $G_i$ containing $G_{i-1}$ is normal in $G$ and either $G_i$ is periodic or for all $g \in G$ and $x \in G_i$ we have $x^g \in xG_{i-1} \cup x^{-1}G_{i-1}$. Clearly $\mathfrak{P}_n$ is subgroup closed. We prove the following two lemmas.

**Lemma 1.** $L \mathfrak{P}_n = \mathfrak{P}_n$ for all $n \geq 1$.

**Lemma 2.** $\mathfrak{P}_n \subseteq \mathfrak{P}_{2n-2} \subseteq \mathfrak{P}_{2n-2}$ for all $n \geq 2$. 
As an immediate consequence of Lemmas 1 and 2 we have the following result:

**Theorem 2.** \( \mathfrak{P}_n \subseteq \mathfrak{P}_{n-2} \subseteq \mathfrak{P}_{2n-2} \) for all \( n \geq 2 \).

Thus \( \rho(n) \leq 2n - 2 \) for all \( n \geq 2 \). In particular \( L\mathfrak{P}_2 = \mathfrak{P}_2 \). It is trivial that \( L\mathfrak{P}_1 = \mathfrak{P}_1 \). Note that \( L\mathfrak{P}_n \neq \mathfrak{P}_{2n-2} \) for all \( n \geq 3 \) since \( \mathfrak{P}_n \) contains every finite supersoluble group of paraheight \( m \) and there exist finite supersoluble groups of every paraheight, see [2]. Also \( \mathfrak{P}_n \neq \mathfrak{P}_n \) for all \( n \geq 3 \) since the group \( G \times S \) of the previous section lies in \( L\mathfrak{P}_{m+2} \) but not \( L\mathfrak{P}_{m+2} \subseteq \mathfrak{P}_{m+2} \) for every \( m \geq 1 \).

**Proof of Lemma 1.** Let \( G \in L\mathfrak{P}_n \). If \( X \) is any finite subset of \( G \) let \( S_X \) denote the set of all \( n \)-tuples \( (X_0, X_1, \ldots, X_n) \) of subsets of \( X \) such that the \( \mathfrak{P}_n \)-group \( H = \langle X \rangle \) has a series \( \{H_i\} \) of length \( n \) as in the definition of \( \mathfrak{P}_n \) such that \( X_i = X \cap H_i \) for \( i = 0, 1, \ldots, n \). Clearly \( S_X \) is finite and non-empty. If \( Y \) is a finite subset of \( G \) containing \( X \) define \( \gamma_X^Y: S_Y \to S_X \) by

\[
\gamma_X^Y: (Y_0, Y_1, \ldots, Y_n) \mapsto (X \cap Y_0, X_1 \cap Y_1, \ldots, X \cap Y_n).
\]

Clearly \( (S_X, \gamma_X^Y: X \subseteq Y \subseteq G, Y \text{ finite}) \) is an inverse system of finite non-empty sets over a directed set and as such its inverse limit is not empty, e.g. [3] Section 1.K.

Let \( \{(X_0, X_1, \ldots, X_n)\} \in \varprojlim S_X \) where \( (X_0, X_1, \ldots, X_n) \in S_X \). For \( i = 0, 1, 2, \ldots, n \) set \( G_i = \bigcup X_i \). In the usual way

\[
\langle 1 \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G
\]

is a series of normal subgroups of \( G \) such that for \( i = 1, 2, \ldots, n \) and for each \( X \) we have

\[
G_i' \subseteq G_{i-1} \quad \text{and} \quad X_i = X \cap G_i.
\]

Let \( x, y \in G_j \) and \( g \in G \). If \( x \) has finite order, \( l \) say set \( X = \{x, y, g, x^{-1}x^g, x^{-2}x^g, \ldots, x^{-l}x^g\} \). Otherwise set \( X = \{x, y, g, x^{-1}x^g, xx^g\} \). There exists a series \( \{H_i\} \) of \( H = \langle X \rangle \) as in the definition of \( \mathfrak{P}_n \) such that \( X_i = X \cap H_i \) for each \( i \). In particular we have \( x \in H_j \) and \( g \in H \). If \( x \) has finite order there exists a positive \( r = r(x) \) not exceeding \( l \) such that \( x^{-r}x^g \in X \cap H_{j-1} = X_{j-1} \subseteq G_{j-1} \). If \( x \) has infinite order \( x^r x^g \in X \cap H_{j-1} \subseteq G_{j-1} \) for \( r = 1 \) or \(-1 \) by the definition of \( \mathfrak{P}_n \). Thus in either case \( \langle x \rangle G_{j-1} \) is normal in \( G \).

This shows that \( G \in \mathfrak{P}_n \). To see that \( G \in \mathfrak{P}_n \) recall the element \( y \in X \). If \( G_j \) is not periodic choose \( y \) to be an element of \( G_j \) of infinite order. Then \( y \) is an element of \( H_j \) of infinite order and hence either \( xx^g \) or \( x^{-1}x^g \) lies in \( H_{j-1} \) again by the definition of \( \mathfrak{P}_n \). It follows easily that \( G \in \mathfrak{P}_n \).
Lemma 3. Let $A \subseteq B \subseteq G'$ be normal subgroups $G$ such that $A$ and $B/A$ have $G$-paraheight 1. Suppose that $A$ is torsionfree and $B/A$ is periodic. Then $B$ contains a normal periodic subgroup $T$ of $G$ with $B/T$ torsion free such that $T$ and $B/T$ have $G$-paraheight at most 1.

Proof. Since $B \subseteq G'$ we have that $A$ is central in $B$. By Schur's Theorem $B'$ is periodic and thus trivial. Hence $B$ is abelian. Let $T$ denote the torsion subgroup of $B$. Trivially $A \cap T = \langle 1 \rangle$ so $T$ is $G$-isomorphic to a subgroup of $B/A$ and consequently has $G$-paraheight at most 1. Let $b \in B$ and $g \in G$. For some positive integer $r$ we have $b^r \in A$. Thus $(b^r)^g$ is either $b^r$ or $b^{-r}$. Hence either $(bb^o)^r = 1$ or $(b^{-1}b^o)^r = 1$. In either case we have $b^o \in bT \cup b^{-1}T$ and it follows that $B/T$ has $G$-paraheight 1.

Proof of Lemma 2. Only $\mathfrak{P}_n \subseteq \mathfrak{P}_{2n+2}$ requires proof. Let $G \in \mathfrak{P}_n$ and let

$$\langle 1 \rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

be a series as in the definition of parasoluble group of paraheight at most $n$. Let $T_i$ denote the torsion subgroup modulo $G_{i-1}$ of $G_i$ for $1 \leq i \leq n$ and consider the series

$$\langle 1 \rangle = G_0 \subseteq T_1 \subseteq G_1 \subseteq \cdots \subseteq T_{n-1} \subseteq G_{n-1} \subseteq G_n = G. \quad (\dagger)$$

If $G$ is not in $\mathfrak{P}_n$ then for some $i < n$ the factor $G_i/G_{i-1}$ is periodic. In this case series $(\dagger)$ has a repetition, after the deletion of which series $(\dagger)$ has length $2n - 2$. Using Lemma 3 we can push the periodic factors of $(\dagger)$ down to the bottom and obtain a series still of length $2n - 2$ that satisfies the properties of the series in the definition of $\mathfrak{P}_n$. Thus $\mathfrak{P}_n \subseteq \mathfrak{P}_n \cup \mathfrak{P}_{2n-2}$ and the Lemma follows.

References